A multi-population traffic flow model on networks accounting for vehicle automation

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joint work with Adriano Festa (Polytechnic of Turin, Italy)
User (Wardrop) equilibrium

- inefficient network utilization
- Braess’ paradox
Navigation Apps Are Turning Quiet Neighborhoods Into Traffic Nightmares

The corner of Fort Lee Road and Broad Avenue in Leonia, N.J. With traffic apps suggesting shortcuts for commuters through the borough, officials have decided to take a stand. Bryan Anselm for The New York Times

Circulation : Waze, l’appli qui agace les riverains

Le succès de l’appli préférée des automobilistes génère quelques effets pervers. En détournant une partie du trafic sur des routes secondaires, elle déplace les nuisances et suscite la colère chez les riverains des villes traversées.

L’application Waze fait régulièrement passer par les petits villages pour éviter les bouchons. LeParisien

L.W. Foderaro (2017)

Outline of the talk

1. Conservation laws on networks
2. A multi-class model on networks
3. Examples
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Conservation laws on networks

Networks

Finite collection of directed arcs $I_\ell = [a_\ell, b_\ell]$ connected by nodes

1 [Holden-Risebro 1995; Garavello-Piccoli 2006]
LWR model\(^2\)

Non-linear transport equation: PDE for mass conservation

\[
\partial_t \rho + \partial_x f(\rho) = 0 \quad x \in \mathbb{R}, t > 0
\]

- \(\rho = \rho(t, x) \in [0, \rho_{\text{max}}]\) mean traffic density
- \(f(\rho) = \rho v(\rho)\) flux function

Empirical flux-density relation: fundamental diagram

\(^2\)\cite{Lighthill-Whitham 1955; Richards 1956}
Riemann problem at junctions

\[
\begin{aligned}
\begin{cases}
\partial_t \rho_\ell + \partial_x f_\ell(\rho_\ell) = 0 \\
\rho_\ell(0, x) = \rho_{\ell,0}
\end{cases}
\end{aligned}
\]

\[\ell = 1, \ldots, n + m\]

Riemann solver: \(\mathcal{RS}_J : (\rho_1,0, \ldots, \rho_{n+m},0) \mapsto (\bar{\rho}_1, \ldots, \bar{\rho}_{n+m})\) s.t.

- conservation of cars: \(\sum_{i=1}^{n} f_i(\bar{\rho}_i) = \sum_{j=n+1}^{n+m} f_j(\bar{\rho}_j)\)
- waves with negative speed in incoming roads
- waves with positive speed in outgoing roads

Consistency condition:

\[
\mathcal{RS}_J(\mathcal{RS}_J(\rho_1,0, \ldots, \rho_{n+m},0)) = \mathcal{RS}_J(\rho_1,0, \ldots, \rho_{n+m},0) \quad (CC)
\]
Riemann problem at junctions

\[
\begin{aligned}
\left\{ \begin{array}{l}
\partial_t \rho_\ell + \partial_x f_\ell(\rho_\ell) = 0 \\
\rho_\ell(0, x) = \rho_{\ell,0} \\
\ell = 1, \ldots, n + m
\end{array} \right.
\end{aligned}
\]

Riemann solver: \( \mathcal{R}S_J : (\rho_{1,0}, \ldots, \rho_{n+m,0}) \mapsto (\bar{\rho}_1, \ldots, \bar{\rho}_{n+m}) \) s.t.

- conservation of cars: \( \sum_{i=1}^{n} f_i(\bar{\rho}_i) = \sum_{j=n+1}^{n+m} f_j(\bar{\rho}_j) \)
- waves with negative speed in incoming roads
- waves with positive speed in outgoing roads

Consistency condition:

\( \mathcal{R}S_J(\mathcal{R}S_J(\rho_{1,0}, \ldots, \rho_{n+m,0})) = \mathcal{R}S_J(\rho_{1,0}, \ldots, \rho_{n+m,0}) \) (CC)

Set \( \bar{\gamma}_\ell = f_\ell(\bar{\rho}_\ell) \)
Demand & Supply

Incoming roads $i = 1, \ldots, n$:

$$\gamma_i^{\max}(\rho_{i,0}) = \begin{cases} f_i(\rho_{i,0}) & \text{if } 0 \leq \rho_{i,0} < \rho^{cr} \\ f_i^{\max} & \text{if } \rho^{cr} \leq \rho_{i,0} \leq 1 \end{cases}$$

