Novel Control Concepts for Heterogeneous Systems

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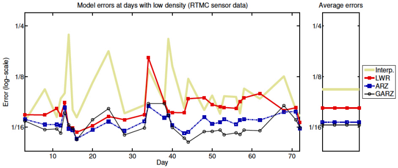
Workshop IV: Social Dynamics beyond Vehicle Autonomy
1st December 2020

Focus on Some Aspects of Social Dynamics beyond Vehicle Autonomy at Interface to possible Mathematical Questions

- Heterogeneous traffic and surroundings conditions are likely to be impossible to describe by deterministic models
- Different scales are present in integrated social and traffic models due to e.g. typical speeds or group sizes
- LV5 autonomy requires possibly provable stable control mechanisms to guarantee traffic safety



Possible Mathematical Descriptions and Questions

Aspect

- Heterogeneous traffic and surroundings conditions are likely to be impossible to describe by deterministic models
- LV5 autonomy requires possibly provable stable control mechanisms to guarantee traffic safety
- Different scales are present in integrated social and traffic models differentiating between typical speeds and group sizes

Mathematical terms

- Uncertainty quantification methods for different traffic models, here: hyperbolic flow models
- Stability / Optimality / Robustness of suitable closed loop control strategies, here: MPC
- Control across scales in nonlinear setting, here: optimal control for swarming type

Aim is to highlight possible contributions or mathematical methods

(Vehicular)Traffic Modeling at Different Scales due to Spatial, Temporal or Agent Aggregation

Hierarchy of traffic models

Microscopic scale

Bando's model (Section 3.2):

$$\dot{x}_i = v_i$$
$$\dot{v}_i = \frac{1}{\epsilon}\left(U_{\text{eq}}(\rho_i) - v_i\right)$$

It does not consider the interaction term and it is unstable if $\epsilon > \frac{1}{2|U'_{\text{eq}}(\rho)|}$

Follow-the-leader model (Section 4.1):

$$\dot{x}_i = v_i = w_i - h(\rho_i)$$

and

$$\dot{v}_i = C_\gamma \frac{(v_{i+1} - v_i)}{(x_{i+1} - x_i)^{\gamma+1}} + \frac{1}{\epsilon}\left(V_{\text{eq}}(\rho_i) - v_i\right)$$

or

$$\dot{w}_i = \frac{1}{\epsilon}\left(V_{\text{eq}}(\rho_i) + h(\rho_i) - w_i\right)$$

Mesoscopic scale

Classical BGK model (Section 3.2):

$$\partial_t f(t,x,v) + v\partial_x f(t,x,v) = \frac{1}{\epsilon}\left(M_f(v;\rho) - f(t,x,v)\right)$$

Sub-characteristic condition
$\int_V v^2 \partial_\rho M_f(v;\rho)\mathrm{d}v \geq \left(\int_V \partial_\rho M_f(v;\rho)\mathrm{d}v\right)^2$
never satisfied in dense traffic

Spatially homogeneous model (Section 2):

$$\partial_t f(t,v) = Q[f,f](t,v)$$

The solution of $Q[f,f] = 0$ provides the distribution at equilibrium $M_f(v;\rho)$

Closure $U_{\text{eq}}(\rho) = \int_V f \mathrm{d}v$ and $Q_{\text{eq}}(\rho) = \rho U_{\text{eq}}(\rho)$

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The sub-characteristic condition is conditionally verified in dense traffic

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Payne-Whitham type model (Section 3.2):

$$\partial_t \rho + \partial_x \rho u = 0$$
$$\partial_t \rho u + \partial_x \int_V v^2 f(t,x,v)\mathrm{d}v = \frac{1}{\epsilon}\left(Q_{\text{eq}}(\rho) - \rho u\right)$$

Sub-characteristic condition
$U'_{\text{eq}}(\rho)^2 \rho^2 < 0$
never satisfied

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Aw-Rascle and Zhang model (Section 3.3):

$$\partial_t \rho + \partial_x (q - \rho h(\rho)) = 0$$
$$\partial_t q + \partial_x \left(\frac{q^2}{\rho} - q h(\rho)\right) = \frac{1}{\epsilon}\left(Q_{\text{eq}}(\rho) + \rho h(\rho) - q\right)$$

where $q = \rho u + \rho h(\rho)$.

