Stability of the phase motion in race-track microtons

Y. A. Kubyshin (UPC), O. Larreal (U. Zulia, Venezuela), T. M-Seara (UPC), R. Ramirez-Ros (UPC)

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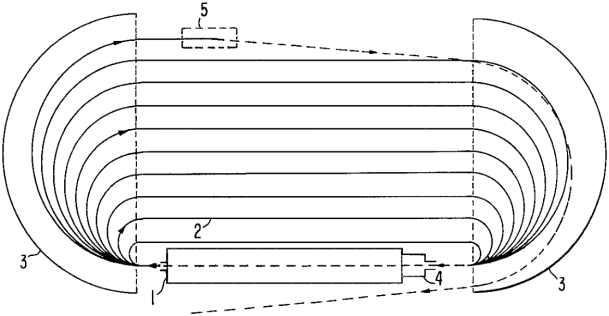
Outline

Schematic view of our RTM model



1: Accelerating structure (AS), 2: Drift space, 3: End magnets,
4: Electron source, and 5: Extraction magnet.

The dynamical variables (ϕ_n, E_n)

- We will take as variables the full particle energy E and the phase ϕ of the accelerating field (radiofrequency field) when the particle is at some (arbitrary) fixed point of the accelerating structure (AS) (for instance the midle point).
- ϕ is usually called the particle phase.
- We call (ϕ_n, E_n) the values of these variables at the n-th turn.

Some simplifications

- We assume that the AS has zero lengh: the gain of energy is given by $\Delta_{max} \cos\phi$, where:
  Δ_{max} is the maximum energy gain ($\Delta_{max} = V_0 e$, V_0 is the voltage, e is the elementary charge),
  ϕ is the phase of the particle.
- We assume that the injected electrons are already ultra-relativistic, so that the velocity of the particles in the beam is equal to the velocity of light c.
- The end magnets will be considered as hard-edge dipole magnets so that the fringe-field effects are not taken into account.

The evolution of the dynamical variables (ϕ_n, E_n)

Duration of the n-th revolution of the beam:

$$T_n = \frac{2l + 2\pi r_n}{c} = \frac{2l}{c} + \frac{2\pi E_n}{ec^2 B},$$

l: separation between the magnets (straight section length)
r_n is the beam trajectory radius in the end magnets (given by $E_n = ecBr_n$, e is the elementary charge, B is the magnetic field induction in the end magnets, E_n energy of the beam).
The beam dynamics in the phase-energy variables is governed by the difference equations

$$\phi_{n+1} = \phi_n + 2\pi T_n / T_{\mathrm{RF}}, \qquad E_{n+1} = E_n + \Delta_{\max} \cos \phi_{n+1},$$

T_{RF}: Period of the accelerating electromagnetic field

Resonance conditions

The microtron is designed in such a way that the phase ϕ_s of the ideal sinchronous particle when passes through the AS is always the same (modulo 2π) at any turn.

We call ϕ_s the synchronous phase.

- Recall that $T_n = T(E_n)$ be the duration of the n-th turn.
- Fix two positive integers m and k and impose the *resonance conditions* on the *synchronous trajectory*, and labeled as 's'.

$$\frac{T_{1,\mathrm{s}}}{T_{\mathrm{RF}}} = m, \qquad \frac{T_{n+1,\mathrm{s}} - T_{n,\mathrm{s}}}{T_{\mathrm{RF}}} = k,$$

For simplicity, we will assume that $k = 1$ in the rest of the talk.

Synchronous trajectory

Then the dynamics of the synchronous trajectory is

$$\phi_{n,s} = \phi_s + 2\pi n\,[m + k(n-1)] \equiv \phi_s \ (\text{mod } 2\pi), \quad E_{n,s} = E_s + n\Delta_s,$$

so its energy undergoes a constant gain at each turn:

$$\Delta_s = \Delta_{max} \cos \phi_s$$

The synchronous trajectory $(\phi_{n,s}, E_{n,s})$ is a solution of the difference equations

$$\phi_{n+1} = \phi_n + 2\pi T_n / T_{RF}, \qquad E_{n+1} = E_n + \Delta_{max} \cos \phi_{n+1},$$

Main goals

The “real particles” oscillate around the ideal synchronous one.

We want to give an accurate description of the longitudinal acceptance (region of stability of these oscillations) in terms of the synchronous phase ϕ_s, that will be taken as parameter.

We will obtain:

- An analytical description of the acceptance for small values of ϕ_s and for ϕ_s near some ressonant values with small error.
- An accurate numerical method which gives the acceptance for values of ϕ_s with physical interest.

Model map

We introduce the variables

$$\psi_n = \phi_n - \phi_s, \qquad w_n = 2\pi k(E_n - E_{n,s})/\Delta_s,$$

that describe the phase and energy deviation of an arbitrary trajectory from the synchronous one. Then the beam dynamics is modeled by the map $(\psi_1, w_1) = f(\psi, w)$, defined by

$$\begin{cases} \psi_1 &= \psi + w, \\ w_1 &= w + 2\pi(\cos\psi_1 - 1) - \mu \sin\psi_1, \end{cases}$$

where $\mu = 2\pi \tan\phi_s$ is a parameter, ψ (respectively, w) is the deviation of the phase (respectively, energy) of an arbitrary trajectory from the phase (respectively, energy) of the synchronous trajectory, which corresponds to the fixed point

$$p_s = (\psi_s, w_s) = (0, 0).$$

Stability domain (or Acceptance) $\mathcal{A}$

We will study the size and the shape of the stability domain

$$\mathcal{A} = \mathcal{A}_\mu = \{p \in \mathbf{T} \times \mathbf{R} : (w_n)_{n \in \mathbf{Z}} \text{ is bounded}\},$$

where $p_n = (\psi_n, w_n) = f^n(p)$. We will also study the size and the shape of the connected component $\mathcal{D}_\mu \subset \mathcal{A}_\mu$ containing p_s.

Remark

We have experimentally checked that

$$\mathcal{A}_\mu \subset [-0.45, 0.35] \times [-0.8, 0.8], \qquad \forall \mu \in (0, 4.6)$$

Linear behavior at $p_s = (0,0)$

From the linear part of the map f at the fixed point $p_s = (0,0)$ we have that it is:

- Hyperbolic iff $\mu \notin [0,4]$ ($\phi_s \notin [0, 32.48°]$);
- Parabolic iff $\mu \in \{0,4\}$ ($\phi_s \in \{0, 32.48°\}$); and
- Elliptic iff $\mu \in (0,4)$ ($\phi_s \in (0, 32.48°)$), in which case the eigenvalues of of the linear part of the map f at the fixed point p_s are $e^{\pm\theta i}$, where $\cos\theta = 1 - \mu/2$.
  - If $\theta/2\pi$ is a rational number we are at a resonance.
  - First, second, third, and fourth order resonances at:
    $\mu = 0$ ($\phi_s = 0°$) where $\theta = 0$,
    $\mu = 4$ ($\phi_s = 32.48°$), where $\theta = \pi$,
    $\mu = 3$ ($\phi_s = 25.52°$) where $\theta = 2\pi/3$
    $\mu = 2$ ($\phi_s = 17.66°$) where $\theta = \pi/2$.

