Application of Hamiltonian Instability and Diffusion

Workshop on 'Beam Dynamics'
IPAM, January 23 - 27, 2017

Marian Gidea

Yeshiva University - New York

Collaborators: Jaime Burgos, Maciej Capinski, Amadeu Delshams, Rafael de la Llave, Jean-Pierre Marco, Pau Roldan, Tere Seara

Research partially supported by the NSF and the Alfred P. Sloan foundation

Outline

Arnold Diffusion

- Arnold diffusion problem (1964): For typical[1], integrable[2] Hamiltonian systems of $n > 2$ degrees of freedom, when applying small, 'generic'[3] perturbations, there are trajectories for which the action I-variable changes $O(1)$:
  $H_\varepsilon(I, \phi) = H_0(I) + \varepsilon H_1(I, \phi)$,
  $(I, \phi) \in \mathbb{R}^n \times \mathbb{T}^d$,
  $\exists\, (I(t), \phi(t))$ s.t. $\|I(T) - I(0)\| > C$ for all ε sufficiently small
- Energy interpretation: Small forcing applied to a frictionless mechanical system can produce large energy growth
- Optical interpretation: In a periodic optical medium whose index of refraction is arbitrarily close to 1, there exist rays which change direction by a prescribed finite angle (Kaloshin and Levi,2008)

[1] strictly convex and superlinear
[2] in the sense of Liouville-Arnold
[3] from an open-dense, or cusp residual set in C^r, with r large

Arnold Diffusion

- A priori unstable case: the unperturbed Hamiltonian possesses a differentiable family of invariant tori that have hyperbolic invariant manifolds

$$H_\varepsilon(I,\phi,p,q,t) = \underbrace{\overbrace{\frac{I^2}{2}}^{rotator} + \overbrace{\left(\frac{1}{2}p^2 + \cos(q) - 1\right)}^{pendulum}}_{unperturbed} + \underbrace{\varepsilon H_1(p,q,I,\phi,t)}_{perturbation}$$

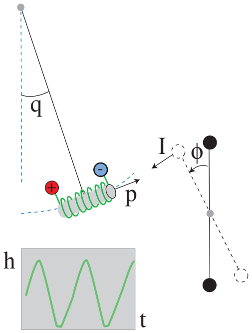
where $(I,\phi,p,q,t) \in M = \mathbb{R}\times\mathbb{T}\times\mathbb{R}\times\mathbb{T}\times\mathbb{T}$

- A priori stable case: the phase space of the unperturbed Hamiltonian is foliated by Lagrangean invariant tori

$$H_\varepsilon(I,\phi) = H_0(I) + \varepsilon H_1(I,\phi)$$

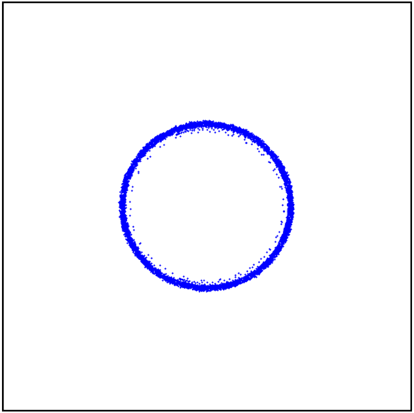
Example

- Example
  - Chain of weakly coupled oscillators (penduli, rotators)



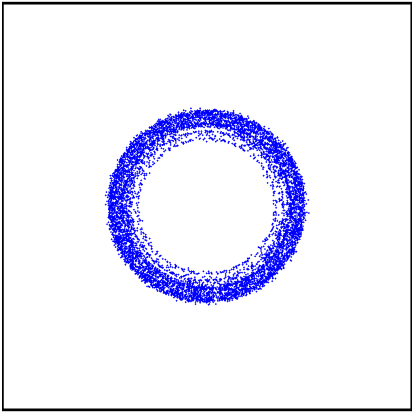
Example

- Example
  - Chain of weakly coupled oscillators (penduli, rotators)



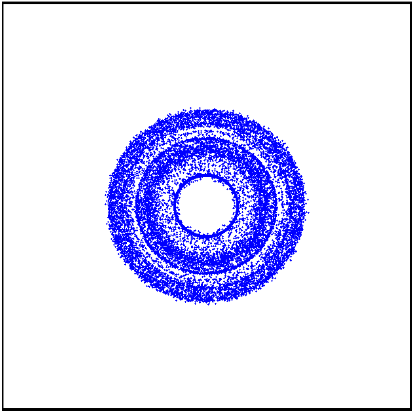
Example

- Example
  - Chain of weakly coupled oscillators (penduli, rotators)



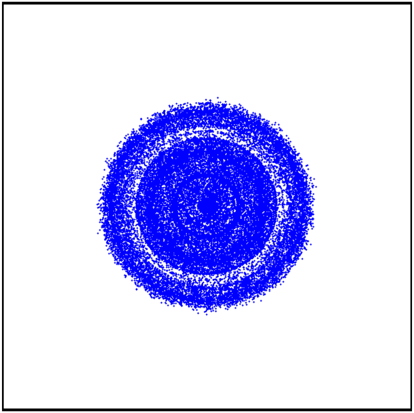
Example

- Example
  - Chain of weakly coupled oscillators (penduli, rotators)



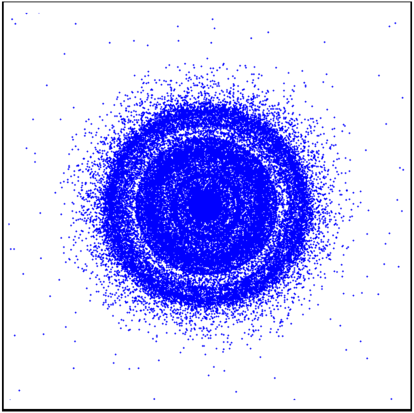
Example

- Example
  - Chain of weakly coupled oscillators (penduli, rotators)



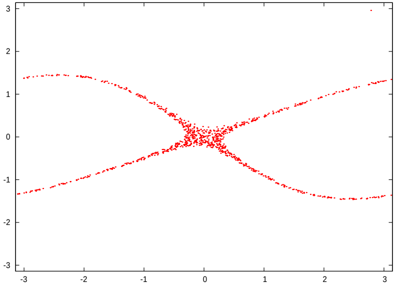
Example

- Example
  - Chain of weakly coupled oscillators (penduli, rotators)



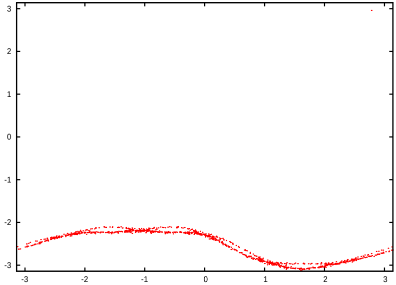
Example

- Example
  - The coupled standard map
    $I_{n+1} = I_n + K\sin(\theta_n) + \varepsilon\sin(\theta_n + \phi_n)$
    $\theta_{n+1} = \theta_n + I_{n+1}$
    $J_{n+1} = J_n + K'\sin(\phi_n) + \varepsilon\sin(\theta_n + \phi_n)$
    $\phi_{n+1} = \phi_n + J_{n+1}$
    $K = 0.85,\ K' = 0.5,\ \varepsilon = 10^{-2}$
    $n = 10^3$ and $n = 10^6$



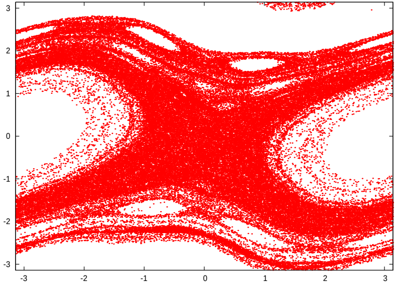
Example

- Example
  - The coupled standard map
    $I_{n+1} = I_n + K\sin(\theta_n) + \varepsilon\sin(\theta_n + \phi_n)$
    $\theta_{n+1} = \theta_n + I_{n+1}$
    $J_{n+1} = J_n + K'\sin(\phi_n) + \varepsilon\sin(\theta_n + \phi_n)$
    $\phi_{n+1} = \phi_n + J_{n+1}$
    $K = 0.85,\ K' = 0.5,\ \varepsilon = 10^{-2}$
    $n = 10^3$ and $n = 10^6$