Outgoing roads $j = n + 1, \ldots, n + m$:

$$\gamma_j^{\max}(\rho_{j,0}) = \begin{cases} f_j^{\max} & \text{if } 0 \leq \rho_{j,0} \leq \rho^{cr} \\ f_j(\rho_{j,0}) & \text{if } \rho^{cr} < \rho_{j,0} \leq 1 \end{cases}$$

\[\text{Admissible fluxes at junction: } \Omega_{\ell} = [0, \gamma_{\ell}^{\max}]\]
Demand & Supply \textsuperscript{3}

Incoming roads $i = 1, \ldots, n$:

$$\gamma^\text{max}_i (\rho_{i,0}) = \begin{cases} f_i (\rho_{i,0}) & \text{if } 0 \leq \rho_{i,0} < \rho^{\text{cr}}_i \\ f^\text{max}_i & \text{if } \rho^{\text{cr}}_i \leq \rho_{i,0} \leq 1 \end{cases}$$

Outgoing roads $j = n + 1, \ldots, n + m$:

$$\gamma^\text{max}_j (\rho_{j,0}) = \begin{cases} f^\text{max}_j & \text{if } 0 \leq \rho_{j,0} \leq \rho^{\text{cr}}_j \\ f_j (\rho_{j,0}) & \text{if } \rho^{\text{cr}}_j < \rho_{j,0} \leq 1 \end{cases}$$

Admissible fluxes at junction: $\Omega_\ell = [0, \gamma^\text{max}_\ell]$
Priority Riemann Solver$^4$

(A) **distribution matrix** of traffic from incoming to outgoing roads

\[ A = \{a_{ji}\} \in \mathbb{R}^{m \times n} : \quad 0 \leq a_{ji} \leq 1, \quad \sum_{j=n+1}^{n+m} a_{ji} = 1 \]

(B) priority vector

\[ P = (p_1, \ldots, p_n) \in \mathbb{R}^n : \quad p_i > 0, \quad \sum_{i=1}^{n} p_i = 1 \]

(C) feasible set

\[ \Omega = \left\{ (\gamma_1, \cdots, \gamma_n) \in \prod_{i=1}^{n} \Omega_i : A \cdot (\gamma_1, \cdots, \gamma_n)^T \in \prod_{j=n+1}^{n+m} \Omega_j \right\} \]

$^4$[DelleMonache-Goatin-Piccoli, CMS 2018]
Algorithm 1 Recursive definition of \( \mathcal{PRS} \)

Set \( J = \emptyset \) and \( J^c = \{1, \ldots, n\} \setminus J \).

while \( |J| < n \) do

\( \forall i \in J^c \rightarrow h_i = \max\{h : h p_i \leq \gamma_{i}^{max}\} = \frac{\gamma_{i}^{max}}{p_{i}}, \)

\( \forall j \in \{n+1, \ldots, n+m\} \rightarrow h_j = \sup\{h : \sum_{i \in J} a_{ji} Q_{i} + h(\sum_{i \in J^c} a_{ji} p_{i}) \leq \gamma_{j}^{max}\} \).

Set \( h = \min_{i,j} \{h_i, h_j\} \).

if \( \exists j \text{ s.t. } h_j = h \) then

Set \( Q = h P \) and \( J = \{1, \ldots, n\} \).

else

Set \( I = \{i \in J^c : h_i = h\} \) and \( Q_i = h p_i \) for \( i \in I \).\n
Set \( J = J \cup I \).

end if

end while
PRS in practice

2 × 2 junction \((n = 2, m = 2)\):

Define the spaces of the incoming fluxes
**PRS in practice**

$2 \times 2$ junction ($n = 2, m = 2$):

1. Define the spaces of the incoming fluxes
2. Consider the demands
**PRS in practice**

2 × 2 junction \((n = 2, \ m = 2)\):

1. Define the spaces of the incoming fluxes
2. Consider the demands
3. Trace the supply lines

\[
\gamma_{4}^{\text{max}} = a_{4,1} \gamma_{1} + a_{4,2} \gamma_{2}
\]

\[
\gamma_{3}^{\text{max}} = a_{3,1} \gamma_{1} + a_{3,2} \gamma_{2}
\]
**PRS in practice**

2 × 2 junction \((n = 2, m = 2)\):

1. Define the spaces of the incoming fluxes
2. Consider the demands
3. Trace the supply lines
4. The feasible set is given by \(\Omega\)

\[
\gamma_{2}^{\text{max}} = a_{4,1} \gamma_{1} + a_{4,2} \gamma_{2}
\]

\[
\gamma_{4}^{\text{max}} = a_{3,1} \gamma_{1} + a_{3,2} \gamma_{2}
\]
**PRS in practice**