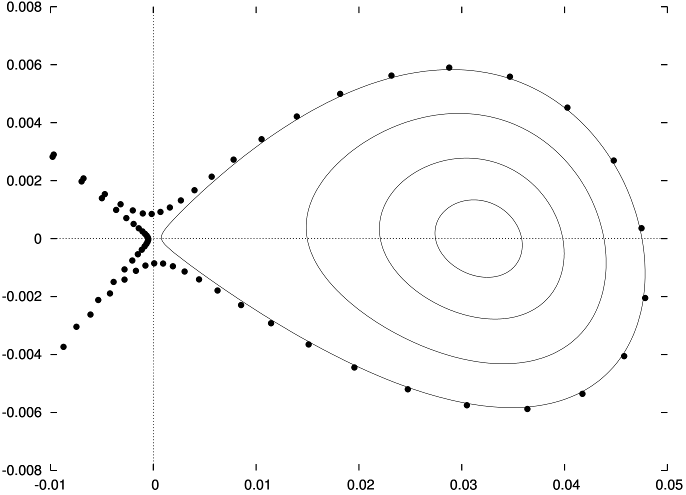
Linear behavior at $p_h = (\psi_h, w_h) = (-2\phi_s, 0)$

The map has another fix point at $p_h = (-2\phi_s, 0)$ which is hyperbolic for any $\mu > 0$.
The eigenvalues of the linear part of the map f at the fixed point p_h are $e^{\pm h}$, where $\cosh h = 1 + \mu/2$.

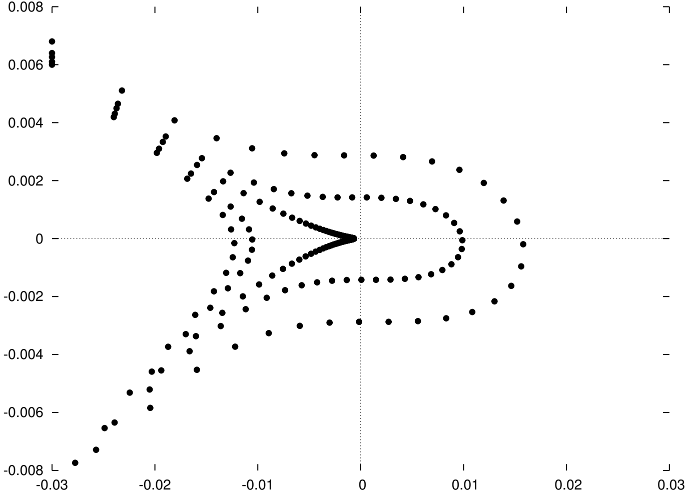
Definition

The quantity $\theta/2\pi \in (0, 1/2)$ is the *rotation number* of the elliptic fixed point p_s. The quantity $h > 0$ is the *characteristic exponent* of the hyperbolic fixed point p_h.

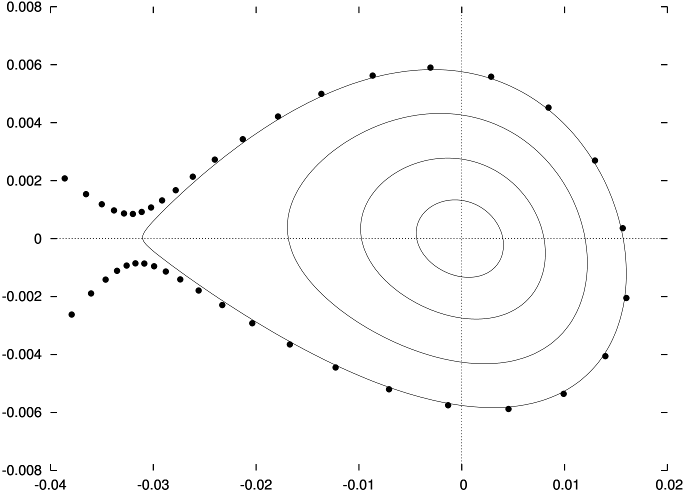
Phase portrait for $\mu = -0.1$ ($\phi_s = -0,9°$)



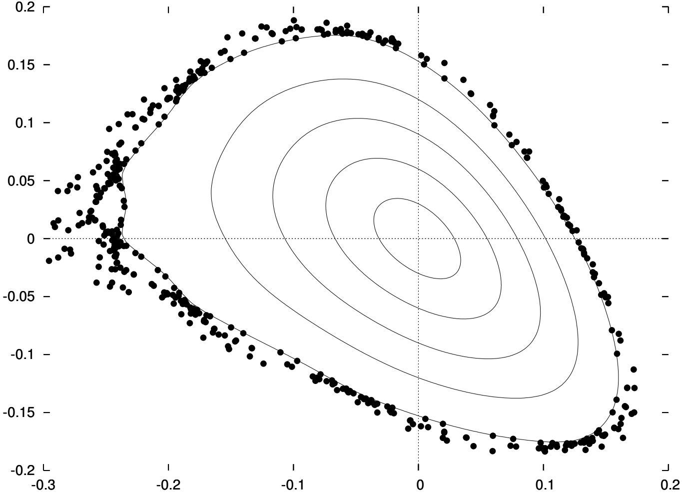
Phase portrait for $\mu = 0$ ($\phi_s = 0°$)



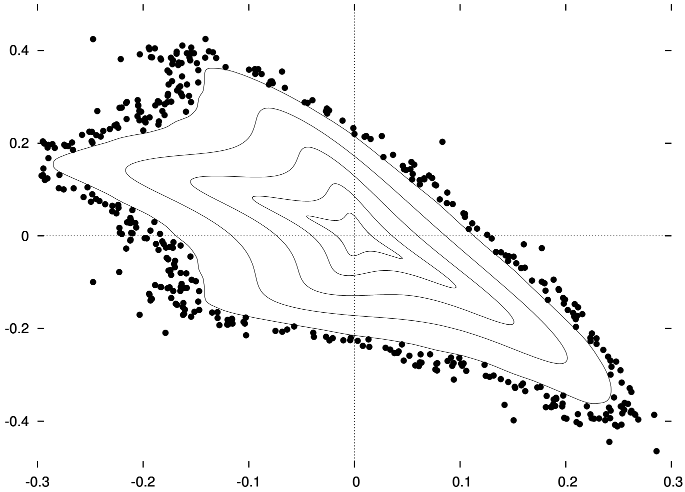
Phase portrait for $\mu = 0.1$ ($\phi_s = 0,9^\circ$)