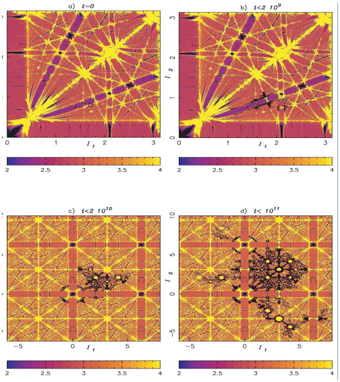
Example

- Example
  - The coupled standard map
    $I_{n+1} = I_n + K\sin(\theta_n) + \varepsilon\sin(\theta_n + \phi_n)$
    $\theta_{n+1} = \theta_n + I_{n+1}$
    $J_{n+1} = J_n + K'\sin(\phi_n) + \varepsilon\sin(\theta_n + \phi_n)$
    $\phi_{n+1} = \phi_n + J_{n+1}$
    $K = 0.85,\ K' = 0.5,\ \varepsilon = 10^{-2}$
    $n = 10^3$ and $n = 10^6$



Arnold Diffusion

- Remarks:
  - diffusion occurs along resonances
  - only few trajectories diffuse a large distance
  - diffusion speed is very slow
  - there is also symbolic dynamics (trajectories with prescribed itineraries)
- Upshot:
  - one can use chaos to control individual trajectories



Credit: Guzzo, Lega and Froeschele

Arnold Diffusion

- Issues with some theoretical results:
  - 'typical' integrable Hamiltonian system may not be generic
  - 'generic' classes of perturbations may be small sets; cannot verify whether a given perturbation is 'generic'
  - perturbation size needs to be very small
  - non-constructive arguments for diffusion

Geometric method

- The dynamics is organized by invariant manifolds:
  - there exist one or several normally hyperbolic invariant manifolds (NHIM)[4]
  - there is an *inner dynamics*, restricted to each NHIM
  - there is an *outer dynamics*, along the homoclinic/heteroclinic intersections of the stable and unstable manifolds of the NHIMs
  - pseudo-orbits formed by the two-dynamics can be *shadowed* by true orbits

[4] NHIM:

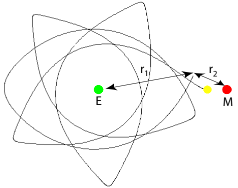
- $\Phi : M \to M$, C^1-smooth flow
- $TM = T\Lambda \oplus E^u \oplus E^s$
- The exponential expansion (contraction) rates of $D\Phi^t$ on $T\Lambda$ are dominated by the expansion (contraction) rates of $D\Phi^t$ on E^u (E^s, resp.)

Geometric method

- Advantages:
  - *explicit construction* of diffusing trajectories
  - *quantitative information* – e.g., diffusion time
  - deal with *general type of integrable Hamiltonians*
  - conditions on generic perturbations are *explicit* and *verifiable by finite precision calculations in concrete systems*
  - deal with *larger size perturbations*
  - deal with perturbations that are not periodic/quasiperiodic – e.g., mildly recurrent, random

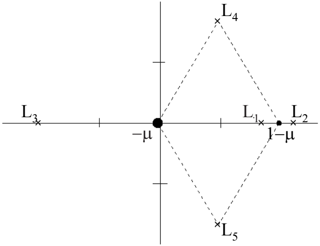
Spatial circular restricted three-body problem

- Analogy between single particle dynamics and planetary motion
- Main goal: use chaos to design zero-cost trajectories with prescribed itineraries
- Motion of satellite in the Sun-Earth system: $H(x, y, z, \dot{x}, \dot{y}, \dot{z}) = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - \omega(x, y, z)$, $\omega(x, y, z) = \frac{1}{2}(x^2 + y^2) + \frac{1-\mu}{r_1} + \frac{\mu}{r_2}$
- Equilibria: L_1, L_2, L_3 - center$\times$center$\times$saddle L_4, L_5 - center$\times$center$\times$center
- For $H = h$, near L_1 there is a 3-dim'l NHIM $\tilde{\Lambda}$ containing many invariant 2-dim'l tori
- Construct diffusing orbits that move from one torus to another in specific ways



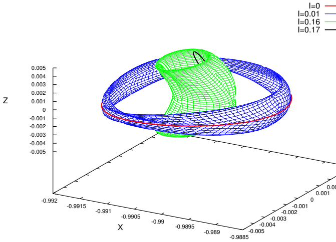
Spatial circular restricted three-body problem

- Analogy between single particle dynamics and planetary motion
- Main goal: use chaos to design zero-cost trajectories with prescribed itineraries
- Motion of satellite in the Sun-Earth system: $H(x, y, z, \dot{x}, \dot{y}, \dot{z}) = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - \omega(x, y, z)$, $\omega(x, y, z) = \frac{1}{2}(x^2 + y^2) + \frac{1-\mu}{r_1} + \frac{\mu}{r_2}$
- Equilibria: L_1, L_2, L_3 - center$\times$center$\times$saddle L_4, L_5 - center$\times$center$\times$center
- For $H = h$, near L_1 there is a 3-dim'l NHIM $\tilde{\Lambda}$ containing many invariant 2-dim'l tori
- Construct diffusing orbits that move from one torus to another in specific ways



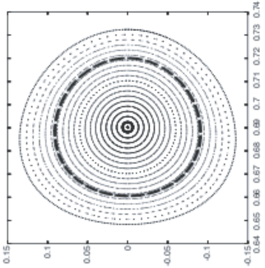
Spatial circular restricted three-body problem

- Analogy between single particle dynamics and planetary motion
- Main goal: use chaos to design zero-cost trajectories with prescribed itineraries
- Motion of satellite in the Sun-Earth system:
  $H(x, y, z, \dot{x}, \dot{y}, \dot{z}) =$
  $\frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - \omega(x, y, z),$
  $\omega(x, y, z) = \frac{1}{2}(x^2 + y^2) + \frac{1-\mu}{r_1} + \frac{\mu}{r_2}$
- Equilibria: L_1, L_2, L_3 - center$\times$center$\times$saddle L_4, L_5 - center$\times$center$\times$center
- For $H = h$, near L_1 there is a 3-dim'l NHIM $\tilde{\Lambda}$ containing many invariant 2-dim'l tori
- Construct diffusing orbits that move from one torus to another in specific ways



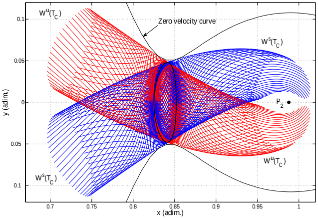
Spatial circular restricted three-body problem

- Analogy between single particle dynamics and planetary motion
- Main goal: use chaos to design zero-cost trajectories with prescribed itineraries
- Motion of satellite in the Sun-Earth system: $H(x, y, z, \dot{x}, \dot{y}, \dot{z}) = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - \omega(x, y, z)$, $\omega(x, y, z) = \frac{1}{2}(x^2 + y^2) + \frac{1-\mu}{r_1} + \frac{\mu}{r_2}$
- Equilibria: L_1, L_2, L_3 - center×center×saddle L_4, L_5 - center×center×center
- For $H = h$, near L_1 there is a 3-dim'l NHIM $\tilde{\Lambda}$ containing many invariant 2-dim'l tori
- Construct diffusing orbits that move from one torus to another in specific ways



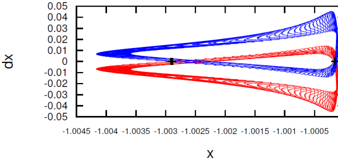
Spatial circular restricted three-body problem

- Analogy between single particle dynamics and planetary motion
- Main goal: use chaos to design zero-cost trajectories with prescribed itineraries
- Motion of satellite in the Sun-Earth system: $H(x, y, z, \dot{x}, \dot{y}, \dot{z}) = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - \omega(x, y, z)$, $\omega(x, y, z) = \frac{1}{2}(x^2 + y^2) + \frac{1-\mu}{r_1} + \frac{\mu}{r_2}$
- Equilibria: L_1, L_2, L_3 - center×center×saddle L_4, L_5 - center×center×center
- For $H = h$, near L_1 there is a 3-dim'l NHIM $\tilde{\Lambda}$ containing many invariant 2-dim'l tori
- Construct diffusing orbits that move from one torus to another in specific ways



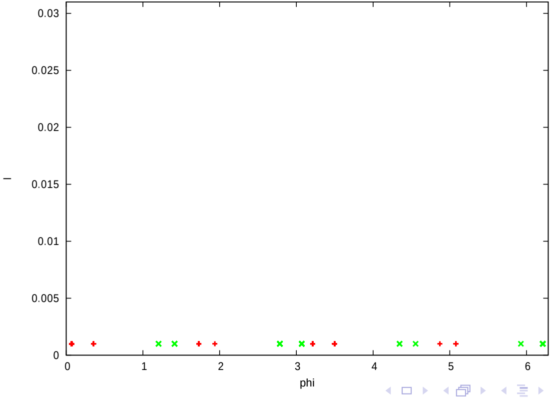
Spatial circular restricted three-body problem

