Random N-Particle Klimontovich-Maxwell System: Probabilistic Analysis, Fluctuations from Mean and Ecker Hierarchy [1]

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Outline

Random N-Particle Klimontovich-Maxwell System

1. RKMS: Random IVP for relativistic Klimontovich density, K, coupled to Maxwell equations
2. $\overline{\text{RKMS}}$: Expected value of RKMS (with respect to random initial conditions) and Vlasov Maxwell approximation. Landau Damping and Fields Medal to Villani (KAM like proof?).
3. Key objects: N-particle pdf, Ψ_N, with permutation symmetry, and marginal, 1 and 2-particle pdfs, Ψ_1 and Ψ_2. Mean $\overline{K}$ of K is Ψ_1.
4. Probabilistic analysis of fluctuations: Coarse graining $\rightarrow$ fluctuations as a sum of identically distributed, but dependent, Bernoulli random variables. Set up for dependent generalizations of WLLN, SLLN, CLT and LD. $K - \overline{K}$ size measured by $\mathcal{F}(\Psi_1, \Psi_2)$
5. Ecker hierarchy: Kinetic Theory for $\overline{\text{RKMS}}$. We discuss Levels I and II for Ψ_1, Ψ_2 and associated generalized macroscopic Maxwell equations. Analogue of BBGKY in relativistic (retarded time) case.
6. Fluctuation System: Difference between RKMS and $\overline{\text{RKMS}}$, a study of $\widetilde{K} = K - \overline{K}$ and a linearization.
7. Summary: We have framework for analysis of mean $\overline{K}$ and fluctuations with some modest results. Many open questions

Random Lorentz Maxwell System

Random IVP

1. Relativistic Particle-Field Equations of Motion

$$\dot{\mathbf{R}}_i = \mathbf{v}(\mathbf{P}_i), \qquad \dot{\mathbf{P}}_i = \mathbf{F}_{\text{ext}}(\mathbf{R}_i, \mathbf{P}_i, t) + \mathbf{F}_i^{Lor}(\mathbf{R}_i, \mathbf{P}_i, t),$$

$$\partial_t \mathbf{B}_i = -\nabla_r \times \mathbf{E}_i, \qquad \partial_t \mathbf{E}_i = c^2 \nabla_r \times \mathbf{B}_i - c Z_0 \mathbf{J}_i^K(r,t),$$

$$\mathbf{J}_i^K(r,t) = q \sum_{j \neq i}^{N} \mathbf{v}(\mathbf{P}_j(t)) \delta(r - \mathbf{R}_j(t)) \quad \text{No Self Force}$$

2. Lorentz force law: $\mathbf{F}_i^{Lor}(r,p,t) := q\big[\mathbf{E}_i(r,t) + \mathbf{v}(p) \times \mathbf{B}_i(r,t)\big]$
$\mathbf{v}(p) = p/m\gamma(p)$ where $\gamma(p) = 1 + p \cdot p/m^2c^2$

3. Identically Distributed Random ICs for particles:
$W_i^0 := (\mathbf{R}_i(0), \mathbf{P}_i(0)), \;\; W^0 := (W_1^0, \cdots, W_N^0)$

4. Field ICs independent of i.

5. Notation for Random ODE solution:
$W_i(t; W^0) := (\mathbf{R}_i(t; W^0), \mathbf{P}_i(t; W^0)) \in \mathbb{R}^6, \;\; W_i(0; W^0) = W_i^0$
$W(t; W^0) := (W_1(t; W^0), \cdots, W_N(t; W^0)) \in \mathbb{R}^{6N}$

- No QED. Radiation reaction? Example: FEL Pendulum in $\mathbf{F}_{\text{ext}}$

- Random 6-D Klimontovich phase space density K

$$K(r,p,t;W^0) = \frac{1}{N}\sum_{i=1}^{N} K_i(r,p,t;W^0), \text{ where}$$

$$K_i(r,p,t;W^0) = \delta(r - \mathbf{R}_i(t;W^0))\delta(p - \mathbf{P}_i(t;W^0))$$