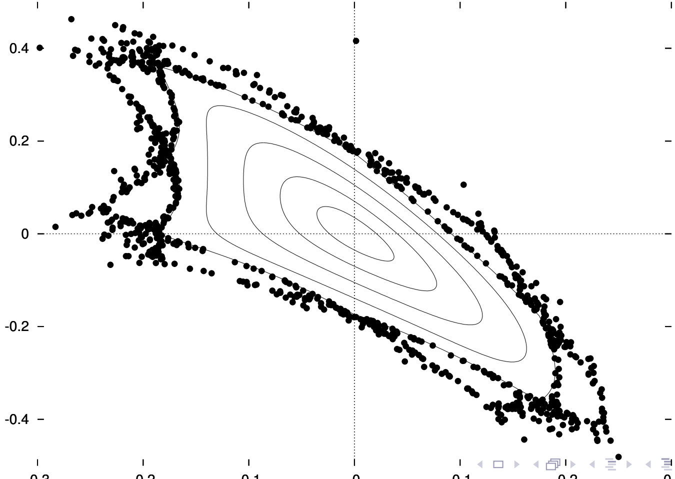
Phase portrait for $\mu = 1$ ($\phi_s = 9^\circ$)



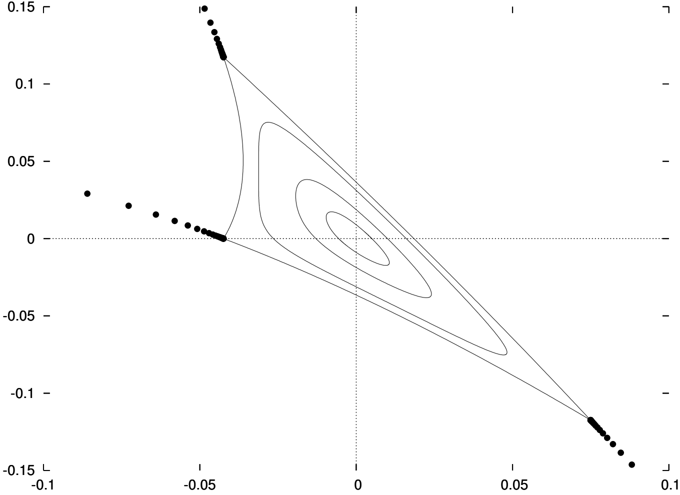
Phase portrait for $\mu = 2$ ($\phi_s = 17.66^\circ$)



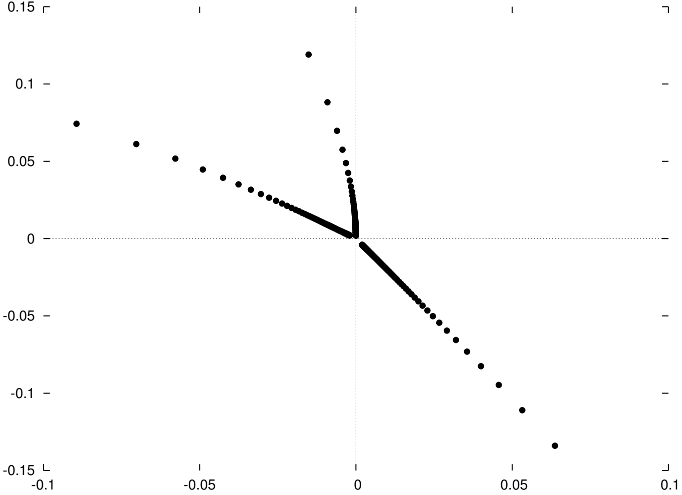
Phase portrait for $\mu = \mu_r \simeq 2.537706055658\ldots$ ($\phi_s = 21.9^\circ$)



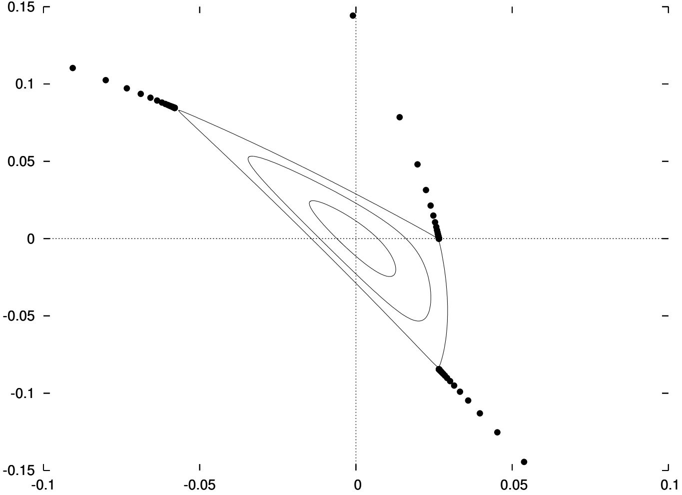
Phase portrait for $\mu = 2.9$ ($\phi_s = 24.77°$)



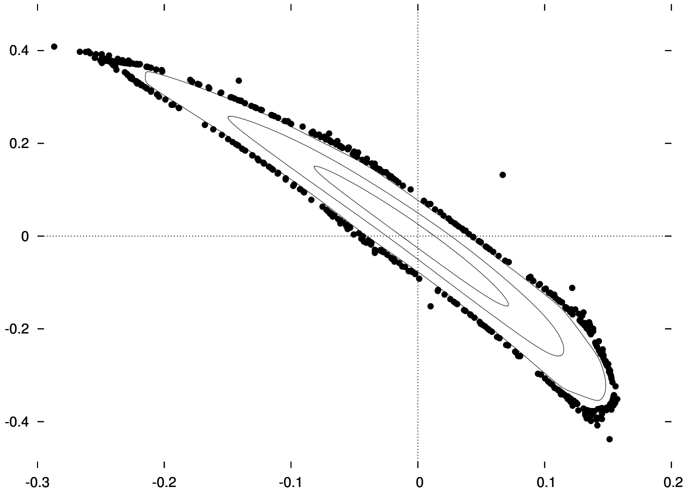
Phase portrait for $\mu = 3$ ($\phi_s = 25.52^\circ$)



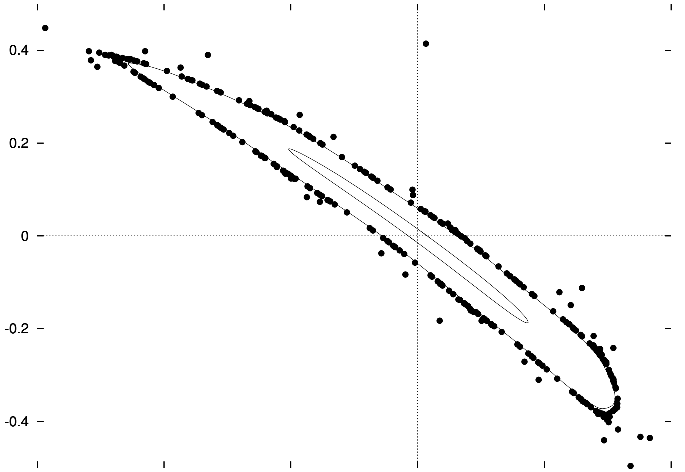
Phase portrait for $\mu = 3.1$ ($\phi_s = 26.26°$)