- Analogy between single particle dynamics and planetary motion
- Main goal: use chaos to design zero-cost trajectories with prescribed itineraries
- Motion of satellite in the Sun-Earth system:
  $H(x,y,z,\dot{x},\dot{y},\dot{z}) =$
  $\frac{1}{2}(\dot{x}^2+\dot{y}^2+\dot{z}^2) - \omega(x,y,z)$,
  $\omega(x,y,z) = \frac{1}{2}(x^2+y^2) + \frac{1-\mu}{r_1} + \frac{\mu}{r_2}$
- Equilibria: L_1, L_2, L_3 - center$\times$center$\times$saddle L_4, L_5 - center$\times$center$\times$center
- For $H = h$, near L_1 there is a 3-dim'l NHIM $\tilde{\Lambda}$ containing many invariant 2-dim'l tori
- Construct diffusing orbits that move from one torus to another in specific ways



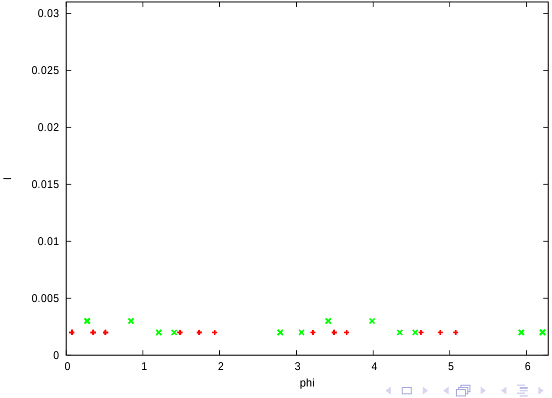
Spatial circular restricted three-body problem

- Compute a Poincaré surface of section Σ and the first return map $\Lambda = \tilde{\Lambda} \cap \Sigma$ — 2-dimensional NHIM
- $\Lambda = \{(I, \phi)\}$ — I out-of-plane amplitude
- Inner dynamics — *twist map*
- Outer dynamics — *scattering map*



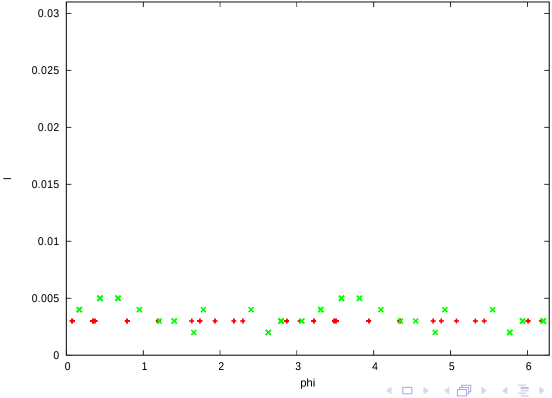
Spatial circular restricted three-body problem

- Compute a Poincaré surface of section Σ and the first return map $\Lambda = \tilde{\Lambda} \cap \Sigma$ — 2-dimensional NHIM
- $\Lambda = \{(I, \phi)\}$ — I out-of-plane amplitude
- Inner dynamics — *twist map*
- Outer dynamics — *scattering map*



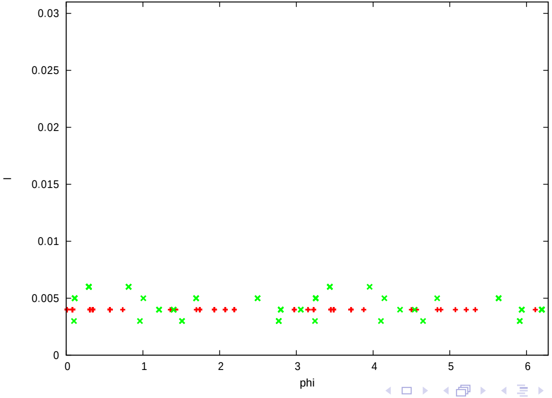
Spatial circular restricted three-body problem

- Compute a Poincaré surface of section Σ and the first return map $\Lambda = \tilde{\Lambda} \cap \Sigma$ — 2-dimensional NHIM
- $\Lambda = \{(I, \phi)\}$ — I out-of-plane amplitude
- Inner dynamics — *twist map*
- Outer dynamics — *scattering map*



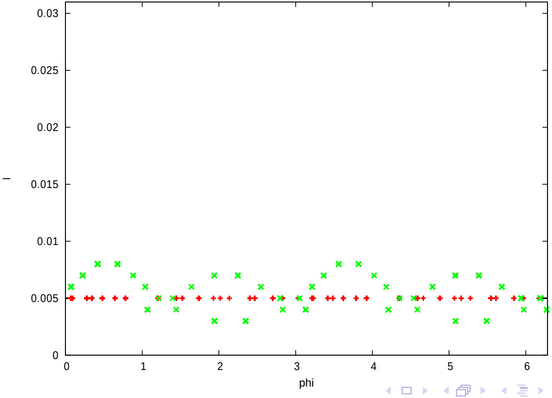
Spatial circular restricted three-body problem

- Compute a Poincaré surface of section Σ and the first return map $\Lambda = \tilde{\Lambda} \cap \Sigma$ — 2-dimensional NHIM
- $\Lambda = \{(I, \phi)\}$ — I out-of-plane amplitude
- Inner dynamics — *twist map*
- Outer dynamics — *scattering map*



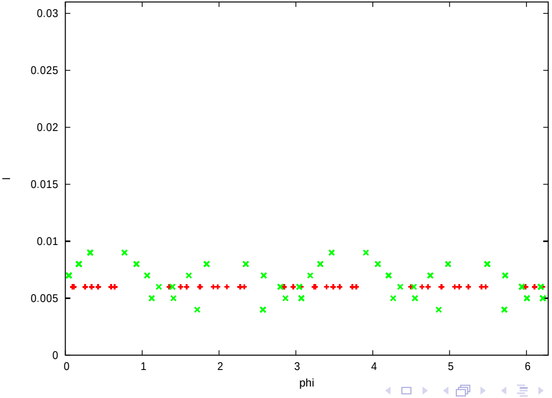
Spatial circular restricted three-body problem

- Compute a Poincaré surface of section Σ and the first return map $\Lambda = \tilde{\Lambda} \cap \Sigma$ — 2-dimensional NHIM
- $\Lambda = \{(I, \phi)\}$ — I out-of-plane amplitude
- Inner dynamics — *twist map*
- Outer dynamics — *scattering map*



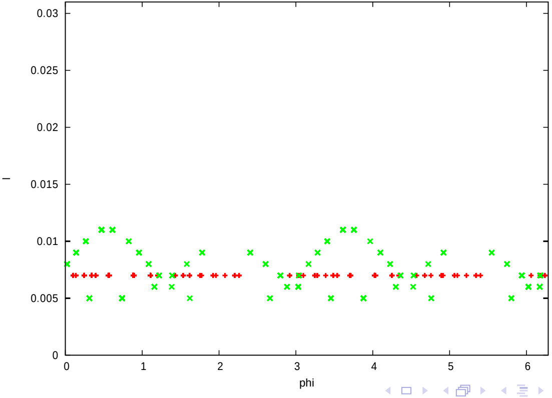
Spatial circular restricted three-body problem

- Compute a Poincaré surface of section Σ and the first return map $\Lambda = \tilde{\Lambda} \cap \Sigma$ — 2-dimensional NHIM
- $\Lambda = \{(I, \phi)\}$ — I out-of-plane amplitude
- Inner dynamics — *twist map*
- Outer dynamics — *scattering map*



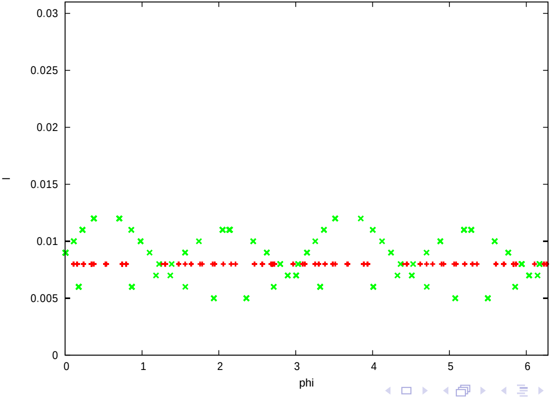
Spatial circular restricted three-body problem

- Compute a Poincaré surface of section Σ and the first return map $\Lambda = \tilde{\Lambda} \cap \Sigma$ — 2-dimensional NHIM
- $\Lambda = \{(I, \phi)\}$ — I out-of-plane amplitude
- Inner dynamics — *twist map*
- Outer dynamics — *scattering map*