- Random Klimontovich Maxwell System (RKMS): Our Basic Model

$$K_{it} + \mathcal{L}^{\text{ext}} K_i + \nabla_p \cdot \left[\mathbf{F}_i^{Lor}(r,p,t;W^0) K_i\right] = 0$$

$$\partial_t \mathbf{B}_i = -\nabla_r \times \mathbf{E}_i, \;\; \partial_t \mathbf{E}_i = c^2 \nabla_r \times \mathbf{B}_i - cZ_0 \mathbf{J}_i^K(r,t;W^0),$$

$$\mathcal{L}^{\text{ext}} = \mathbf{v}(p) \cdot \nabla_r + \mathbf{F}_{\text{ext}}(r,p,t) \cdot \nabla_p$$

$$\mathbf{J}_i^K(r,t;W^0) = q \sum_{j \neq i}^{N} \int dp \mathbf{v}(p) K_j(r,p,t;W^0)$$

- Random Initial Condition
$K_i(r,p,0;W^0) = \delta(r - \mathbf{R}_i(0))\delta(p - \mathbf{P}_i(0))$, $W_i^0 := (\mathbf{R}_i(0), \mathbf{P}_i(0))$

Joint pdf, Ψ_N, of N particle motion

Permutation Symmetry and one and two particle pdfs

1. Joint pdf Ψ_N of the $W_i(t;W^0) := \big(\mathbf{R}_i(t;W^0),\ \mathbf{P}_i(t;W^0)\big)$ is $\Psi_N(w,t) := \Pr\{W_1(t;W^0) = w_1, \cdots, W_N(t;W^0) = w_N\}$.
2. Permutation Symmetry (PS): $\Psi_N(w,t) = \Psi_N(P_{ij}(w),t)$, where $P_{ij}(w)$ interchanges the i and j components of w.
   Follows from assumption that $W_i(0;W^0)$ are identically distributed.
3. PS $\Rightarrow$ one and two particle pdfs independent of particle number
   $\Psi_1(v,t) = \int \Psi_N(v, w_2, \cdots, w_N, t) dw_2 \cdots dw_N$.
   $\Psi_2(v,v',t) = \int \Psi_N(v, v', w_3 \cdots, w_N, t) dw_3 \cdots dw_N$.
   Here $v = (r,p)$ and $v' = (r',p')$
4. Expected Value of $\mathcal{O}(W^0)$: $\overline{\mathcal{O}} \equiv \overline{\mathcal{O}(W^0)} := \int dw \Psi_N(w,0)\mathcal{O}(w)$
   E.g., $\overline{\mathbf{E}}_i(r,t) := \overline{\mathbf{E}_i(r,t;W^0)}$
5. PS $\Rightarrow$ $\overline{K}_i(r,p,t) = \Psi_1(r,p,t)$, $\overline{K_i(r,p,t;W^0)K_j(r',p',t;W^0)} = \Psi_2(r,p,r',p',t)$, $\overline{\mathbf{E}}_i = \overline{\mathbf{E}}_1$, $\overline{\mathbf{B}}_i = \overline{\mathbf{B}}_1$, $\overline{\mathbf{F}_i K_i} = \overline{\mathbf{F}_1 K_1}$, $\overline{\mathbf{J}}_i^K = \overline{\mathbf{J}}_1^K$
   That is, they are all independent of particle number.

There is no closed form evolution law for Ψ_N, but surely the joint N particle-field pdf does have one. What does it look like?

Taking Expected Value of RKMS gives $\overline{\text{RMKS}}$ defined by:

$$\Psi_{1t} + \mathcal{L}^{\text{ext}}\Psi_1 + \nabla_p \cdot \left[\overline{\mathbf{F}_1^{Lor}(r,p,t;W^0)K_1(r,p,t;W^0)}\right] = 0$$

Similar equation for $\Psi_2(r,p,r',p',t), etc.$

$$\partial_t\overline{\mathbf{B}}_1 = -\nabla_r \times \overline{\mathbf{E}}_1, \;\; \partial_t\overline{\mathbf{E}}_1 = c^2\nabla_r \times \overline{\mathbf{B}}_1 - cZ_0\overline{\mathbf{J}}_1^K(r,t)$$

$$\mathbf{F}_1^{Lor}(r,p,t;W^0) := q\left[\mathbf{E}_1(r,t;W^0) + \mathbf{v}(p) \times \mathbf{B}_1(r,t;W^0)\right]$$

$$\overline{\mathbf{J}}_1^K(r,t) = qN^* \int dp\mathbf{v}(p)\Psi_1(r,p,t), \;\; N^* = N - 1$$