Sub-characteristic condition $U'_{\text{eq}} \geq -h'(\rho)$ conditionally verified in dense traffic

- Diagram of some examples of traffic models
- Depending on type of interaction coupling on different levels possible e.g. individual or fluid

Modeling Influence of Heterogeneous Conditions by Methods of Uncertainty Quantification

Exemplify on fluid–like (macroscopic) traffic flow model for density $\rho = \rho(t, x)$, velocity $v = v(t, x)$, property $z = z(t, x)$

$$\partial_t \begin{pmatrix} \rho \\ z \end{pmatrix} + \partial_x \begin{pmatrix} \rho v \\ z v \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{\tau} \left(\rho \left(U_{eq}(\rho) - v \right) \right) \end{pmatrix}, \; z = \rho v + \rho p(\rho)$$

- Model effect of heterogeneous conditions by random variable ξ with given probability distribution $\mathcal{P}$
- Example: relaxation parameter $\tau = \tau_0 + \xi$ or initial or boundary conditions
- Simplest case: Random variable does not have underlying dynamic: introduces parametric uncertainty also called ξ: $\rho = \rho(t, x, \xi)$ [1]

Efficient model predictions $\mathbb{E}(\rho(t, x))$ and $Var(\rho(t, x))$?

[1] and similar for the functions v, z

Generalized Polynomial Chaos Expansion or Karuhn–Loeve Expansion

Stochastic hyperbolic equation for $\rho = \rho(t, x, \xi)$ with uncertain relaxation time

$$\partial_t \begin{pmatrix} \rho \\ z \end{pmatrix} + \partial_x \begin{pmatrix} \rho v \\ z v \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{\tau(\xi)} \left(\rho \left(U_{eq}(\rho) - v \right) \right) \end{pmatrix}$$

- Multi-Level Monte Carlo, Stochastic Collocation, Moment methods, ...
- (Truncated) Series expansion in ξ base functions $\phi_i(\xi), i = 0, \ldots$ orthogonal w.r.t. to weight associate to $\mathcal{P}$

$$\rho(t, x, \xi) = \sum_{i=0}^{K} \rho_i(t, x) \phi_i(\xi)$$

- E.g. for Legendre polynomials $\mathbb{E}(\rho(t, x)) = \rho_0(t, x)$ and $Var(\rho(t, x)) = \sum_{i \geq 1} \rho_i(t, x)$

Mathematical Questions on Well-Posedness

Stochastic hyperbolic equation for $\rho = \rho(t, x, \xi)$ transformed to System of transport equations of size $2(K+1)$ for coefficients $\vec{\rho} = (\rho_0, \ldots, \rho_K)$:

$$\partial_t \begin{pmatrix} \vec{\rho} \\ \vec{z} \end{pmatrix} + \partial_x \int F(\vec{\rho}, \vec{z}, \vec{v}) \vec{\phi} dP = \int S(\vec{\rho}, \vec{z}, \vec{v}) \vec{\phi} dP$$

- $K = 0$ is the deterministic fluid model
- Formulation assumes the solution ρ belongs to the space of $2K+1$ base functions in ξ
- Requires to define Galerkin projection of F, i.e. (ρv) possible, requires a priori computation of a tensor

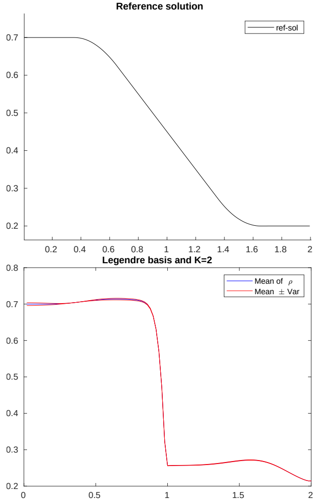
$$M_{\ell,i,k} = \int \phi_\ell(\xi)\phi_i(\xi)\phi_k(\xi) dP$$

Mathematical Questions on Well-Posedness (cont'd)

Stochastic hyperbolic equation for $\rho = \rho(t, x, \xi)$ transformed to system of transport equations of size $2(K+1)$ for coefficients

$$\partial_t \begin{pmatrix} \vec{\rho} \\ \vec{z} \end{pmatrix} + \partial_x \int F(\vec{\rho}, \vec{z}, \vec{v}) \vec{\phi} dP = \int S(\vec{\rho}, \vec{z}, \vec{v}) \vec{\phi} dP$$

- $K = 1$ and gPC expansion of (ρ, z, v) leads to loss of hyperbolicity
- Explicit example available having complex eigenvalues of the projected flux function
- Local Lax–Friedrichs scheme converges but yields qualitatively wrong solution



Hyperbolic gPC Formulation

Stochastic hyperbolic equation for $\rho = \rho(t, x, \xi)$ transformed to system of transport equations of size $2(K+1)$ for coefficients