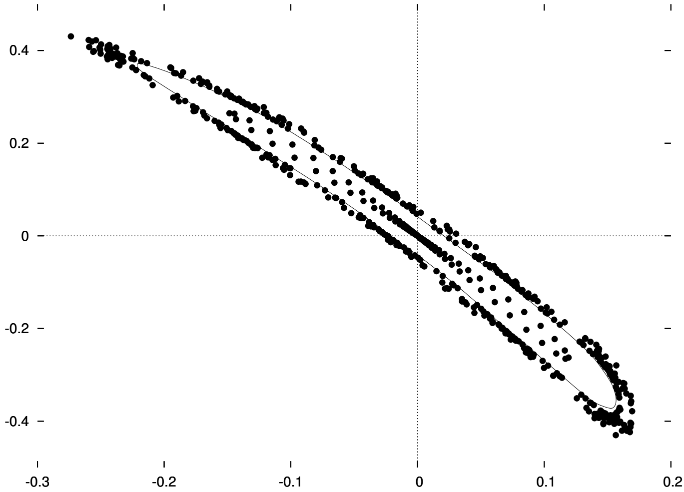
Phase portrait for $\mu = 3.9$ ($\phi_s = 31.82°$)



Phase portrait for $\mu = 4$ ($\phi_s = 32.48^\circ$)



Phase portrait for $\mu = 4.05$ ($\phi_s = 32.80^\circ$)



Local stability: Statement

The linear type and the local stability are related as follows. The hyperbolic type implies local instability, whereas the elliptic type is generically locally stable, but local instability may take place in degenerate cases. The parabolic type is the hardest one, but it can be studied using results from Levi-Civita and Simó.

Theorem

The synchronous trajectory is locally stable if and only if

$$\mu \in (0,4] \setminus \{3\}.(\phi_s(0,32.48^o] \setminus \{25.52^o\}$$

Local instability at the saddle-center bifurcation: $\mu = 0$ ($\phi_s = 0$); $\theta = 0$

If $\mu = 0$, the map has the form

$$\begin{cases} w_1 &= w - \pi(\psi + w)^2 + O_4(\psi, w), \\ \psi_1 &= \psi + w. \end{cases}$$

$\frac{\partial^2 w_1}{\partial \psi^2}(0,0) = -2\pi \neq 0$, one can apply the Levi-Civita criterion (*).

Hence, the origin is locally unstable.

(*)T. Levi-Civita, Sopra alcuni criteri di instabilità, Annali di Matematica Ser. III, 5 (1901) 221–307.

Local stability in the elliptic case: generic values

If $\mu \in (0,4) \setminus \{2,3\}$, as $\theta \neq \pi/2, 2\pi/3$, then there exists an area preserving polynomial change of variables which brings f into its third order Birkhoff normal form

$$z_1 = e^{i(\theta+\tau|z|^2)} z + O(|z|^4),$$

τ is the twist coefficient and can be computed analytically. In particular, $\tau = \tau(\mu)$ has just one root in $(0,4)$; namely,

$$\mu_r := 2.537706055658189018165133406\ldots$$

Therefore, for $\mu \neq \mu_r$ Moser Twist Theorem gives the existence of invariant curves near the origin.

Hence, the origin is locally stable.

Local stability at $\mu = \mu_r$

If $\mu = \mu_r$, we should compute the five order Birkhoff normal form

$$z_1 = e^{i(\theta_r + \tau_1 |z|^2 + \tau_2 |z|^4)} z + O(|z|^6),$$

where $\tau_1 = 0$ and $\tau_2 \in \mathbf{R}$ is the second Birkhoff coefficient. This is cumbersome, so we have numerically checked that

$$\rho(\psi, 0) = \theta_r / 2\pi + \rho_2 \psi^4 + O(\psi^5),$$

for some $\rho_2 \approx -200$, which implies that $\tau_2 \neq 0$, and so the origin is locally stable. Here, $\rho(p)$ denotes the rotation number around the elliptic fixed point p_s of the point p.

Hence, the origin is locally stable.

Local stability at the fourth order resonance: $\mu = 2$ ($\phi_s = 17.66^o$)

If $\mu = 2$ we have that $\theta = \pi/2$,
The change $x = \psi + w$ and $y = \psi$, puts f in the form

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} -y - a(x) \\ x \end{pmatrix} = R_{\pi/2} \begin{pmatrix} x \\ y + a(x) \end{pmatrix},$$

where $a(x) = a_2 x^2 + a_3 x^3 + O(x^4)$, with $a_2 = \pi$, $a_3 = -1/3$.
We know from Simó's criterion (*) that the origin is locally unstable if and only if

$$0 < a_3 \leq a_2^2.$$

Hence, the origin is locally stable.

(*) C. Simó, Stability of degenerate fixed points of analytic area preserving mappings, Astérisque, 98–99, Soc. Math. France, Paris, 1982.

Local instability at the third order resonance: $\mu = 3$ ($\phi_s = 25,52^o$)

If $\mu = 3$, we have that $\theta = 2\pi/3$

$(\psi_3, w_3) = f^3(\psi, w)$ is near the identity and can be written as:

$$\psi_3 = \psi - \frac{\partial G}{\partial w_3}(w_3, \psi) \qquad \text{and} \qquad w_3 = w + \frac{\partial G}{\partial \psi}(w_3, \psi).$$

with $G(w_3, \psi) = \pi\psi(\psi + w_3)(2\psi + w_3) + O_4(\psi, w_3)$.
Simó's criterion states that the origin is locally stable iff $G(w_3, \psi)$ has a strict extremun at the origin.

Hence, the origin is locally unstable.

Local stability at the second order resonance: $\mu = 4$ ($\phi_s = 32,48^o$)

If $\mu = 4$, we have that $\theta = \pi$.

The change $x = 2\psi + w$ and $y = \psi$ puts f in the form

$$\begin{cases} x_1 &= -x + b(x-y), \\ y_1 &= x - y, \end{cases}$$

where $b(u) = b_2 u^2 + b_3 u^3 + O(u^4)$, with $b_2 = -\pi$, $b_3 = 2/3$. There is stability if $b_2 \neq 0$, $b_3 \neq 0$, and $2b_3 + b_2^2 > 0$.

Hence, the origin is locally stable.

The border of the acceptance: the "last" invariant curve around p_2

We know that for $\phi_s \neq 0\,, 25, 52^o$:

The elliptic point p_s is surrounded by invariant curves.

Questions:

- How to compute these curves?
- How to compute the last one?

We provide to different methods to compute the last invariant curve that will give us the size and shape of $\mathcal{D}_\mu \subset \mathcal{A}_\mu$.