- This is not a closed system.
- Particles and fields uncorrelated $\Rightarrow \overline{\mathbf{F}_1^{Lor}K_1} = \overline{\mathbf{F}}_1^{Lor}\Psi_1 \Rightarrow \overline{\text{RMKS}}$ becomes the (closed) Vlasov Maxwell system and $\Psi_2(r,p,r',p',t) = \Psi_1(r,p,t)\Psi_1(r',p',t)$
- Particles and fields correlated: Need approximate evolution law for $\overline{\mathbf{F}_1^{Lor}K_1} \rightarrow$ Ecker Hierarchy.

Direct Comparison of K and $\overline{K} = \Psi_1$ - 1

Probabilistic Context and sum of identically distributed Berrnoulli random variables

1. Compare K and $\overline{K} = \Psi_1$. Problem: K is either 0 or ∞.
2. So, compare K_A and $\overline{K}_A$, $A \subset \mathbb{R}^6$,

$$K_A(t, W^0) := \int_A drdpK(r,p,t;W^0) = \frac{1}{N} \cdot \# \text{ particles in } A$$

$$\overline{K}_A(t) = \int_A drdp\Psi_1(r,p,t) =: \Psi_{1A}(t),$$

3. $K_A(t;W^0) = \frac{1}{N}\sum_1^N X_i(t;W^0)$ where $X_i = 1$ if ith particle $\in$ A, 0 otherwise (X_i are dependent, but ID, Bernoulli (DID) RVs). $\Rightarrow$ $N \cdot K_A =$ sum of DID RVs with moments

$$\overline{X}_i(t) = \overline{X_i^2}(t) = \Psi_{1A}(t), \;\; \text{since } X_i = X_i^2$$

$$\overline{X_iX_j} = \int_{A\times A} drdpdr'dp'\Psi_2(r,p,r',p',t) =: \Psi_{2A}(t), \; i \neq j$$

4. Ready for probabilistic analysis of large sum of identically distributed, but not independent, random variables.

Direct Comparison of K and $\overline{K} = \Psi_1$ - 2

A weak law of large numbers calculation

1. Modest Result

$$\overline{(K_A - \overline{K}_A)^2} = \overline{K_A^2} - \overline{K}_A^2 = \frac{1}{N}\sum_{i,j}\overline{X_i X_j} - \Psi_{1A}^2 =$$

$$\frac{1}{N}(\Psi_{1A} - \Psi_{2A}) + (\Psi_{2A} - \Psi_{1A}^2) = \text{KL Individual} + \text{KL Collective?}$$

2. So, Find Ψ_1 and Ψ_2 !!!
3. Particle-Field's uncorrelated $\Rightarrow$ Vlasov Maxwell for Ψ_1 and $\Psi_2 = \Psi_1\Psi_1' \Rightarrow \Psi_{2A} = \Psi_{1A}^2$ thus $\overline{(K_A - \overline{K}_A)^2} = \frac{1}{N}(\Psi_{1A} - \Psi_{1A}^2)$
4. Particle-Field's correlated Ecker Hierarchy for Ψ_1, Ψ_2 evolution

   Useful tools:

   - Probability theory for sums of dependent RVs - e.g., SLLN, Large Deviations
   - Dynamical Systems theory - e.g., chaos, mixing, ergodicity

1. Our mean field equation for $\overline{K} = \Psi_1$ is
$\Psi_{1t} + \mathcal{L}^{\text{ext}}\Psi_1 + \nabla_p \cdot \left[\overline{\mathbf{F}_1^{Lor}(r,p,t;W^0)K_1(r,p,t;W^0)}\right] = 0$
Need approximate evolution law for term in red
2. The Ecker trick[2], at levels I and II, is to define

$$\check{\mathbf{E}}_{\mathrm{I}}(r,r',p',t) := \overline{\mathbf{E}_1(r,t;W^0)K_1(r',p',t;W^0)}$$
$$\check{\mathbf{E}}_{\mathrm{II}}(r,r',p',r'',p'',t) := \overline{\mathbf{E}_1(r,t;W^0)K_1(r',p',t;W^0)K_1(r'',p'',t;W^0)}$$

and similarly for $\check{\mathbf{B}}_{\mathrm{I}}(r,r',p',t)$ and $\check{\mathbf{B}}_{\mathrm{II}}(r,r',p',r'',p'',t)$
3. The first two levels of the Ecker hierarchy are (non-closed) evolution laws for the triplets $(\Psi_1, \check{\mathbf{E}}_{\mathrm{I}}, \check{\mathbf{B}}_{\mathrm{I}})$ and $(\Psi_2, \check{\mathbf{E}}_{\mathrm{II}}, \check{\mathbf{B}}_{\mathrm{II}})$