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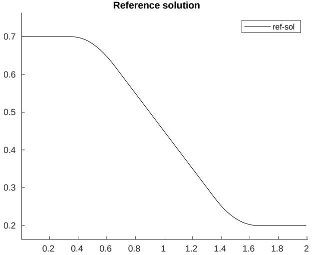
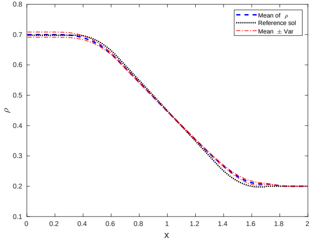
- Expand only (ρ, z) **not** v and recompute coefficient in v
- gPC projection of $v = \frac{z}{\rho} - p(\rho)$ requires to solve the (linear) system

$$\mathrm{P}(\vec{\rho})\vec{v} = \vec{z} - \mathrm{P}^{\gamma+1}(\rho)$$

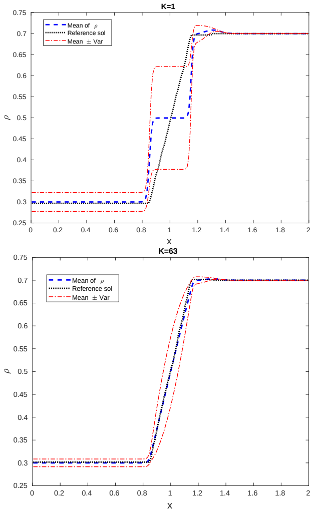
with $\mathrm{P}(\rho) = \vec{\rho}^{\,T} M$.





Results on the gPC Formulation

- $P(\rho)$ is positive definite for a subclass of base functions
- Hyperbolicity: Expansion in (ρ, z) and projection yields a strictly hyperbolic system for $\rho_0 > 0$
- Consistency: for deterministic $\tau \to 0$ recover the gPC formulation of the scalar (hyperbolic) LWR model
- Numerically: Observe convergence to reference for $K \to \infty$



Solution to Riemann problem $K = 1$ and $K = 63$ modes

Recall Example of Hierarchy of Traffic Models

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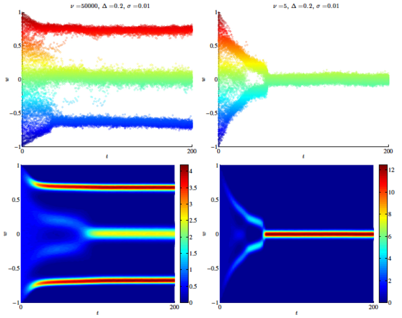
- Results on Uncertainty Quantification on Mesoscopic Level Exist
- Open or closed loop control consistent across the scales

Optimal Control Across Scales in Nonlinear Dynamics

Where may this problem arise?

- Control strategies for N (interacting) agents like large crowds of people, suitably many vehicles on large–scale highway system
- Goal is a consistent control for **any** number of agents including $N = \infty$
- Exemplified on the case of typical swarming type interactions with state x with single control u



uncontrolled and controlled on particle and meanfield

$$\frac{d}{dt}x_i = \frac{1}{N}\sum_{j=1}^{N} p(x_i, x_j, u), \;\; u = \text{ argmin }_{\tilde{u}} \int_0^T \frac{1}{N}\sum_{j=1}^{N} \phi(x_j, \tilde{u})dt$$

Finite (P) and corresponding Infinite (MF) Optimal Control Problem

$$(P)\ u = \text{ argmin }_{\tilde{u}} \int_0^T \frac{1}{N} \sum_j \phi(x_i, u) dt \text{ s.t. } \frac{d}{dt} x_i = \frac{1}{N} \sum_j p(x_i, x_j, u)$$

$$(MF)\ u = argmin_{\tilde{u}} \int_0^T \int \phi(x, u) \mu(x, t) dx \text{ s.t. } 0 = \partial_t \mu + div_x \left(\mu G_\mu \right)$$

ODE

- Pontryagins Maximum Principle and adjoint variables $z_i \in \mathbb{R}^K$
- BBGKY hierarchy leads to joint meanfield distribution $g = g(t, x, z)$ in state x and adjoint z for $N = \infty$

MF

- L^2–first–order optimality conditions

$$0 = \partial_t \mu + div_x \left(\mu G_\mu \right),$$
$$0 = -\partial_t \lambda - \nabla_x \lambda^T G_\mu -$$
$$- \int \mu(y, t) \nabla_x \lambda(y, t)^T p(y, x, u) dy + \phi$$

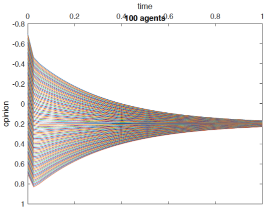
Necessary conditions for consistent control?