First method: interpolation by a Hamiltonian flow

In several cases, the map f (or some power) is near the identity in suitable variables.

Then, one can interpolate our map f by a Hamiltonian flow. Given $n \in \mathbf{N}$, there exists $H_n(\psi, w)$ such that if we consider the Hamiltonian system

$$\begin{aligned} \dot{\psi} &= \frac{\partial H_n}{\partial w}(\psi, w) \\ \dot{w} &= \frac{\partial H_n}{\partial \psi}(\psi, w) \end{aligned}$$

the invariant curves of f can be approximated by

$$H_n(\psi, w) = E, \ \ E \in \mathbf{R}$$

First method: interpolation by a Hamiltonian flow

Method:

- Choose an error e_n
- Look for the Hamiltonian H_n
- Look for E_{max} such that $H_n(\psi, w) = E_{max}$ gives the last invariant curve.

This method is very accurate near the resonant values of ϕ_s and gives an analytic formula for the boundary.

Near the saddle-center bifurcation: $0 < \mu \ll 1$

The scaling $x = \psi/\mu$ and $y = w/\mu^{3/2}$ transforms f into

$$\tilde{f} = \mathrm{I} + \mu^{1/2}\tilde{f}_1 + \mathrm{O}(\mu) = \phi_{\tilde{H}_1}^{\mu^{1/2}} + \mathrm{O}(\mu) = \phi^1_{\mu^{1/2}\tilde{H}_1} + \mathrm{O}(\mu),$$

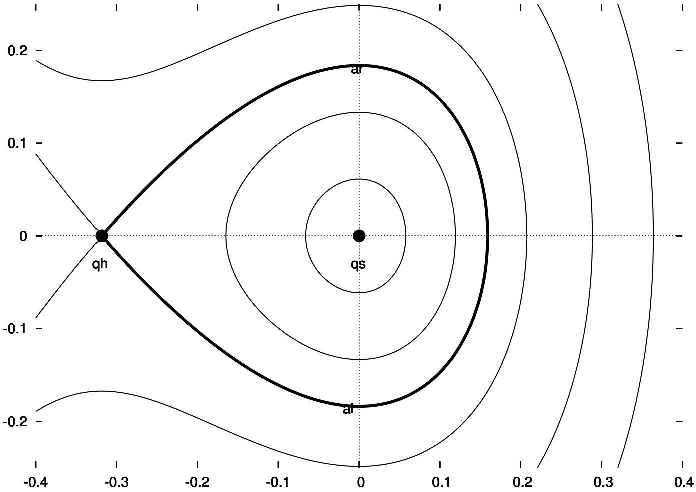
where the *limit Hamiltonian* is

$$\tilde{H}_1(x,y) = (x^2 + y^2)/2 + \pi x^3/3 - 1/6\pi^2.$$

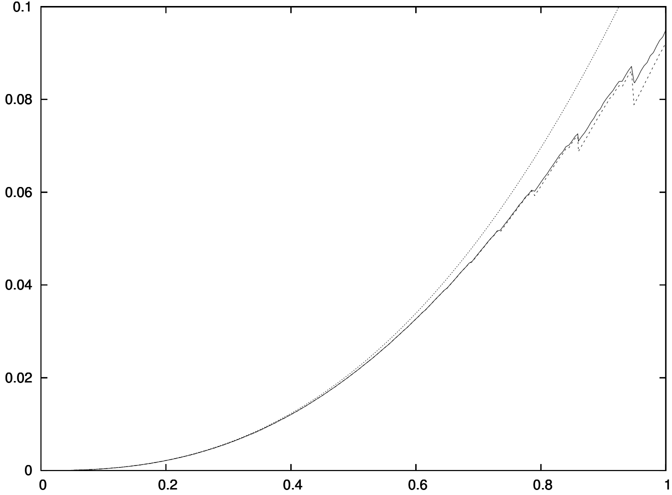
This Hamiltonian has an elliptic point $\tilde{q}_\mathrm{s} = (0,0)$ and a saddle point $\tilde{q}_\mathrm{h} = (-1/\pi, 0)$, whose invariant curves coincide giving rise to a separatrix that encloses a domain of area $6/5\pi^2$. Thus,

$$|\mathcal{A}_\mu|, |\mathcal{D}_\mu| = 6\mu^{5/2}/5\pi^2 + \mathrm{O}(\mu^3), \qquad \text{as } \mu \to 0^+.$$

Phase portrait of the limit Hamiltonian $\tilde{H}_1$



The areas $|\mathcal{A}_\mu|$, $|\mathcal{D}_\mu|$, and $6\mu^{5/2}/5\pi^2$ versus μ



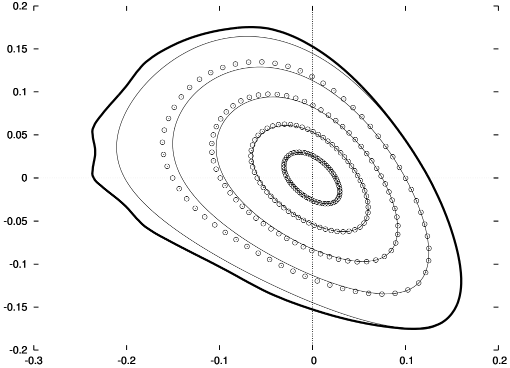
Interpolating Hamiltonians for $0 < \mu \ll 1$

We can refine the limit Hamiltonian $\tilde{H}^{[1]}(x, y; \mu) = \mu^{1/2}\tilde{H}_1(x, y)$ in order to obtain some approximating Hamiltonians

$$\tilde{H}^{[n]}(x, y; \mu) = \sum_{j=1}^{n} \mu^{j/2}\tilde{H}_j(x, y),$$

such that $\tilde{f} = \phi^1_{\tilde{H}^{[n]}} + \mathrm{O}(\mu^{(n+1)/2})$.
This approximation will be valid for biger values of μ

Invariant curves and level curves of $H^{[4]}(\psi, w; \mu)$ for $\mu = 1$



At the fourth order resonance: $\mu = 2$

If $\mu = 2$ and we perform the change $x = \psi + w$ and $y = \psi$, then the map f verifies

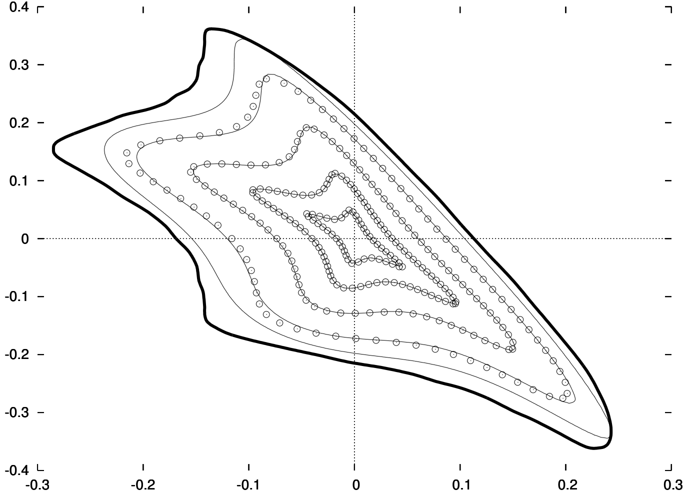
$$\tilde{f}^4 = \mathrm{I} + O_3(x, y).$$

