IPAM Long Program *Mathematical Challenges and Opportunities for Autonomous Vehicles*
Workshop III: "Large Scale Autonomy: Connectivity and Mobility Networks"

A multi-population traffic flow model on networks accounting for vehicle automation

Paola Goatin

INRIA Sophia Antipolis - Méditerranée
paola.goatin@inria.fr

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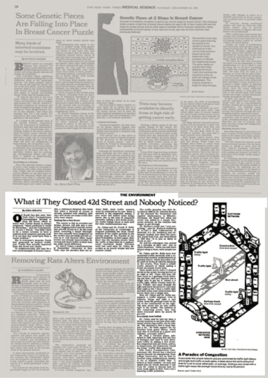
joint work with Adriano Festa (Polytechnic of Turin, Italy)

User (Wardrop) equilibrium

- inefficient network utilization
- Braess' paradox


MEDICAL SCIENCE

Some Genetic Pieces Are Falling Into Place In Breast Cancer Puzzle

Genetic Flaws at 2 Steps In Breast Cancer

THE ENVIRONMENT

What if They Closed 42d Street and Nobody Noticed?

Removing Rats Alters Environment

A Paradox of Congestion

Navigation apps side-effects

The New York Times

Navigation Apps Are Turning Quiet Neighborhoods Into Traffic Nightmares

The corner of Fort Lee Road and Broad Avenue in Leonia, N.J. With traffic apps suggesting shortcuts for commuters through the borough, officials have decided to take a stand. Bryan Anselm for The New York Times

L.W. Foderaro (2017)

J.-G. B. (2018)

leparisien.fr/info-paris-ile-de-france-oise/transports/circulation-waze-l-appli-qui-agace...

MENU   Le Parisien   S'ABONNER

Circulation : Waze, l'appli qui agace les riverains

Le succès de l'appli préférée des automobilistes génère quelques effets pervers. En détournant une partie du trafic sur des routes secondaires, elle déplace les nuisances et suscite la colère chez les riverains des villes traversées.

L'application Waze fait régulièrement passer par les petits villages pour éviter les bouchons. LP/Arnaud Dumontier

Outline of the talk

1. Conservation laws on networks
2. A multi-class model on networks
3. Examples

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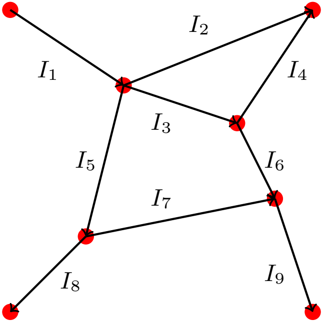
Conservation laws on networks[1]

Networks

Finite collection of directed arcs $I_\ell =]a_\ell, b_\ell[$ connected by nodes



[1] [Holden-Risebro 1995; Garavello-Piccoli 2006]

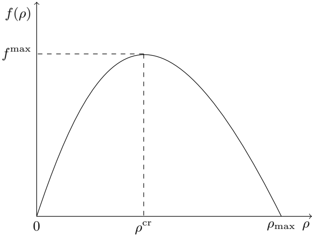
LWR model[2]

Non-linear transport equation: PDE for mass conservation

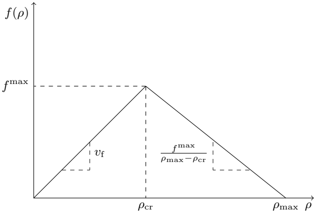
$$\partial_t \rho + \partial_x f(\rho) = 0 \qquad x \in \mathbb{R}, t > 0$$

- $\rho = \rho(t,x) \in [0, \rho_{\max}]$ mean traffic density
- $f(\rho) = \rho v(\rho)$ flux function

Empirical flux-density relation: fundamental diagram



Greenshields '35

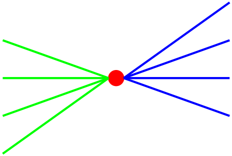


Newell-Daganzo

[2] [Lighthill-Whitham 1955; Richards 1956]

Riemann problem at junctions

$$\begin{cases} \partial_t \rho_\ell + \partial_x f_\ell(\rho_\ell) = 0 \\ \rho_\ell(0,x) = \rho_{\ell,0} \end{cases}$$
$$\ell = 1, \dots, n+m$$



Riemann solver: $\mathcal{RS}_J : (\rho_{1,0}, \dots, \rho_{n+m,0}) \longmapsto (\bar\rho_1, \dots, \bar\rho_{n+m})$ s.t.