Necessary Conditions for Consistent Controls (P) vs (MF)

Lemma. Decompose $g(t,x,z) = \mu(x,t)\mu_c(z,x,t)$. Then, the finite and infinite problem are consistent provided that

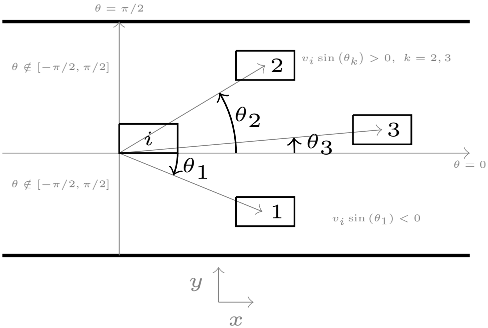
$$\nabla_x \lambda(t,x) = \int z\mu_c(z,x,t)dz$$

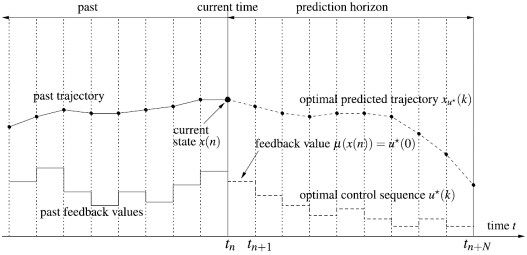
- Multiplier λ in the L^2 sense is gradient of the expectation of conditional probability density of BBGKY hierarchy
- Mesoscopic formulation allows to analyse consistent numerical discretization
- Extension to Game Theoretic Setting possible, e.g. Stackelberg game with infinitely many players



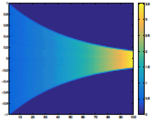
Including Uncertainty Using MPC Framework

- Open loop control maybe not suitable for on-board/online control of LV5 cars
- Sudden, unexpected events need to be accounted for by suitable control measures
- Receding horizon or Model–Predictive Control

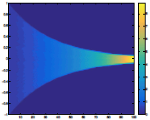




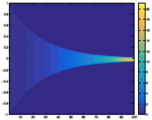
Performance of Receding Horizon Control Depending on Prediction Horizon



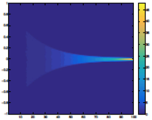
(a) $N = 2$



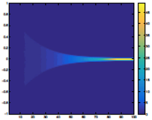
(b) $N = 3$



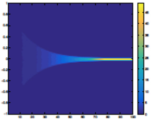
(c) $N = 4$



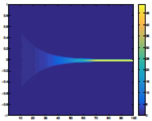
(d) $N = 5$



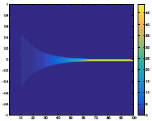
(e) $N = 6$



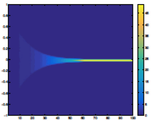
(f) $N = 7$



(g) $N = 8$



(h) $N = 9$



(i) $N = 10$

Evolution of state dynamics under receding horizon control with different prediction horizons N on Meanfield

Analytical Performance Bounds Available

$$V^*(\tau, Y) = \min_u \int_\tau^T h(X) + \frac{\nu}{2} u^2 ds, \;\; x_i'(t) = f(x_i(t), X_{-i}(t)) + u$$

$$V^{MPC}(\tau, y) = \int_\tau^T h(X^{MPC}) + \frac{\nu}{2} (u^{MPC})^2 ds, \;\; (x_i^{MPC})'(t) = f(X^{MPC}(t)) + u^{MPC}$$

- V^* value function for (full) optimal control u and initial data $X(\tau) = Y$
- V^{MPC} receding horizon control dynamics and corresponding value function
- Grüne [2009]: Finitely many particles. There exists $0 < \alpha < 1$ such that

$$V^{MPC}(\tau, y) \leq \frac{1}{\alpha} V^*(\tau, y)$$

- α depends on size of horizon and growth conditions of f

Bounds are Independent of Number of Particles

- Result extends to mesoscopic level under **same assumptions** ; also: value of α independent of number of particles
- Growth conditions are fulfilled e.g. simple swarming models
- Allows to control **a priori** prediction horizon
- Open: Stability on mesoscopic level

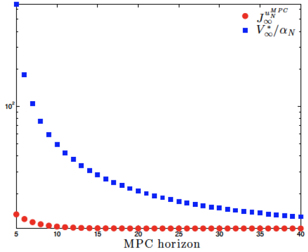


Figure 2. Value of the cost functional $J_T^{u_N^{MPC}}(X_0)$ for controls obtained using a MPC str with control horizon N (red) and presentation of the optimal costs $V_T^*(X_0)$ multiplied by where α_N is computed as in [28, Theorem 5.4]. For $N \leq 4$ no estimate of the type (2.11) cou established.

Quality of the estimate $V^{MPC}(\tau, y) \leq \frac{1}{\alpha} V^*(\tau, y)$.

blue=true/α, red=receding horizon

Summary

- Modeling heterogeneous aspects by random variables leads to interesting mathematical questions for example on macroscopic level
- Similar question of uncertainty on kinetic scale have been investigated but links are not fully explored
- For control actions some links between microscopic and kinetic scale are established
- Receding horizon methods for treating time dependent uncertainty are also scale invariant

Present results here are in collaboration with S. Gerster, E. Iacomini, C. Ringhofer and M. Zanella.

Thank you for your attention.

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