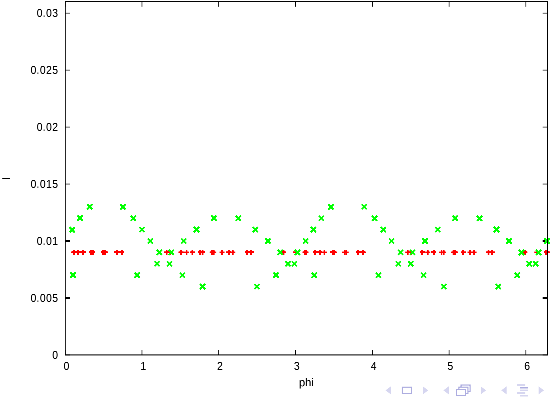
Spatial circular restricted three-body problem

- Compute a Poincaré surface of section Σ and the first return map $\Lambda = \tilde{\Lambda} \cap \Sigma$ — 2-dimensional NHIM
- $\Lambda = \{(I, \phi)\}$ — I out-of-plane amplitude
- Inner dynamics — *twist map*
- Outer dynamics — *scattering map*



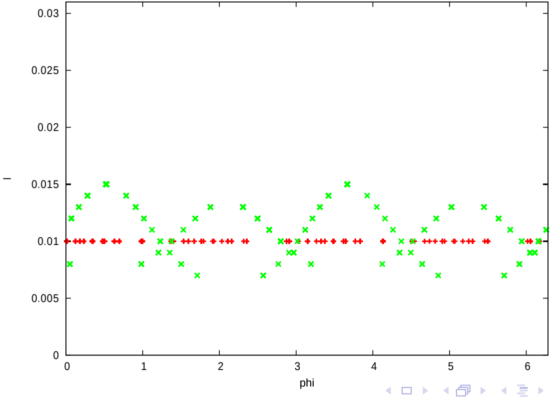
Spatial circular restricted three-body problem

- Compute a Poincaré surface of section Σ and the first return map $\Lambda = \tilde{\Lambda} \cap \Sigma$ — 2-dimensional NHIM
- $\Lambda = \{(I, \phi)\}$ — I out-of-plane amplitude
- Inner dynamics — *twist map*
- Outer dynamics — *scattering map*



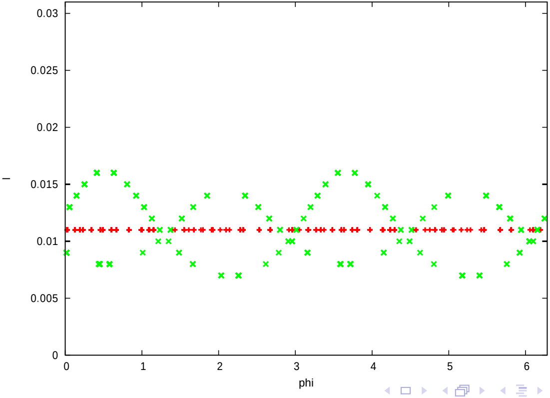
Spatial circular restricted three-body problem

- Compute a Poincaré surface of section Σ and the first return map $\Lambda = \tilde{\Lambda} \cap \Sigma$ — 2-dimensional NHIM
- $\Lambda = \{(I, \phi)\}$ — I out-of-plane amplitude
- Inner dynamics — *twist map*
- Outer dynamics — *scattering map*



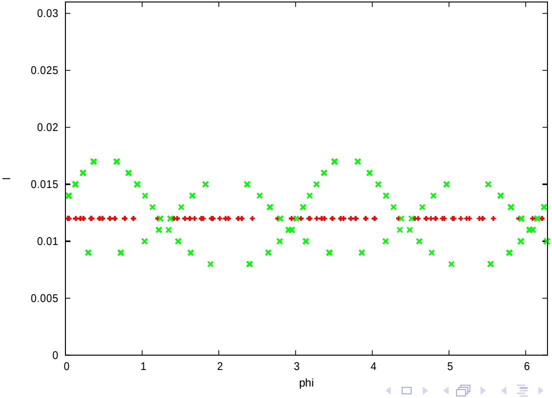
Spatial circular restricted three-body problem

- Compute a Poincaré surface of section Σ and the first return map $\Lambda = \tilde{\Lambda} \cap \Sigma$ — 2-dimensional NHIM
- $\Lambda = \{(I, \phi)\}$ — I out-of-plane amplitude
- Inner dynamics — *twist map*
- Outer dynamics — *scattering map*



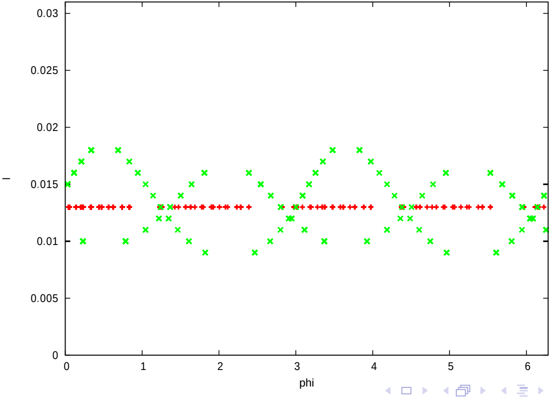
Spatial circular restricted three-body problem

- Compute a Poincaré surface of section Σ and the first return map $\Lambda = \tilde{\Lambda} \cap \Sigma$ — 2-dimensional NHIM
- $\Lambda = \{(I, \phi)\}$ — I out-of-plane amplitude
- Inner dynamics — *twist map*
- Outer dynamics — *scattering map*



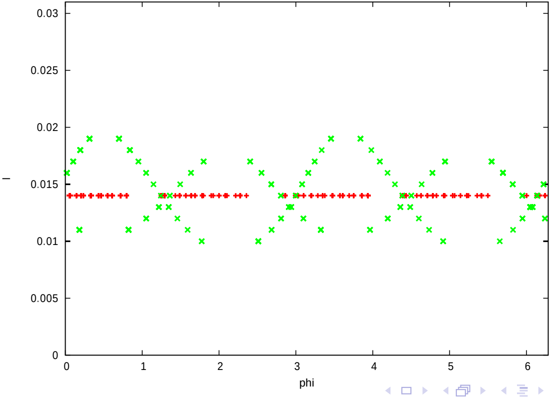
Spatial circular restricted three-body problem

- Compute a Poincaré surface of section Σ and the first return map $\Lambda = \tilde{\Lambda} \cap \Sigma$ — 2-dimensional NHIM
- $\Lambda = \{(I, \phi)\}$ — I out-of-plane amplitude
- Inner dynamics — *twist map*
- Outer dynamics — *scattering map*



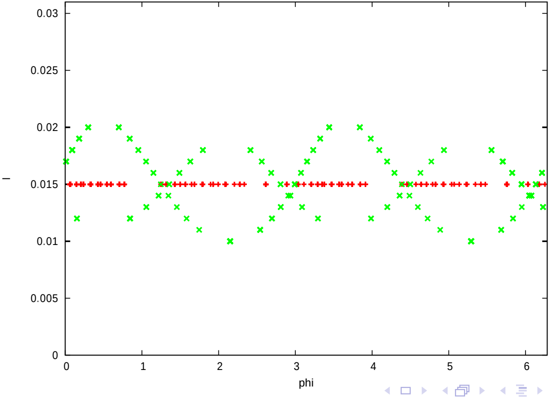
Spatial circular restricted three-body problem

- Compute a Poincaré surface of section Σ and the first return map $\Lambda = \tilde{\Lambda} \cap \Sigma$ — 2-dimensional NHIM
- $\Lambda = \{(I, \phi)\}$ — I out-of-plane amplitude
- Inner dynamics — *twist map*
- Outer dynamics — *scattering map*



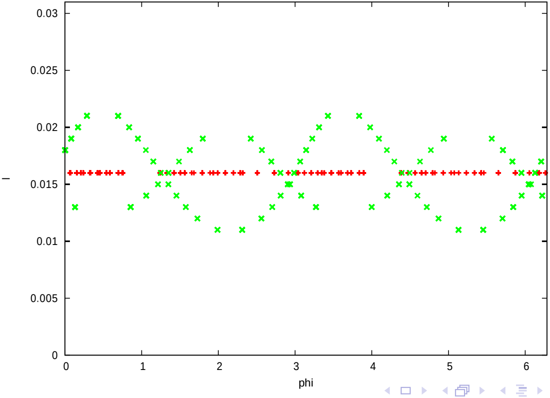
Spatial circular restricted three-body problem