We can procced and find a unique Hamiltonian of the form

$$\tilde{H}^{[n]}(x, y) = \tilde{H}_4(x, y) + \cdots + \tilde{H}_n(x, y),$$

such that $\tilde{f}^4 = \phi^1_{\tilde{H}^{[n]}} + O_n(x, y)$.
We can obtain explicit formulas for $\tilde{H}^{[i]}$

$\mu = 2$: Invariant curves and level curves of $H^{[6]}(x, y)$



Near the third order resonance: $\mu \simeq 3$

If $\mu = 3 + \epsilon$ with $0 < |\epsilon| \ll 1$, then f is transformed by the scaling $(x, y) = \pi(\psi, w)/\epsilon$ into a map $\tilde{f}$ such that

$$\tilde{f}^3 = \phi^1_{\epsilon \tilde{H}_1} + O(\epsilon^2),$$

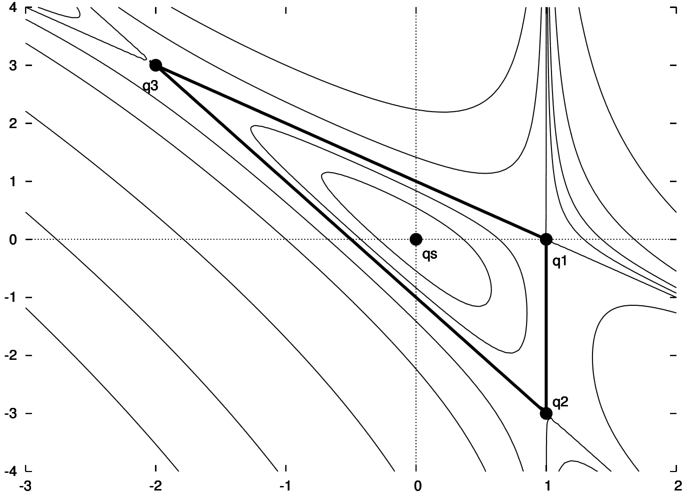
where the *limit Hamiltonian* is

$$\tilde{H}_1(x, y) = 3x^2 + 3xy + y^2 - 2x^3 - 3x^2y - xy^2 - 1.$$

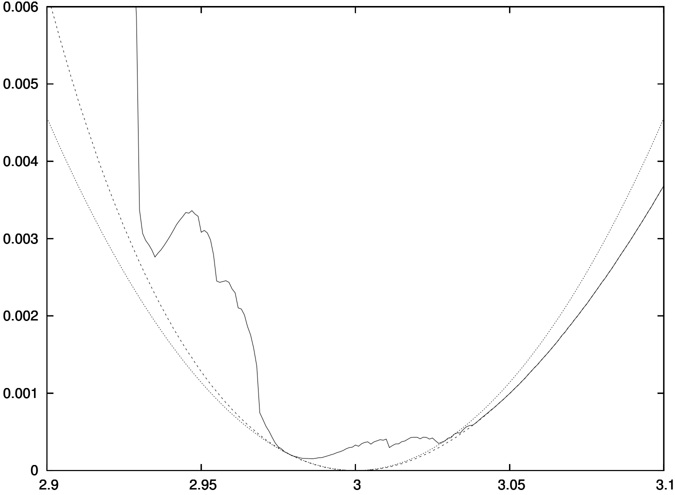
This Hamiltonian has one elliptic point: $\tilde{q}_s = (0, 0)$, and three saddle points: $\tilde{q}_1 = (1, 0)$, $\tilde{q}_2 = (1, -3)$, and $\tilde{q}_3 = (-2, 3)$, whose separatrices enclose a triangle of area $9/2$. Hence,

$$|\mathcal{D}_{3+\epsilon}| = 9\epsilon^2/2\pi^2 + O(\epsilon^3), \qquad \text{as } \epsilon \to 0.$$

$\mu = 3 + \epsilon$: Phase portrait of the limit Hamiltonian $\tilde{H}_1$



The areas $|\mathcal{A}_\mu|$, $|\mathcal{D}_\mu|$, and $9(\mu - 3)^2/2\pi^2$ versus μ



Second method: computation of rotation numbers.

We compute the "last" invariant curve for the rest of values of ϕ_s using a different idea.

Given any $p = (\psi, w) \in \mathcal{A}$, let φ_n be the “lifted argument” of $f^n(p)$ with respect to the elliptic point p_s. If the limit

$$\rho = \rho(p) := \frac{1}{2\pi} \lim_{n \to +\infty} \frac{\varphi_n - \varphi_0}{n}$$

exists, then we say that $\rho(p)$ is the *rotation number* of the point p under the map f around the elliptic point p_s. We note that

$$\lim_{p \to p_s} \rho(p) = \theta / 2\pi,$$

where $\theta/2\pi$ is the rotation number of the elliptic fixed point p_s.
There exist algorithms to compute $\rho(p)$ in an efficient way (*).

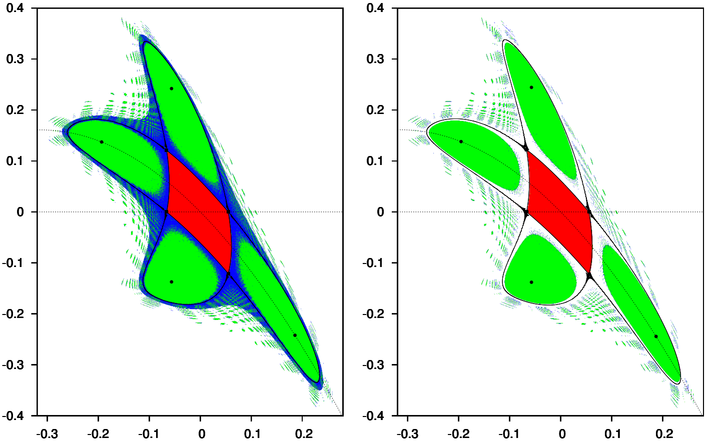
A. Luque and J. Villanueva, Quasi-periodic frequency analysis using averaging-extrapolation methods, SIAM J. Appl. Dyn. Sist., 13 (2013) 1–46.