[2] Günter Ecker, Theory of Fully Ionized Plasmas, Academic Press, 1972

Equations for $\Psi_1(r,p,t), \check{\mathbf{E}}_{\mathrm{I}}(r,r',p',t)$ and $\check{\mathbf{B}}_{\mathrm{I}}(r,r',p',t)$ are

$$\Psi_{1t} + \mathcal{L}^{\text{ext}}\Psi_1 + \nabla_p \cdot \left[\check{\mathbf{E}}_{\mathrm{I}}(r,r,p,t) + \mathbf{v}(p) \times \check{\mathbf{B}}_{\mathrm{I}}(r,r,p,t)\right] = 0$$

$$\check{\mathbf{E}}_{\mathrm{I}t} = c^2\nabla_r \times \check{\mathbf{B}}_{\mathrm{I}} - cZ_0qN^* \int dp\Psi_2(r,p,r',p',t) - \mathbf{v}(p') \cdot \nabla_{r'}\check{\mathbf{E}}_{\mathrm{I}} - QT(\mathbf{E}_1)$$

$$\check{\mathbf{B}}_{\mathrm{I}t} = \nabla_r \times \check{\mathbf{E}}_{\mathrm{I}} - \mathbf{v}(p') \cdot \nabla_{r'}\check{\mathbf{B}}_{\mathrm{I}} - QT(\mathbf{B}_1)$$

$$\nabla_r \cdot \check{\mathbf{B}}_{\mathrm{I}} = 0, \quad \nabla_r \cdot \check{\mathbf{E}}_{\mathrm{I}} = cZ_0qN^* \int dp\Psi_2(r,p,r',p',t).$$

1. The arguments of $\check{\mathbf{E}}_{\mathrm{I}}$ and $\check{\mathbf{B}}_{\mathrm{I}}$ in the Ψ_1 equation are (r,r,p,t) rather than (r,r',p',t).
2. Ψ_2 enters the picture
3. $QT(\mathbf{E}_1) = \overline{\mathbf{E}_1(\mathbf{F}_1' \cdot \nabla_{p'}K_1')}, QT(\mathbf{B}_1) = \overline{\mathbf{B}_1(\mathbf{F}_1' \cdot \nabla_{p'}K_1')}$ are quadratic in the fields rather than linear in $\overline{\mathbf{F}_1^L K_1}$ in $\overline{\text{RKMS}}$

We are in the process of analyzing these Level I equations as well as those for Level II.

The Ecker Hierarchy for $\overline{\text{RKMS}}$: Evolution of $\overline{\mathbf{F}_1^{Lor} K_1}$ - 3

Approximate closed system at Level I

Drop quadratic terms and assume particles uncorrelated $(\Psi_2 = \Psi_1 \Psi_1') \Rightarrow$ closed system for $(\Psi_1, \check{\mathbf{E}}_{\mathrm{I}}, \check{\mathbf{B}}_{\mathrm{I}})$:

$$\Psi_{1t} + \mathcal{L}^{\text{ext}} \Psi_1 + \nabla_p \cdot \left[\check{\mathbf{E}}_{\mathrm{I}}(r,r,p,t) + \mathbf{v}(p) \times \check{\mathbf{B}}_{\mathrm{I}}(r,r,p,t) \right] = 0$$

$$\check{\mathbf{B}}_{\mathrm{I}t} = \nabla_r \times \check{\mathbf{E}}_{\mathrm{I}} - (\mathbf{v}(p') \cdot \nabla_{r'}) \check{\mathbf{B}}_{\mathrm{I}}$$