- conservation of cars: $\sum_{i=1}^{n} f_i(\bar\rho_i) = \sum_{j=n+1}^{n+m} f_j(\bar\rho_j)$
- waves with negative speed in incoming roads
- waves with positive speed in outgoing roads

Consistency condition:

$$\mathcal{RS}_J\big(\mathcal{RS}_J(\rho_{1,0}, \dots, \rho_{n+m,0})\big) = \mathcal{RS}_J(\rho_{1,0}, \dots, \rho_{n+m,0}) \qquad \textbf{(CC)}$$

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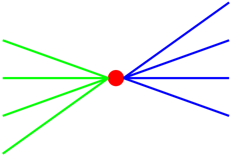

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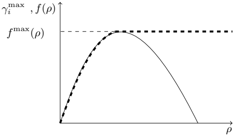
Set $\bar\gamma_\ell = f_\ell(\bar\rho_\ell)$

Demand & Supply [3]

Incoming roads $i = 1, \ldots, n$:

$$\gamma_i^{\max}(\rho_{i,0}) = \begin{cases} f_i(\rho_{i,0}) & \text{if } 0 \le \rho_{i,0} < \rho^{\mathrm{cr}} \\ f_i^{\max} & \text{if } \rho^{\mathrm{cr}} \le \rho_{i,0} \le 1 \end{cases}$$



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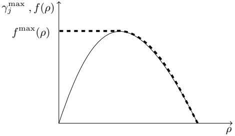

[3] [Lebacque]

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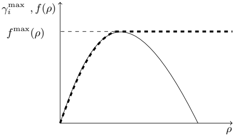

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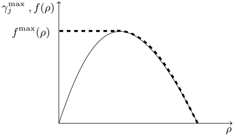


Admissible fluxes at junction: $\Omega_\ell = [0, \gamma_\ell^{\max}]$

[3] [Lebacque]

Priority Riemann Solver[4]

(A) distribution matrix of traffic from incoming to outgoing roads

$$A = \{a_{ji}\} \in \mathbb{R}^{m \times n} : \quad 0 \leq a_{ji} \leq 1, \sum_{j=n+1}^{n+m} a_{ji} = 1$$

(B) priority vector

$$P = (p_1, \ldots, p_n) \in \mathbb{R}^n : \quad p_i > 0, \sum_{i=1}^{n} p_i = 1$$

(C) feasible set

$$\Omega = \left\{ (\gamma_1, \cdots, \gamma_n) \in \prod_{i=1}^{n} \Omega_i : A \cdot (\gamma_1, \cdots, \gamma_n)^T \in \prod_{j=n+1}^{n+m} \Omega_j \right\}$$

[4] [DelleMonache-Goatin-Piccoli, CMS 2018]

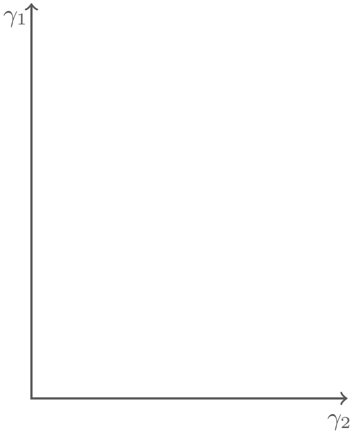
Priority Riemann Solver

Algorithm 1 Recursive definition of $\mathcal{PRS}$

Set $J = \emptyset$ and $J^c = \{1, \ldots, n\} \setminus J$.
while $|J| < n$ **do**
  $\forall i \in J^c \rightarrow h_i = \max\{h : h\, p_i \leq \gamma_i^{max}\} = \frac{\gamma_i^{max}}{p_i}$,
  $\forall j \in \{n+1 \ldots, n+m\} \rightarrow h_j = \sup\{h : \sum_{i \in J} a_{ji} Q_i + h(\sum_{i \in J^c} a_{ji} p_i) \leq \gamma_j^{max}\}$.
  Set $\hbar = \min_{ij}\{h_i, h_j\}$.
  if $\exists\ j$ s.t. $h_j = \hbar$ **then**
    Set $Q = \hbar\, P$ and $J = \{1, \ldots, n\}$.
  else
    Set $I = \{i \in J^c : h_i = \hbar\}$ and $Q_i = \hbar\, p_i$ for $i \in I$.
    Set $J = J \cup I$.
  end if
end while

$\mathcal{PRS}$ in practice

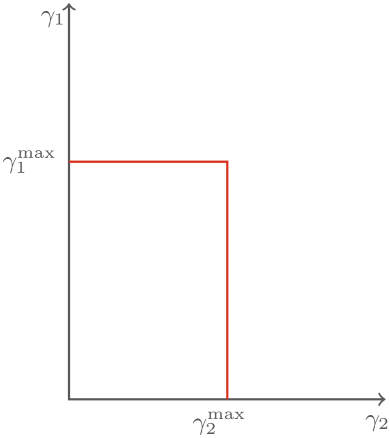
2×2 junction ($n = 2$, $m = 2$):