2 × 2 junction \((n = 2, \, m = 2)\):

\[
\begin{align*}
\gamma^2_1 & = a_{3,1} \gamma_1 + a_{3,2} \gamma_2 \\
\gamma^2_2 & = a_{4,1} \gamma_1 + a_{4,2} \gamma_2
\end{align*}
\]

- Define the spaces of the incoming fluxes
- Consider the demands
- Trace the supply lines
- The feasible set is given by \(\Omega\)
- Trace the priority line
PRS in practice

2 × 2 junction \((n = 2, m = 2)\): 

\[
\gamma_2 = a_{2,1} \gamma_1 + a_{2,2} \gamma_2 \\
\gamma_3 = a_{3,1} \gamma_1 + a_{3,2} \gamma_2 \\
\gamma_4 = a_{4,1} \gamma_1 + a_{4,2} \gamma_2 \\
\]

1. Define the spaces of the incoming fluxes
2. Consider the demands
3. Trace the supply lines
4. The feasible set is given by \(\Omega\)
5. Trace the priority line

Different situations can occur
**(PRS): optimal point**

P intersects the supply lines in $\partial \Omega$
PRS: optimal point

P intersects the supply lines outside $\Omega$
General Riemann Solver at junctions:

- no restriction on $A$
- no restriction on the number of roads
- priorities come before flux maximization
- compact algorithm to compute solutions
- general existence result via Wave-Front-Tracking
Outline of the talk

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2. A multi-class model on networks
3. Examples
Multi-class model on networks

\( \rho^c_\ell \) density of vehicles of class \( c = 1, \ldots, N_c \) on link \( I_\ell \)
\( \rho_\ell = \sum_c \rho^c_\ell \) total traffic density on link \( I_\ell \)

\[
\partial_t \rho^c_\ell + \partial_x (\rho^c_\ell v_\ell (\rho_\ell)) = 0 \quad x \in I_\ell, t > 0,
\]

[Garavello-Piccoli, CMS 2005; Cristiani-Priuli, NHM 2015; Samanayarake&al, Tr. Sci. 2018]
Multi-class model on networks $^5$

$\rho^c_\ell$ density of vehicles of class $c = 1, \ldots, N_c$ on link $I_\ell$

$\rho_\ell = \sum_c \rho^c_\ell$ total traffic density on link $I_\ell$

$$\partial_t \rho^c_\ell + \partial_x (\rho^c_\ell v_\ell (\rho_\ell)) = 0 \quad x \in I_\ell, t > 0,$$

Summing on $c = 1, \ldots, N_c$ we get

$$\partial_t \rho_\ell + \partial_x (\rho_\ell v_\ell (\rho_\ell)) = 0 \quad x \in I_\ell, t > 0,$$

$^5$[Garavello-Piccoli, CMS 2005; Cristiani-Priuli, NHM 2015; Samanayarake\&al, Tr. Sci. 2018]
Multi-class junction conditions

1. **Compose the total distribution matrix.**
   \[ A^c = \{ a^c_{ji} \}_{i,j} \] distribution matrices for each class \( c = 1, \ldots, N_c \). Set
   \[
   A := \{ a_{ji} \}, \quad \text{where} \quad a_{ji} := \sum_{c=1}^{N_c} a^c_{ji} \frac{\rho^c_i}{\rho_i}
   \] (1)
   weighted distribution matrix for the *total density* of the populations at the junction.

2. **Compute the fluxes** \((\bar{\gamma}_1, \ldots, \bar{\gamma}_{n+m})\)
   using the selected Riemann solver \( \mathcal{RS}_J = \mathcal{RS}_J^A \) corresponding to (1).

3. **Distribute the fluxes among the various classes.**
   The incoming and outgoing fluxes for each class are given by
   \[
   \bar{\gamma}^c_i = \frac{\rho^c_i}{\rho_i} \bar{\gamma}_i, \quad i = 1, \ldots, n, \quad \bar{\gamma}^c_j = \sum_{i=1}^{n} a^c_{ji} \bar{\gamma}^c_i, \quad j = n + 1, \ldots, n + m.
   \]
Goal: minimize the weighted distance from the target $T^c$

Value function

$$u^c_\ell(y) = \inf \{d^c(y, x) : x \in T^c\}$$

where

$$d^c(y, x) = \inf \left\{ \int_y^{L_\ell} \frac{1}{g^c(z, t, \rho^\ell(z, t))} dz + \sum_i \int_0^{L_i} \frac{1}{g^c(z, t, \rho_{\ell_i}(z, t))} dz \right\}$$