- Compute a Poincaré surface of section Σ and the first return map $\Lambda = \tilde{\Lambda} \cap \Sigma$ — 2-dimensional NHIM
- $\Lambda = \{(I, \phi)\}$ — I out-of-plane amplitude
- Inner dynamics — *twist map*
- Outer dynamics — *scattering map*



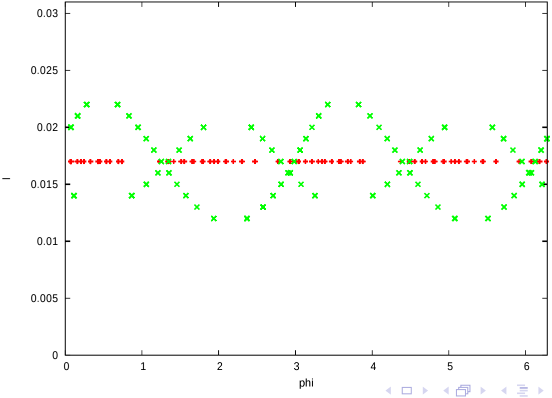
Spatial circular restricted three-body problem

- Compute a Poincaré surface of section Σ and the first return map $\Lambda = \tilde{\Lambda} \cap \Sigma$ — 2-dimensional NHIM
- $\Lambda = \{(I, \phi)\}$ — I out-of-plane amplitude
- Inner dynamics — *twist map*
- Outer dynamics — *scattering map*



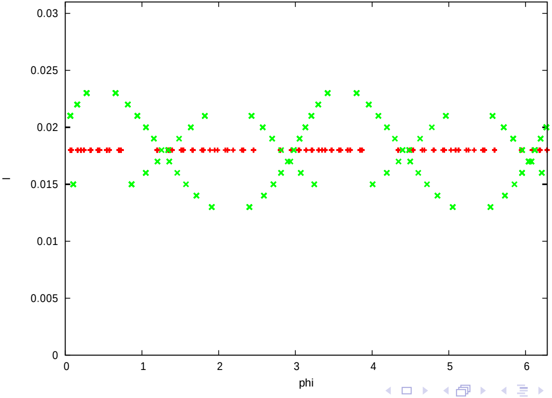
Spatial circular restricted three-body problem

- Compute a Poincaré surface of section Σ and the first return map $\Lambda = \tilde{\Lambda} \cap \Sigma$ — 2-dimensional NHIM
- $\Lambda = \{(I, \phi)\}$ — I out-of-plane amplitude
- Inner dynamics — *twist map*
- Outer dynamics — *scattering map*



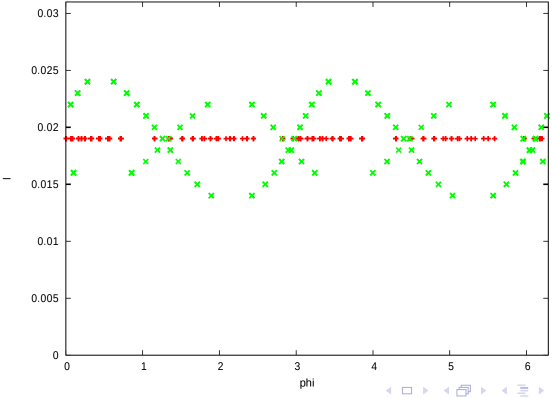
Spatial circular restricted three-body problem

- Compute a Poincaré surface of section Σ and the first return map $\Lambda = \tilde{\Lambda} \cap \Sigma$ — 2-dimensional NHIM
- $\Lambda = \{(I, \phi)\}$ — I out-of-plane amplitude
- Inner dynamics — *twist map*
- Outer dynamics — *scattering map*



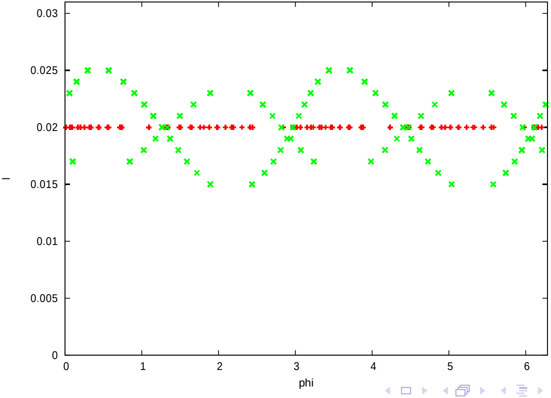
Spatial circular restricted three-body problem

- Compute a Poincaré surface of section Σ and the first return map $\Lambda = \tilde{\Lambda} \cap \Sigma$ — 2-dimensional NHIM
- $\Lambda = \{(I, \phi)\}$ — I out-of-plane amplitude
- Inner dynamics — *twist map*
- Outer dynamics — *scattering map*



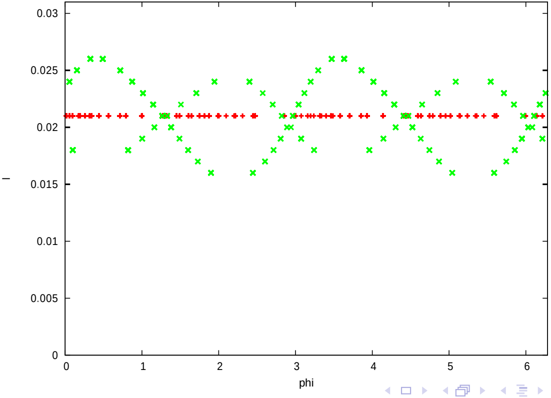
Spatial circular restricted three-body problem

- Compute a Poincaré surface of section Σ and the first return map $\Lambda = \tilde{\Lambda} \cap \Sigma$ — 2-dimensional NHIM
- $\Lambda = \{(I, \phi)\}$ — I out-of-plane amplitude
- Inner dynamics — *twist map*
- Outer dynamics — *scattering map*



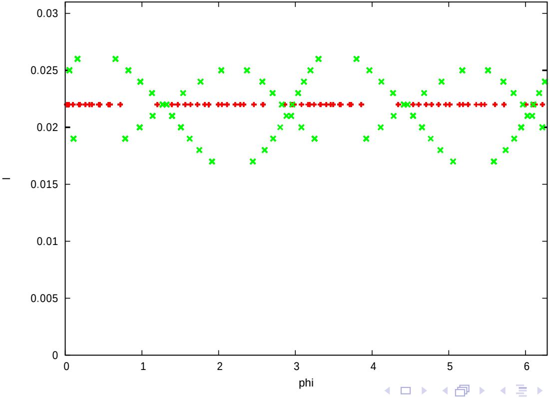
Spatial circular restricted three-body problem

- Compute a Poincaré surface of section Σ and the first return map $\Lambda = \tilde{\Lambda} \cap \Sigma$ — 2-dimensional NHIM
- $\Lambda = \{(I, \phi)\}$ — I out-of-plane amplitude
- Inner dynamics — *twist map*
- Outer dynamics — *scattering map*



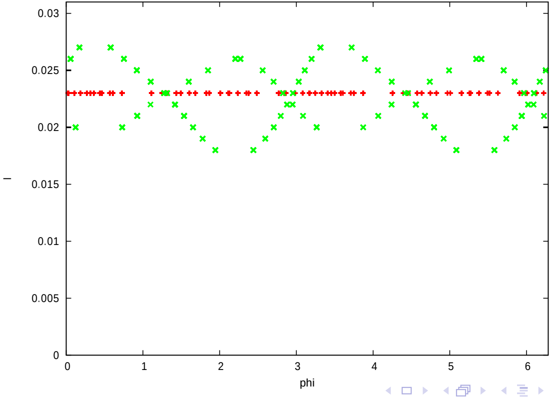
Spatial circular restricted three-body problem

- Compute a Poincaré surface of section Σ and the first return map $\Lambda = \tilde{\Lambda} \cap \Sigma$ — 2-dimensional NHIM
- $\Lambda = \{(I, \phi)\}$ — I out-of-plane amplitude
- Inner dynamics — *twist map*
- Outer dynamics — *scattering map*



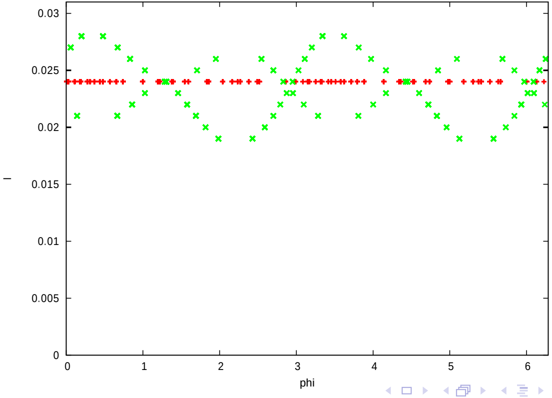
Spatial circular restricted three-body problem