The rotation number (dynamical consequences)

Rotation numbers allow to distinguish the three main bounded dynamical behaviors in APMs:

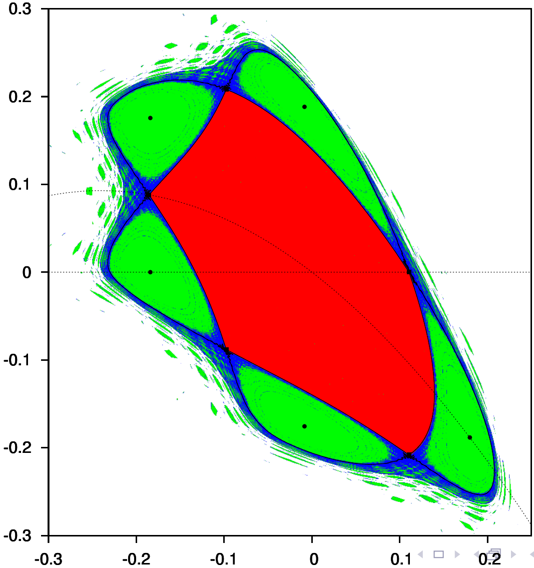
- A IC is a closed invariant curve around p_s where the dynamics is conjugated to a rigid rotation. If p is inside a IC, $\rho(p)$ exists and, generically, is a Diophantine number.
- A (m, n)-periodic chain of elliptic islands is an invariant region with several connected components such that each of them surrounds a (m, n)-periodic elliptic point. If p is inside some periodic chain, then $\rho(p) = m/n$ is rational.
- A chaotic sea is the region between two adjacent ICs without the stable elliptic islands. If p is inside a chaotic sea, then $\rho(p)$ generically does not exist.

Stability regions for $\mu = 2.037$ and $\mu = 2.038$. Chaotics seas, Invariant Curves, elliptic islands,

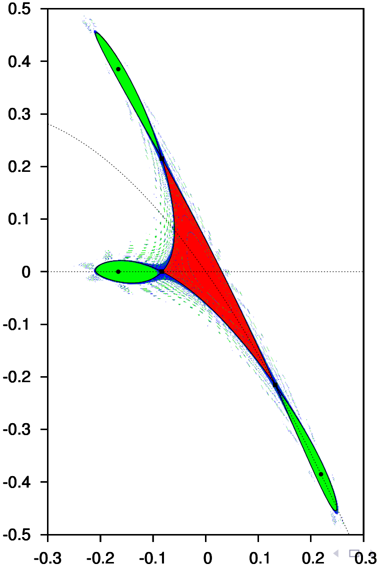


The region $\mathcal{D} \subset \mathcal{A}$.

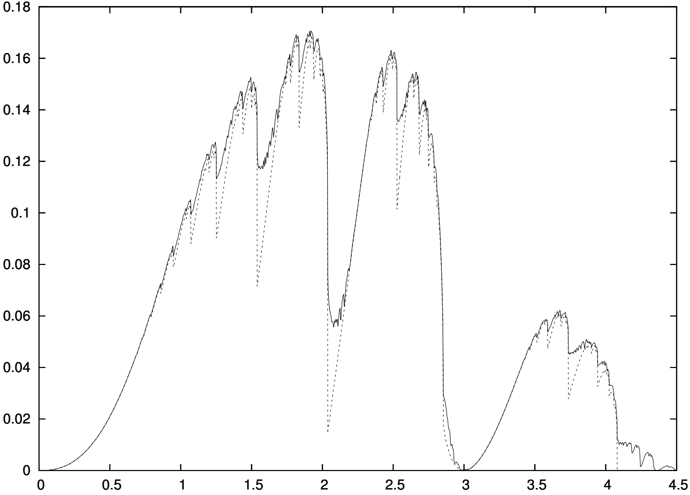
Stability region for $\mu = 1.539$.



Stability region for $\mu = 2.853$.



Areas $|\mathcal{A}_\mu|$ and $|\mathcal{D}_\mu|$ versus μ



Birth and escape of the main elliptic islands: $n < 10$

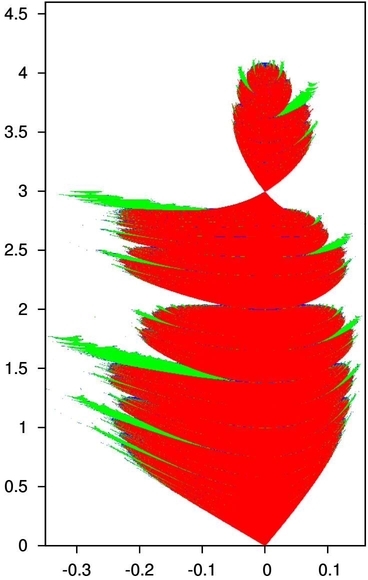
(m, n)	Birth at	Escape at $\mu_\star$ with
$(1, 9)$	$\mu_\bullet \approx 0.468$	$0.859 < \mu_\star < 0.860$
$(1, 8)$	$\mu_\bullet = 2 - \sqrt{2} \simeq 0.586$	$0.948 < \mu_\star < 0.949$
$(1, 7)$	$\mu_\bullet \approx 0.753$	$1.071 < \mu_\star < 1.072$
$(1, 6)$	$\mu_\bullet = 1$	$1.251 < \mu_\star < 1.252$
$(1, 5)$	$\mu_\bullet = \frac{1}{2}(5 - \sqrt{5}) \simeq 1.382$	$1.539 < \mu_\star < 1.540$
$(2, 9)$	$\mu_\bullet \approx 1.653$	$1.835 < \mu_\star < 1.836$
$(1, 4)$	$\mu_\bullet = 2$	$2.037 < \mu_\star < 2.038$
$(2, 7)$	$\mu_\bullet \approx 2.445$	$2.526 < \mu_\star < 2.527$
$(1, 3)$	$\mu_\bullet = 3$	$2.853 < \mu_\star < 2.854$
$(3, 8)$	$\mu_\bullet = 2 + \sqrt{2} \simeq 3.414$	$3.589 < \mu_\star < 3.590$
$(2, 5)$	$\mu_\bullet = \frac{1}{2}(5 + \sqrt{5}) \simeq 3.618$	$3.735 < \mu_\star < 3.736$
$(3, 7)$	$\mu_\bullet \approx 3.802$	$3.942 < \mu_\star < 3.943$
$(4, 9)$	$\mu_\bullet \approx 3.879$	$4.023 < \mu_\star < 4.024$
$(1, 2)$	$\mu_\bullet = 4$	$4.080 < \mu_\star < 4.081$

The sections with the symmetry lines

We gather the sections of the stability domain with the symmetry line $\mathrm{F}ix(r_0) = \{(\psi, w) : w = 0\}$ into the two-dimensional set:

$$\mathcal{S}_0 = \{(\mu, \psi) \in (0, +\infty) \times \mathbf{T} : (\psi, 0) \in \mathcal{A}_\mu\},$$

in order to visualize their evolution in the parameter μ.