$g^c$ being the running cost, thus giving the “shortest path”
Weighted distance from the target $\mathcal{T}^c$: $u^c_\ell$ viscosity solution of

$$
\begin{aligned}
\partial_x u^c_\ell(x) + \frac{1}{g^c(x,t,\rho^c_\ell(x,t))} = 0 & \quad x \in I_\ell \\
\min_{\ell \in \partial \text{Out}(J_k)} u^c_\ell(0) = u^c_\ell(L_l) & \quad J_k \in \mathcal{J} \setminus \mathcal{T}^c, \; l \in \text{Inc}(J_k) \\
u^c_\ell(L_\ell) = 0, & \quad \pi_\ell(L_\ell) \in \mathcal{T}^c
\end{aligned}
$$

where $g^c$ is the running cost ($g^c \equiv 1$ or $g^c = v_\ell$)

$\longrightarrow$ eikonal equation on network

[Schieborn-Camilli 2013; Camilli-Festa-Schieborn 2013]
Strategy modeling on network (cont’d)

Weighted distance from the target $\mathcal{T}^c$: $u^c_\ell$ viscosity solution of

\[
\begin{dcases}
\partial_x u^c_\ell(x) + \frac{1}{g^c(x,t,\rho^c_\ell(x,t))} = 0 & x \in I_\ell \\
\min_{\ell \in \text{Out}(J_k)} u^c_\ell(0) = u^c_\ell(L_l) & J_k \in \mathcal{J} \setminus \mathcal{T}^c, \; l \in \text{Inc}(J_k) \\
\alpha_{\ell l} = 0, & \pi^c_\ell(L_l) \in \mathcal{T}^c.
\end{dcases}
\]

where $g^c$ is the running cost ($g^c \equiv 1$ or $g^c = v^c$)

$\rightarrow$ **eikonal equation on network**

[Schieborn-Camilli 2013; Camilli-Festa-Schieborn 2013]

We set

\[
W^c_k := \left\{ l \in \text{Out}(J_k) : \; u^c_\ell(0) = \min_{j \in \text{Out}(J_k)} u^c_j(0) \right\}
\]

and

\[
A^c_k = \begin{cases} \alpha^c_{ji} = 1/|W^c_k|, & \text{if } j \in W^c_k \\ \alpha^c_{ji} = 0, & \text{otherwise} \end{cases}
\]
System discretization

Conservation laws:

\[
\rho_{c,\nu}^{\ell,+1} = \rho_{c,\nu}^{\ell,1} - \frac{\Delta t}{\Delta x_{\ell,1}} \left( \frac{\rho_{c,\nu}^{\ell,1}}{\rho_{\nu}^{\ell}} F_{\nu}^{\ell} - \gamma_{c,\nu}^{\ell,1} \right)
\]

\[
\rho_{c,\nu}^{\ell,h+1} = \rho_{c,\nu}^{\ell,h} - \frac{\Delta t}{\Delta x_{\ell,h}} \left( \frac{\rho_{c,\nu}^{\ell,h}}{\rho_{\nu}^{\ell,h}} F_{\nu}^{\ell,h} - \frac{\rho_{c,\nu}^{\ell,h-1}}{\rho_{\nu}^{\ell,h-1}} F_{\nu}^{\ell,h-1} \right)
\]

\[
\rho_{c,\nu}^{\ell,N_{\ell}} = \rho_{c,\nu}^{\ell,N_{\ell}} - \frac{\Delta t}{\Delta x_{\ell,N_{\ell}}} \left( \gamma_{c,\nu}^{\ell,N_{\ell}} - \frac{\rho_{c,\nu}^{\ell,N_{\ell}-1}}{\rho_{\nu}^{\ell,N_{\ell}-1}} F_{\nu}^{\ell,N_{\ell}-1} \right)
\]

where

\[
F_{\nu}^{\ell,h} = F_{\ell}(\rho_{\nu}^{\ell,h}, \rho_{\nu}^{\ell,h+1}) := \min \left\{ D_{\ell}(\rho_{\nu}^{\ell,h}), S_{\ell}(\rho_{\nu}^{\ell,h+1}) \right\} \text{ (Godunov scheme)}
\]

\[
\Delta t \leq \min_{\ell,h} \Delta x_{\ell,h}/V_{\ell} \text{ (CFL condition)}
\]

Eikonal equations:

\[
\frac{u_{c,\nu}^{\ell,h+1} - u_{c,\nu}^{\ell,h}}{\Delta x_{\ell,h}} + \frac{1}{g_{\nu}(\rho_{\nu}^{\ell,h})} = 0
\]

\[
u_{c,\nu}^{\ell,N_{\ell}} = \min_{i \in \text{Out}(J_{k})} u_{c,\nu}^{i,1}, \quad x_{\ell,N_{\ell}} = J_{k} \in \mathcal{J}
\]
System discretization (cont’d)