- Compute a Poincaré surface of section Σ and the first return map $\Lambda = \tilde{\Lambda} \cap \Sigma$ — 2-dimensional NHIM
- $\Lambda = \{(I, \phi)\}$ — I out-of-plane amplitude
- Inner dynamics — *twist map*
- Outer dynamics — *scattering map*



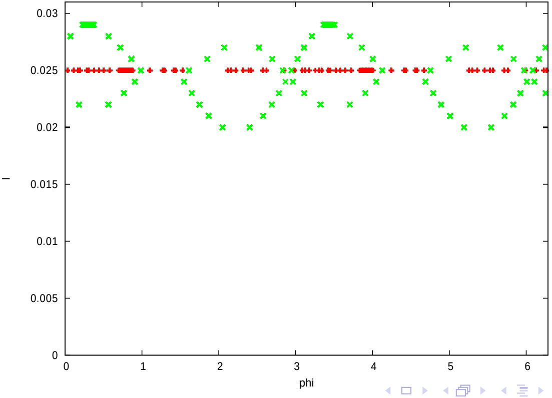
Spatial circular restricted three-body problem

- Compute a Poincaré surface of section Σ and the first return map $\Lambda = \tilde{\Lambda} \cap \Sigma$ — 2-dimensional NHIM
- $\Lambda = \{(I, \phi)\}$ — I out-of-plane amplitude
- Inner dynamics — *twist map*
- Outer dynamics — *scattering map*



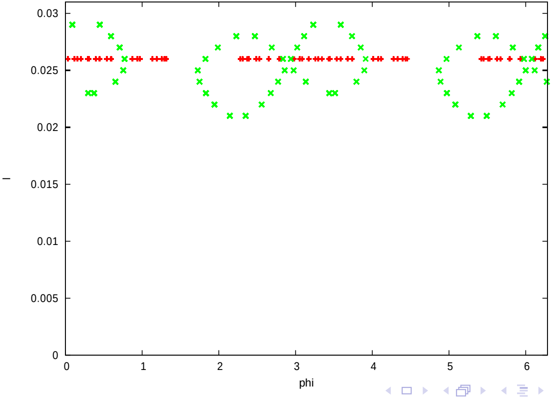
Spatial circular restricted three-body problem

- Compute a Poincaré surface of section Σ and the first return map $\Lambda = \tilde{\Lambda} \cap \Sigma$ — 2-dimensional NHIM
- $\Lambda = \{(I, \phi)\}$ — I out-of-plane amplitude
- Inner dynamics — *twist map*
- Outer dynamics — *scattering map*



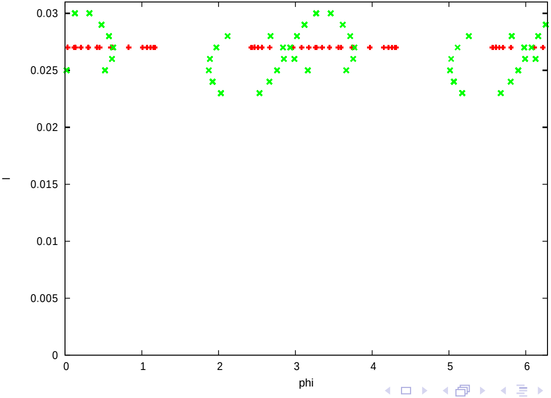
Spatial circular restricted three-body problem

- Compute a Poincaré surface of section Σ and the first return map $\Lambda = \tilde{\Lambda} \cap \Sigma$ — 2-dimensional NHIM
- $\Lambda = \{(I, \phi)\}$ — I out-of-plane amplitude
- Inner dynamics — *twist map*
- Outer dynamics — *scattering map*



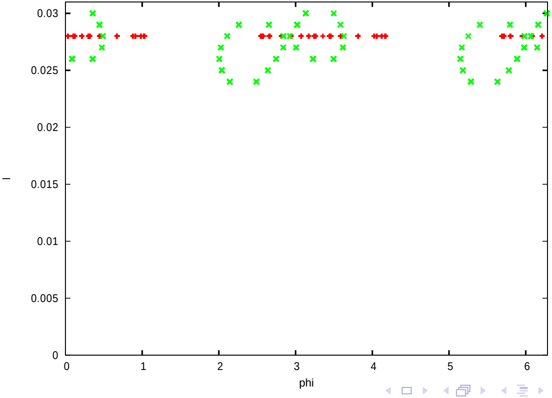
Spatial circular restricted three-body problem

- Compute a Poincaré surface of section Σ and the first return map $\Lambda = \tilde{\Lambda} \cap \Sigma$ — 2-dimensional NHIM
- $\Lambda = \{(I, \phi)\}$ — I out-of-plane amplitude
- Inner dynamics — *twist map*
- Outer dynamics — *scattering map*



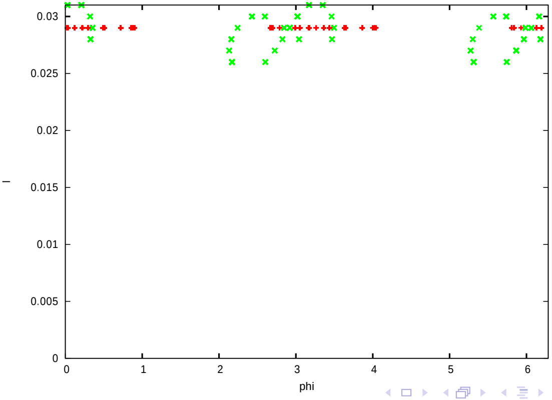
Spatial circular restricted three-body problem

- Compute a Poincaré surface of section Σ and the first return map $\Lambda = \tilde{\Lambda} \cap \Sigma$ — 2-dimensional NHIM
- $\Lambda = \{(I, \phi)\}$ — I out-of-plane amplitude
- Inner dynamics — *twist map*
- Outer dynamics — *scattering map*



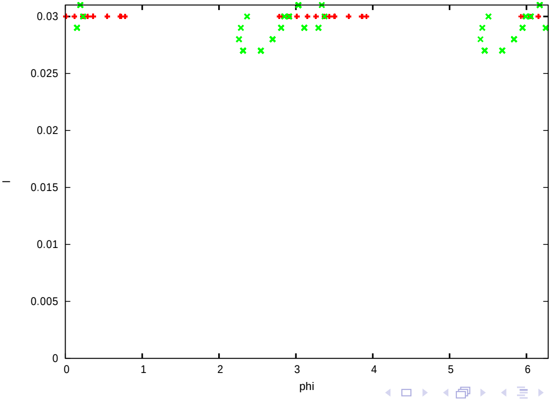
Spatial circular restricted three-body problem

- Compute a Poincaré surface of section Σ and the first return map $\Lambda = \tilde{\Lambda} \cap \Sigma$ — 2-dimensional NHIM
- $\Lambda = \{(I, \phi)\}$ — I out-of-plane amplitude
- Inner dynamics — *twist map*
- Outer dynamics — *scattering map*



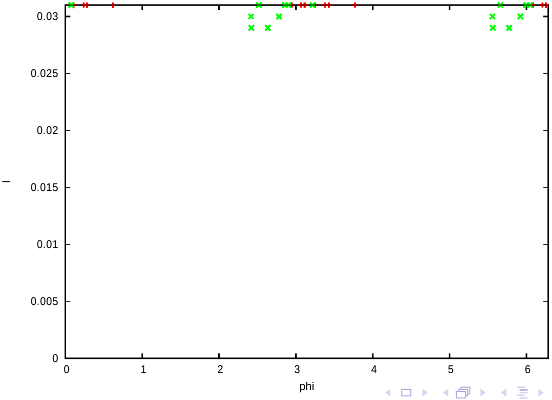
Spatial circular restricted three-body problem

- Compute a Poincaré surface of section Σ and the first return map $\Lambda = \tilde{\Lambda} \cap \Sigma$ — 2-dimensional NHIM
- $\Lambda = \{(I, \phi)\}$ — I out-of-plane amplitude
- Inner dynamics — *twist map*
- Outer dynamics — *scattering map*



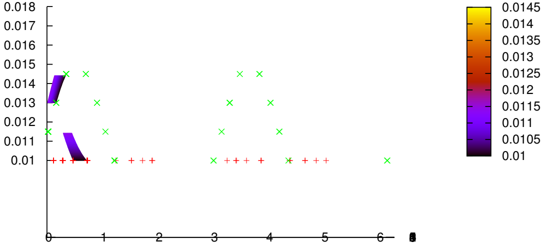
Spatial circular restricted three-body problem