4.5
4
3.5
3
2.5
2
1.5
1
0.5
0
-0.3
-0.2
-0.1
0
0.1

Two "empirical" rules used in particle accelerators and a "practical comment"

1. Values of ϕ_s for which an accelerator can operate are contained in the interval of linear stability of the synchronous trajectory $(0, 32.5^\circ)$.

   True except for the value $\phi_S := \arctan(3/2\pi) \approx 25.5^\circ$, that corresponds to the third order resonance.

2. The optimal values of ϕ_s are close to the middle point of such interval.

   True: The acceptance area reaches its maximal value $|\mathcal{A}| \approx 0.17$ at $\mu \approx 1.912$, which roughly corresponds to $\phi_s \approx 16.9^\circ$

In this work we deal with perpetual stability, although only $2 \cdot 10^7$ turns were considered in our numerical computations of $\mathcal{A}$.
The number of turns made by each particle is typically of just a few tens in real RTMs, 90 in the RTM machine of the MAMI complex at the Institute for Nuclear Physics in Mainz.

Two ideas

1. Invariant curves of hyperbolic points (or PO) as approximate boundaries of stability domains. The area of the lobes between such invariant curves is equal to the flux through certain closed curves composed by arcs of invariant curves. These lobes have an exponentially small area for analytic close-to-the identity maps (Fontich-Simó). This will hapen near the resonant values.
2. Invariant curves as topological obstruction to the existence of RICs. If the unstable invariant curve of some periodic orbit intersects the stable invariant curve of another, then there can be no RICs between both periodic orbits (Olvera, Simó).

First idea: Singular splitting for $0 < \mu \ll 1$

- If $0 < \mu \ll 1$, then the map f is approximated, after a rescaling, by the $\mu^{1/2}$-time flow of the limit Hamiltonian

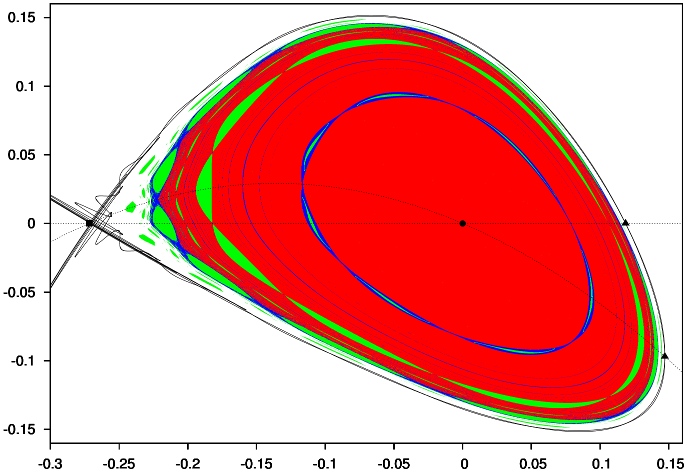
$$\tilde{H}_1(x, y) = (x^2 + y^2)/2 + \pi x^3/3 - 1/6\pi^2.$$

- This Hamiltonian has the homoclinic solution

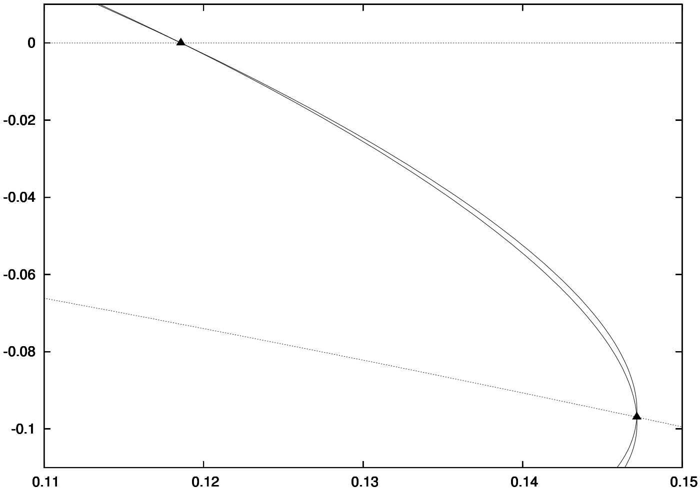
$$x_0(t) = \frac{3}{2\pi \cosh^2(t/2)} - \frac{1}{\pi}, \qquad y_0(t) = \frac{3 \sinh(t/2)}{2\pi \cosh^3(t/2)},$$

  which is analytic in a complex strip of width $d_0 = \pi$.
- Fontich-Simó: The splitting of the separatrices is $O(e^{-c/h})$ for any $0 < c < 2\pi^2$, where h is the characteristic exponent of the hyperbolic fixed point p_h.

Stability domain for $\mu = 0.859$



The lobe for $\mu = 0.859$, with area $A \approx 3.8082 \times 10^{-5}$



Asymptotic formula of the splitting for $0 < \mu \ll 1$

- Let A be the area of the *lobe* delimited by the part of the separatrices between the two primary homoclinic points on the symmetry lines.
- Our numerical experiments strongly suggest that there exist some asymptotic coefficients $a_0, a_1, a_2, \ldots$ such that

$$A \asymp e^{-2\pi^2/h} \sum_{n \geq 0} a_n h^{2n}, \qquad (h \to 0).$$

- $a_0 \approx 1.42098502709189813726617259727 \times 10^5$.
- $a_1 = 0$.
- It can be proved as in (*)

(*) Exponentially small splitting of separatrices in the perturbed McMillan map P. Martín, D. Sauzin, T. M. Seara. DCDS 31(2): 301-371, 2011

First idea: Singular splitting for $\mu \approx 3$

- If $\mu = 3 + \epsilon$ with $0 < |\epsilon| \ll 1$, then the map f^3 is approximated, after a rescaling, by the ϵ-time flow of the limit Hamiltonian

$$\tilde{H}_1(x, y) = 3x^2 + 3xy + y^2 - 2x^3 - 3x^2y - xy^2 - 1.$$

- This Hamiltonian has the heteroclinic solution

$$x_0(t) \equiv 0, \qquad y_0(t) = -3/\left(e^{3t} + 1\right),$$

  which is analytic in a complex strip of width $d_0 = \pi/3$.
- Fontich-Simó: The splitting of the separatrices is $O(e^{-c/h})$ for any $0 < c < 2\pi^2/3$, where h is the characteristic exponent of the hyperbolic fixed points of f^3.

Second idea: Study for $\mu = 1.539$

- The $(1,5)$-periodic elliptic island escapes from the stability domain at some $1.539 < \mu_\star < 1.540$.
- Set $\mu = 1.539$. We have numerically found the following chain of heteroclinic connections between the (m, n)-periodic saddle points on the symmetry lines:

$$(0,1) \to (1,6) \to (2,11) \to (3,16) \to (4,21).$$

- Hence, there is no IC with rotation number $\rho \in (0, 4/21)$.
- Besides, if $\mu \in [1.539, \mu_\star)$, then the rotation number of the LRIC should be the "most irrational" number in the interval

$$(4/21, 1/5).$$