$$\check{\mathbf{E}}_{\mathrm{I}t} = c^2 \nabla_r \times \check{\mathbf{B}} - (\mathbf{v}(p') \cdot \nabla_{r'}) \check{\mathbf{E}}_{\mathrm{I}} - c Z_0 q N^* \int dp \Psi_1(r,p,t)] \Psi_1(r',p',t)$$

$$\nabla_r \cdot \check{\mathbf{B}}_{\mathrm{I}}(r,r',p',t) = 0$$

$$\nabla_r \cdot \check{\mathbf{E}}_{\mathrm{I}}(r,r',p',t) = c Z_0 q N^* \int dp \Psi_1(r,p,t)] \Psi_1(r',p',t).$$

- FEL context: Perhaps will shed light on fluctuations, $\widetilde{K} = K - \Psi_1$.
- (1) (F_1^{Lor}, K_1) uncorrelated $\Rightarrow \Psi_1$ given by VM and $\Psi_2 = \Psi_1 \Psi_1'$
  (2) $(K_1(r,p,t;w_0), K_1(r',p',t;w_0))$ uncorrelated $\Rightarrow \Psi_1$ given by Ecker approximation here.
  Shows assumption that fields and particles are uncorrelated is different from the assumption that particles are uncorrelated.

• Now, we present the system which evolves the fluctuations, $\tilde{K} = K - \Psi_1$.
When this system is combined with the $\overline{\text{RKMS}}$ and approximated we find a closed system which may also shed light on FEL dynamics.
• One Goal: Understand Kim-Lindberg distinction between collective and individual aspects of fluctuations in the context here. See Kwang-Je's talk, Problem 1, and FEL2011 Proceedings.

Fluctuation Equations - 2

Defined from RKMS; $O(N)$ PDEs

1. Recall the expected values of $(K_i, \mathbf{E}_i, \mathbf{B}_i)$ in the RKMS are independent of i and given by $(\Psi_1, \overline{\mathbf{E}}_1, \overline{\mathbf{B}}_1)$ in the $\overline{\text{RKMS}}$
2. Thus fluctuations from the mean $(\widetilde{K}_i, \widetilde{\mathbf{E}}_i, \widetilde{\mathbf{B}}_i)$ are defined by $(K_i, \mathbf{E}_i, \mathbf{B}_i) = (\Psi_1, \overline{\mathbf{E}}_1, \overline{\mathbf{B}}_1) + \epsilon(\widetilde{K}_i, \widetilde{\mathbf{E}}_i, \widetilde{\mathbf{B}}_i)$, where $\epsilon = 1$ is an order parameter. Plugging into RKMS and using $\overline{\text{RKMS}}$ gives

$$\Psi_{1t} + \mathcal{L}^{\text{ext}}\Psi_1 + \nabla_p \cdot \left[\overline{F}_1^{Lor}\Psi_1 + \epsilon^2\overline{\widetilde{\mathbf{F}}_1^{Lor}\widetilde{K}_1}\right] = 0$$

$$\widetilde{K}_{it} + \mathcal{L}^{\text{ext}}\widetilde{K}_i + \nabla_p \cdot (\widetilde{\mathbf{F}}_i^{Lor}\Psi_1 + \overline{\mathbf{F}}^{Lor}\widetilde{K}_i) = -\epsilon\left[\widetilde{\mathbf{F}}_i^{Lor}\widetilde{K}_i - \overline{\widetilde{\mathbf{F}}_i^{Lor}\widetilde{K}_i}\right]$$

$$\mathcal{L}^{\text{ext}} = \mathbf{v}(p) \cdot \nabla_r + \mathbf{F}_{\text{ext}}(r, p, t) \cdot \nabla_p$$

Plus associated Maxwell equations

3. This is a closed system, but no more useful then the RKMS since there are $O(N)$ PDEs.
4. Can't sum over i to get equation for $\widetilde{K}$