- Compute a Poincaré surface of section Σ and the first return map $\Lambda = \tilde{\Lambda} \cap \Sigma$ — 2-dimensional NHIM
- $\Lambda = \{(I, \phi)\}$ — I out-of-plane amplitude
- Inner dynamics — *twist map*
- Outer dynamics — *scattering map*



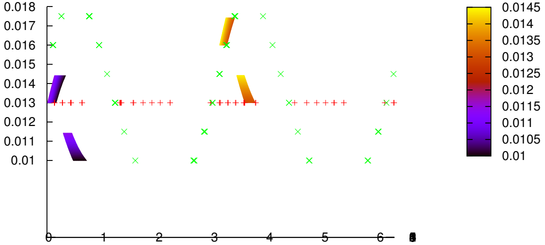
Existence of diffusing orbits

- Construct sequences of 2-dimensional rectangles that are correctly aligned by the outer dynamics (scattering map) and alternately with the inner dynamics (twist map)
- Use topology to conclude the existence of *shadowing orbits*



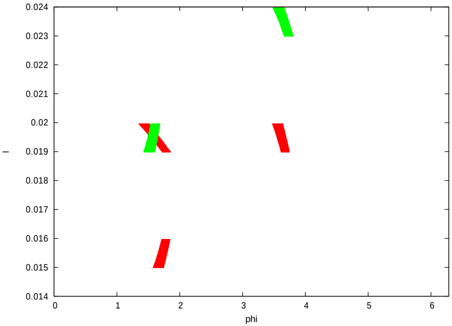
Existence of diffusing orbits

- Construct sequences of 2-dimensional rectangles that are correctly aligned by the outer dynamics (scattering map) and alternately with the inner dynamics (twist map)
- Use topology to conclude the existence of *shadowing orbits*



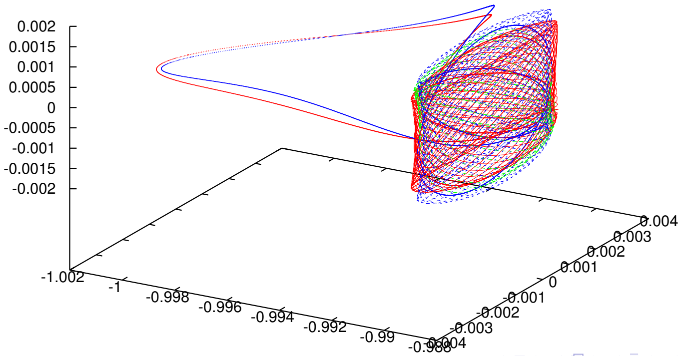
Existence of diffusing orbits

- Construct sequences of 2-dimensional rectangles that are correctly aligned by the outer dynamics (scattering map) and alternately with the inner dynamics (twist map)
- Use topology to conclude the existence of *shadowing orbits*



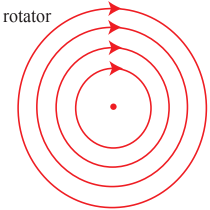
Spatial circular restricted three-body problem

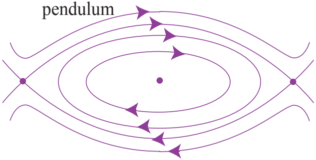
Example: diffusing orbits (2 jumps)



A general result in the a priori unstable case

- Example:
  $H_\varepsilon(p,q,I,\phi,t) =$
  $h_0(I) + \sum_{i=1}^n \pm P_i(p_i,q_i) + \varepsilon H_1(p,q,I,\phi,t)$
  where $P_i(p_i,q_i) = \frac{1}{2}p_i^2 + V_i(q_i)$, and
  $(p,q,I,\phi,t) \in \mathbb{R}^n \times \mathbb{T}^n \times \mathbb{R}^d \times \mathbb{T}^d \times \mathbb{T}^1$.
  Assumptions:
  (A1) V_i, h_0 and H_1 are C^r
  (A2) Each V_i is 1-periodic in q and has a unique non-degenerate global maximum at $(0,0)$; hence, there is a homoclinic orbit to $(0,0)$
  (A3) The perturbation H_1 satisfies some explicit non-degeneracy conditions (depending on H_1 evaluated along the homoclinic family of the unperturbed system)





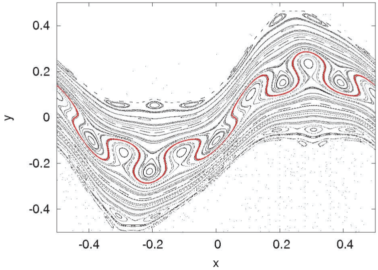
A general result in the a priori unstable case

Theorem (M.G.,de la Llave,Seara,2014)

Assuming the conditions A1-A3, there exists $\varepsilon_0 > 0$, and $\rho > 0$ such that, for each $\varepsilon \in (0, \varepsilon_0)$, there exists a trajectory $x(t)$ such that

$$\|I(x(T)) - I(x(0))\| > \rho \text{ for some } T > 0.$$

- Note:
  - we do not require $\partial^2 h_0/\partial I^2$ to be sign definite or non-degenerate
  - we allow for non-twist dynamics — appear in magnetic field lines in toroidal plasma devices (tokamaks, stellerators), in transport in magnetized plasma, and in satellite dynamics near critical inclination



Credit: del-Castillo-Negrete, Greene, and Morrison

Shadowing Lemma for NHIM's

Shadowing Lemma [M.G.,de la Llave,Seara,2014]

Assume that $f : M \to M$ is a C^r-map, $r \geq 3$, $\Lambda \subseteq M$ is a normally hyperbolic invariant manifold, and $S : U^- \to U^+$ is a scattering map. Assume:

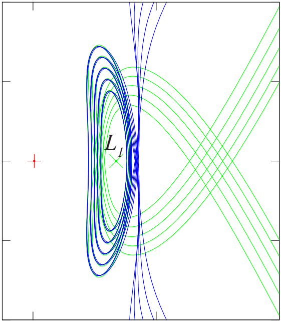
- S area preserving
- almost every point in Λ is recurrent for $f_{|\Lambda}$

Then, given any orbit $\{x_i\}_{i=0,\ldots,n}$ of the scattering map in Λ, i.e. $x_{i+1} = S(x_i)$ for all $i = 0, \ldots, n-1$, for every $\rho > 0$ there exist an orbit $z_{i+1} = f^{k_i}(z_i)$ in M, for some $k_i > 0$, s.t. $d(z_i, x_i) < \rho$ for all $i = 0, \ldots, n$

- Idea of the proof: Apply Poincaré Recurrence to f to return close to the x_i's
- Remark: one can use several scattering maps rather than a single one

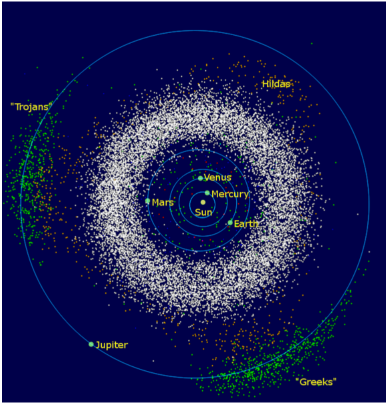
Planar elliptic restricted three-body problem

- Planar elliptic restricted three-body problem: the primaries move on elliptic orbits of eccentricities ε instead of circular orbits
- Model: motion of Oterma comet in the Sun-Jupiter system
- Hamiltonian
  $H_\varepsilon(\mathbf{x}, t) = H_0(\mathbf{x}) + \varepsilon H_1(\mathbf{x}, t)$,
  H_0 is the Hamiltonian of the PCRTBP
- Theorem [Capiński, M.G., de la Llave, 2014]: There exists $\varepsilon_0 > 0$ and $\rho > 0$ such that for each $0 < \varepsilon < \varepsilon_0$ there exists $\mathbf{x}(t)$ such that $\|H_0(\mathbf{x}(T)) - H_0(\mathbf{x}(0))\| > \rho$



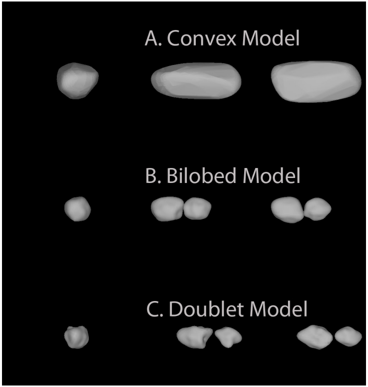
Restricted four-body problem

- Model: motion of moon/spacecraft near a Trojan asteroid in the Sun-Jupiter-(624) Hektor system
- Hill approximation – four equilibria L_1, L_2, L_3, L_4
- The stable and unstable manifolds $W^{s,u}(\lambda_i)$, of the periodic orbit λ_i around L_i, $i = 1, 2$, intersect both in the interior region and in the exterior region
- Complicated dynamics in the inner region – possible explanation for the orbit of Hektor's moon
- The perturbative effect of eccentricity of Jupiter – Arnold diffusion