Junction coupling conditions:

\[ W_{k}^{c,\nu} = \left\{ l \in \text{Out}(J_k) : u_{l,1}^{c,\nu} = \min_{i \in \text{Out}(J_k)} u_{i,1}^{c,\nu} \right\}, \]

\[ A_{k}^{c,\nu} = \left\{ a_{ji}^{c,\nu} \right\}_{ji} : a_{ji}^{c,\nu} = \left\{ \begin{array}{ll} 1/|W_{k}^{c,\nu}|, & \text{if } j \in W_{k}^{c,\nu}, \\ 0, & \text{otherwise}, \end{array} \right. \]

\[ A_{k}^{\nu} = \left\{ \sum_{c=1}^{N_c} a_{ji}^{c,\nu} \rho_{i,N_i}^{c,\nu} \right\}_{ji}, \]

\[ (\bar{\gamma}_{1}, \ldots, \bar{\gamma}_{\ell n_k + m_k}) = \mathcal{R} \mathcal{S} A_{k}^{\nu}(\rho_{1}^{\nu}, \ldots, \rho_{\ell n_k + m_k}^{\nu}) \]

\[ \bar{\gamma}_{i,N_i}^{c,\nu} = \frac{\rho_{i,N_i}^{c,\nu}}{\rho_{i,N_i}^{\nu}} \bar{\gamma}_{i}^{\nu}, \quad i \in \text{Inc}(J_k), \]

\[ \bar{\gamma}_{j,1}^{c,\nu} = \sum_{i=\ell_1}^{\ell n_k} a_{ji}^{c,\nu} \bar{\gamma}_{i}^{c,\nu}, \quad j \in \text{Out}(J_k), \]

Initial and boundary conditions

\[ \rho_{\ell,h}^{c,0} = \frac{1}{\Delta x_{\ell,h}} \int_{x_{\ell,h}}^{x_{\ell,h+1}} \bar{\rho}_{\ell}^{c}(x) dx, \quad u_{\ell,N_{\ell}}^{c,\nu} = \rho_{\ell,N_{\ell}}^{c,\nu} = 0, \quad x_{\ell,N_{\ell}} \in T^{c}, \]
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Example 1\textsuperscript{6}: Pasadena

\textsuperscript{6}[Thai-LaurentBrouty-Bayen, IEEE ITS 2016]
Example 1

\[ v_1^\ell (\rho) = 1 - \rho, \quad g_1^1 (\rho) = 1 \]
\[ v_2^\ell (\rho) = 1 - \rho, \quad g_2^2 (\rho) = 1 - \rho \]

\[ \rho_1^{1,0} = (1 - P) \bar{\rho}, \quad \rho_1^{2,0} = P \bar{\rho}, \quad \bar{\rho} = 0.9 \]

P=0.5
**Example 1**

Total Travel Time in the whole network for each of the two populations and for the whole population depending on the penetration rate of routed vehicles $P$.

\[
TTT(\rho) = \Delta t \Delta x \sum_{\nu=0}^{N_f} \sum_{\ell \in L} \sum_{h=1}^{N_{\ell}} \rho_{\ell,h}^\nu
\]
Example 2: Sophia Antipolis
initial condition $\bar{\rho} = 0.8$

$P=0.5$
Example 2

initial condition $\bar{\rho} = 0.8$

P=0.5

population 1, time=752

population 2, time=752
Example 2

initial condition $\bar{\rho} = 0.8$

$P=0.5$

![Diagram of population 1, time=1504](image1)

non informed

![Diagram of population 2, time=1504](image2)

informed
Example 2

initial condition $\bar{\rho} = 0.8$

$P=0.5$

population 1, time=2256

non informed

population 2, time=2256

informed
Multi-population model accounting for routing choices:

- Can be applied to any Riemann Solver at junction
- Solves eikonal equations on networks
- Reproduces expected behaviour
- Can be extended to route choice based on traffic forecast
- Convergence?

[A. Festa and P. Goatin, Modeling the impact of on-line navigation devices in traffic flows, 2019 IEEE 58th Conference on Decision and Control (CDC), Nice, France (2019), 323-328.]
Related reference


Multi-buffer junction model accounting for variable routing ratios:

- Hamilton-Jacobi formulation of LWR
- Well-posedness by fixed-point theorem
- Suitable for solving optimal control problems as DTA
- Can (in principle) be extended to multi-commodity
Related reference


Multi-buffer junction model accounting for variable routing ratios:

- Hamilton-Jacobi formulation of LWR
- Well-posedness by fixed-point theorem
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