Fluctuation Equations - 3

$\epsilon = 0$ and sum fluctuation equations over $i \to O(1)$ PDEs

- Linearize in fluctuations i.e., set $\epsilon = 0$
- Sum fluctuation equations over i

gives the closed Vlasov-Maxwell system from $\overline{\text{RKMS}}$

$$\Psi_{1t} + \mathcal{L}^{\text{ext}}\Psi_1 + \nabla_p \cdot (\overline{\mathbf{F}}_1^{Lor}\Psi_1) = 0,$$

$$\partial_t \overline{\mathbf{B}}_1 = -\nabla_r \times \overline{\mathbf{E}}_1, \;\; \partial_t \overline{\mathbf{E}}_1 = c^2 \nabla_r \times \overline{\mathbf{B}}_1 - cZ_0 q N^* \int dp \mathbf{v}(p) \Psi_1(r,p,t),$$

coupled to the (summed-over-i) fluctuation equations

$$\widetilde{K}_t + \mathcal{L}^{\text{ext}}\widetilde{K} + \nabla_p \cdot (\widetilde{\mathbf{F}}^{Lor}\Psi_1 + \overline{\mathbf{F}}^{Lor}\widetilde{K}) = 0$$

$$\partial_t \widetilde{\mathbf{B}} = -\nabla_r \times \widetilde{\mathbf{E}}, \;\; \partial_t \widetilde{\mathbf{E}} = c^2 \nabla_r \times \widetilde{\mathbf{B}} - cZ_0 q \frac{N^*}{N} \int dp \mathbf{v}(p) \widetilde{K}(r,p,t;W^0)$$

Here $(\widetilde{K}, \widetilde{\mathbf{E}}, \widetilde{\mathbf{B}}) = \frac{1}{N}\sum(\widetilde{K}_i, \widetilde{\mathbf{E}}_i, \widetilde{\mathbf{B}}_i)$

Can KL distinction be seen here?

1. Random ICs are
$\widetilde{K}(r, p, 0; W^0) = \sum_1^N \delta(r - r_{i0})\delta(p - p_{i0}) - \Psi_1(q, p, 0)$

Summary

1. Klimontovich density, K, defined
2. RKMS defined as a random IVP
3. $\overline{\text{RKMS}}$ defined as, mean RKMS. $\overline{K} = \Psi_1$ introduced as main focus
4. Vlasov Maxwell emerges from $\overline{\text{RKMS}}$ when particles and fields uncorrelated giving $\Psi_1 = \Psi_{Vlasov}$
5. K_A introduced as coarse grained K. Set up for probabilistic analysis in framework of large sums of random variables. Modest result: $\overline{(K_A - \overline{K}_A)^2} = \frac{1}{N}(\Psi_{1A} - \Psi_{2A}) + (\Psi_{2A} - \Psi_{1A}^2)$
6. Ecker hierarchy introduced when particles and fields correlated: Levels I and II gives equations Ψ_1 and Ψ_2 must satisfy.
7. Ecker I with particles uncorrelated, i.e., $\Psi_2 \approx \Psi_1 \Psi_1'$, and quadratic terms dropped, gives closed system approximation to Ψ_1 and thus a correction to Ψ_{Vlasov}
8. Fluctuation equation for $\widetilde{K} = K - \Psi_1$ derived in a linearization. It forms a closed system when coupled to VM and is ready for analysis, e.g. in context of Kim-Lindberg collective and individual aspects of fluctuations.

Some Next Steps

1. Continue work on Ecker Hierarchy
2. Work out above details in Kim-Lindberg model
3. Consider $\mathbf{F}_{\text{ext}}(r, p, t)$ in FEL context. For $\epsilon := \sqrt{E_{\text{typical}}/B_{\text{undulator}}}/\gamma$ small, FEL pendulum behavior emerges in Method of Averaging[3] in a fixed traveling wave Maxwell field (See Poster and PRSTAB 16, 090702 (2013)). Study this situation in the full collective case described here (RKMS) exploring MoA in this coupled ODE-PDE context.
4. Intriguing: Solve Maxwell and insert into particle equations $\rightarrow$ Functional ODEs because of retarded times. What does the N-particle (functional) Liouville equation look like? Is there a generalized Ecker/BBGKY hierarchy for this?
5. Study the joint particle-field pdf, $\Phi(\mathbf{e}, \mathbf{b}, w, t))$. There must be an evolution law and integrating out the fields gives $\Psi_N(w, t)$
6. Sort out Radiation Reaction effects
7. Consider QED extension and quantum fluctuations

[3] Rigorous perturbation methods: MoA $\rightarrow$ Nekhoroshev $\rightarrow$ KAM - See Dumas' KAM Story