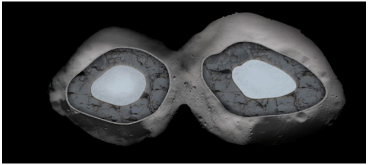
Restricted four-body problem

- Model: motion of moon/spacecraft near a Trojan asteroid in the Sun-Jupiter-(624) Hektor system
- Hill approximation – four equilibria L_1, L_2, L_3, L_4
- The stable and unstable manifolds $W^{s,u}(\lambda_i)$, of the periodic orbit λ_i around L_i, $i = 1, 2$, intersect both in the interior region and in the exterior region
- Complicated dynamics in the inner region – possible explanation for the orbit of Hektor's moon
- The perturbative effect of eccentricity of Jupiter – Arnold diffusion



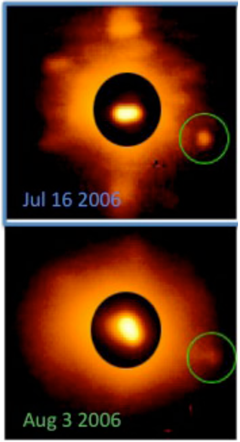
Restricted four-body problem

- Model: motion of moon/spacecraft near a Trojan asteroid in the Sun-Jupiter-(624) Hektor system
- Hill approximation – four equilibria L_1, L_2, L_3, L_4
- The stable and unstable manifolds $W^{s,u}(\lambda_i)$, of the periodic orbit λ_i around L_i, $i = 1, 2$, intersect both in the interior region and in the exterior region
- Complicated dynamics in the inner region – possible explanation for the orbit of Hektor's moon
- The perturbative effect of eccentricity of Jupiter – Arnold diffusion



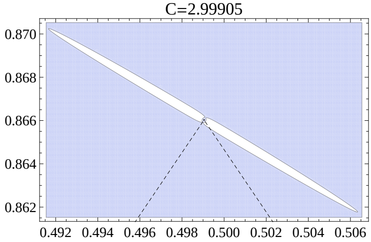
Restricted four-body problem

- Model: motion of moon/spacecraft near a Trojan asteroid in the Sun-Jupiter-(624) Hektor system
- Hill approximation – four equilibria L_1, L_2, L_3, L_4
- The stable and unstable manifolds $W^{s,u}(\lambda_i)$, of the periodic orbit λ_i around L_i, $i = 1, 2$, intersect both in the interior region and in the exterior region
- Complicated dynamics in the inner region – possible explanation for the orbit of Hektor's moon
- The perturbative effect of eccentricity of Jupiter – Arnold diffusion



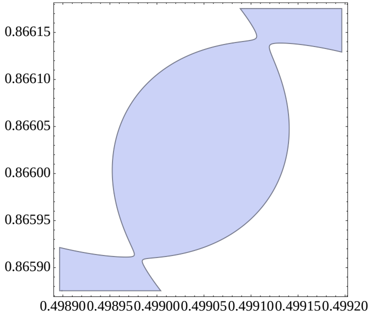
Restricted four-body problem

- Model: motion of moon/spacecraft near a Trojan asteroid in the Sun-Jupiter-(624) Hektor system
- Hill approximation – four equilibria L_1, L_2, L_3, L_4
- The stable and unstable manifolds $W^{s,u}(\lambda_i)$, of the periodic orbit λ_i around L_i, $i = 1, 2$, intersect both in the interior region and in the exterior region
- Complicated dynamics in the inner region – possible explanation for the orbit of Hektor's moon
- The perturbative effect of eccentricity of Jupiter – Arnold diffusion



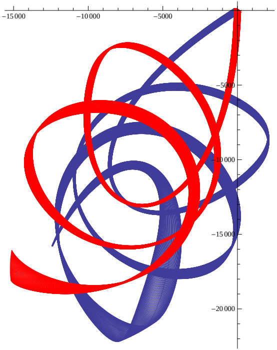
Restricted four-body problem

- Model: motion of moon/spacecraft near a Trojan asteroid in the Sun-Jupiter-(624) Hektor system
- Hill approximation – four equilibria L_1, L_2, L_3, L_4
- The stable and unstable manifolds $W^{s,u}(\lambda_i)$, of the periodic orbit λ_i around L_i, $i = 1, 2$, intersect both in the interior region and in the exterior region
- Complicated dynamics in the inner region – possible explanation for the orbit of Hektor's moon
- The perturbative effect of eccentricity of Jupiter – Arnold diffusion



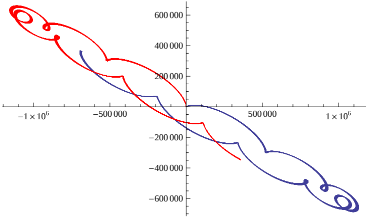
Restricted four-body problem

- Model: motion of moon/spacecraft near a Trojan asteroid in the Sun-Jupiter-(624) Hektor system
- Hill approximation – four equilibria L_1, L_2, L_3, L_4
- The stable and unstable manifolds $W^{s,u}(\lambda_i)$, of the periodic orbit λ_i around L_i, $i = 1, 2$, intersect both in the interior region and in the exterior region
- Complicated dynamics in the inner region – possible explanation for the orbit of Hektor's moon
- The perturbative effect of eccentricity of Jupiter – Arnold diffusion



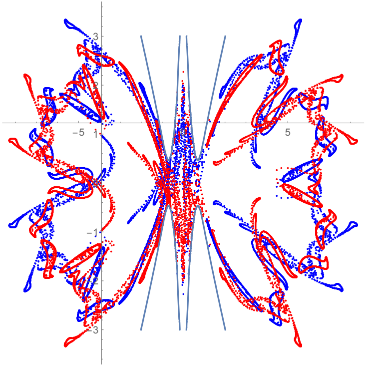
Restricted four-body problem

- Model: motion of moon/spacecraft near a Trojan asteroid in the Sun-Jupiter-(624) Hektor system
- Hill approximation – four equilibria L_1, L_2, L_3, L_4
- The stable and unstable manifolds $W^{s,u}(\lambda_i)$, of the periodic orbit λ_i around L_i, $i = 1, 2$, intersect both in the interior region and in the exterior region
- Complicated dynamics in the inner region – possible explanation for the orbit of Hektor's moon
- The perturbative effect of eccentricity of Jupiter – Arnold diffusion



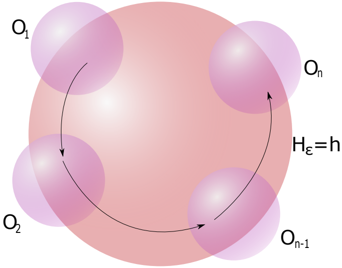
Restricted four-body problem

- Model: motion of moon/spacecraft near a Trojan asteroid in the Sun-Jupiter-(624) Hektor system
- Hill approximation – four equilibria L_1, L_2, L_3, L_4
- The stable and unstable manifolds $W^{s,u}(\lambda_i)$, of the periodic orbit λ_i around L_i, $i = 1, 2$, intersect both in the interior region and in the exterior region
- Complicated dynamics in the inner region – possible explanation for the orbit of Hektor's moon
- The perturbative effect of eccentricity of Jupiter – Arnold diffusion



A general result in the a priori stable case

- Given:
  - $H_\varepsilon(I,\phi) = H_0(I) + \varepsilon H_1(I,\phi)$, C^2-Hamiltoninan, $(I,\phi) \in \mathbb{A}^3 = \mathbb{R}^3 \times \mathbb{T}^3$
  - H_0 strictly convex
- Then:
  - for every $O_1, O_2, \ldots O_n$ open sets in 'action space' $\mathbb{R}^3$, h regular value of H_0, s.t. $O_j \cap \{H = h\} \neq \emptyset$, then for for εH_1 cusp residual there exists $\Phi^t_\varepsilon(x)$ in $\{H_\varepsilon = h\}$ with $I(\Phi^{t_j}_\varepsilon(x)) \in O_j$ for some t_j



A general result in the a priori stable case

- Steps:
  - I. Resonances determine 'chains of cylinders' [Marco,2012,2015]
  - II. Existence of diffusing orbits under certain conditions [M.G. and Marco,2015]
  - III. 'Cusp genericity' of those conditions [Marco,2015]
- Approach for II:
  - Geometric – goes back to Birkhoff's theory on connecting orbits; related approaches
  - Constructive – diffusing orbits can be found explicitly (via an algorithm)