1. Define the spaces of the incoming fluxes

$\mathcal{PRS}$ in practice

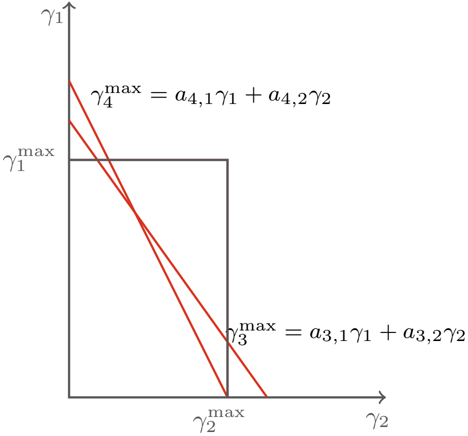
2×2 junction ($n = 2$, $m = 2$):



1. Define the spaces of the incoming fluxes
2. Consider the demands

$\mathcal{PRS}$ in practice

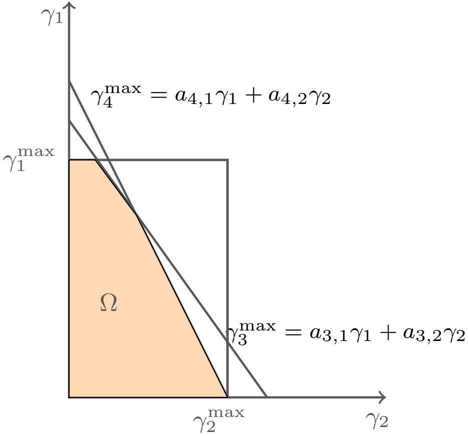
2×2 junction $(n = 2,\ m = 2)$:



1. Define the spaces of the incoming fluxes
2. Consider the demands
3. Trace the supply lines

$\mathcal{PRS}$ in practice

2×2 junction ($n = 2$, $m = 2$):



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3. Trace the supply lines
4. The feasible set is given by Ω

$\mathcal{PRS}$ in practice

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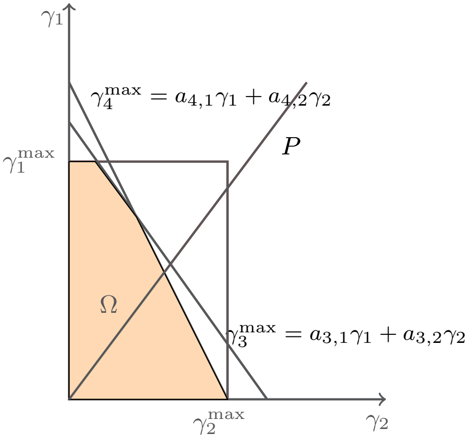

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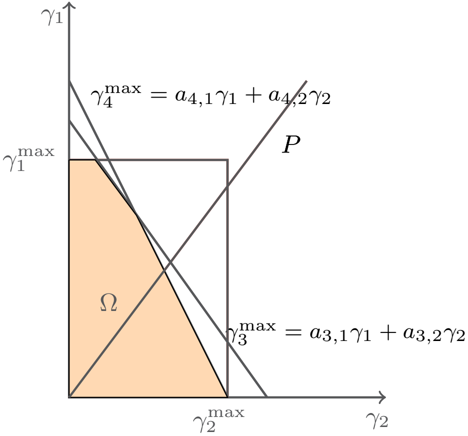
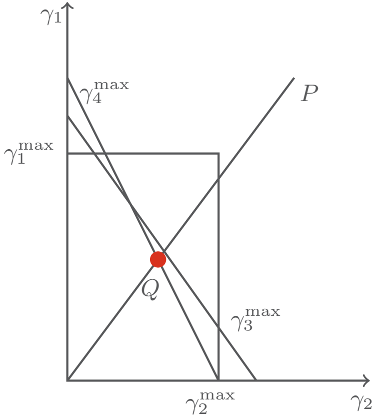


1. Define the spaces of the incoming fluxes
2. Consider the demands
3. Trace the supply lines
4. The feasible set is given by Ω
5. Trace the priority line

Different situations can occur

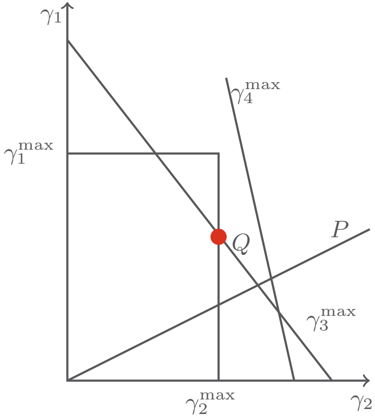
$\mathcal{PRS}$: optimal point

P intersects the supply lines in $\partial\Omega$



$\mathcal{PRS}$: optimal point

P intersects the supply lines outside Ω



$\mathcal{PRS}$: summary

General Riemann Solver at junctions:

- no restriction on A
- no restriction on the number of roads
- priorities come before flux maximization
- compact algorithm to compute solutions
- general existence result via Wave-Front-Tracking

Outline of the talk

Multi-class model on networks[5]

ρ_ℓ^c density of vehicles of class $c = 1, \ldots, N_c$ on link I_ℓ
$\rho_\ell = \sum_c \rho_\ell^c$ total traffic density on link I_ℓ

$$\partial_t \rho_\ell^c + \partial_x(\rho_\ell^c v_\ell(\rho_\ell)) = 0 \qquad x \in I_\ell, t > 0,$$

[5] [Garavello-Piccoli, CMS 2005; Cristiani-Priuli, NHM 2015; Samanayarake&al, Tr. Sci. 2018]

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$$\partial_t \rho_\ell^c + \partial_x(\rho_\ell^c v_\ell(\rho_\ell)) = 0 \qquad x \in I_\ell, t > 0,$$

Summing on $c = 1, \dots, N_c$ we get

$$\partial_t \rho_\ell + \partial_x(\rho_\ell v_\ell(\rho_\ell)) = 0 \qquad x \in I_\ell, t > 0,$$

[5] [Garavello-Piccoli, CMS 2005; Cristiani-Priuli, NHM 2015; Samanayarake&al, Tr. Sci. 2018]

Multi-class junction conditions

1. *Compose the total distribution matrix.*
   $A^c = \left\{a^c_{ji}\right\}_{i,j}$ distribution matrices for each class $c = 1, \dots, N_c$. Set

$$A := \{a_{ji}\}, \quad \text{where} \quad a_{ji} := \sum_{c=1}^{N_c} a^c_{ji} \frac{\rho^c_i}{\rho_i} \tag{1}$$

   weighted distribution matrix for the *total density* of the populations at the junction.

2. *Compute the fluxes* $(\bar{\gamma}_1, \dots, \bar{\gamma}_{n+m})$
   using the selected Riemann solver $\mathcal{RS}_J = \mathcal{RS}^A_J$ corresponding to (1).

3. *Distribute the fluxes among the various classes.*
   The incoming and outgoing fluxes for each class are given by

$$\bar{\gamma}^c_i = \frac{\rho^c_i}{\rho_i}\bar{\gamma}_i, \quad i = 1, \dots, n, \quad \bar{\gamma}^c_j = \sum_{i=1}^{n} a^c_{ji}\bar{\gamma}^c_i, \qquad j = n+1, \dots, n+m.$$

Strategy modeling on network

Goal: minimize the weighted distance from the target $\mathcal{T}^c$

Value function

$$u_\ell^c(y) = \inf\{d_c(y,x) \colon x \in \mathcal{T}^c\}$$

where

$$d_c(y,x) = \inf\left\{\int_y^{L_\ell} \frac{1}{g^c(z,t,\rho_\ell(z,t))}dz + \sum_i \int_0^{L_i} \frac{1}{g^c(z,t,\rho_{\ell_i}(z,t))}dz\right\}$$

g^c being the *running cost*, thus giving the "shortest path"

Strategy modeling on network (cont'd)

Weighted distance from the target $\mathcal{T}^c$: u_ℓ^c viscosity solution of

$$\begin{cases} \partial_x u_\ell^c(x) + \frac{1}{g^c(x,t,\rho_\ell(x,t))} = 0 & x \in I_\ell \\ \min\limits_{\ell \in Out(J_k)} u_\ell^c(0) = u_\ell^c(L_l) & J_k \in \mathcal{J} \setminus \mathcal{T}^c,\ l \in Inc(J_k) \\ u_\ell^c(L_\ell) = 0\,, & \pi_\ell(L_\ell) \in \mathcal{T}^c \end{cases}$$

where g^c is the running cost ($g^c \equiv 1$ or $g^c = v_\ell$)

$\longrightarrow$ **eikonal equation on network**

[Schieborn-Camilli 2013; Camilli-Festa-Schieborn 2013]

Strategy modeling on network (cont'd)

Weighted distance from the target $\mathcal{T}^c$: u_ℓ^c viscosity solution of

$$\begin{cases} \partial_x u_\ell^c(x) + \frac{1}{g^c(x,t,\rho_\ell(x,t))} = 0 & x \in I_\ell \\ \min\limits_{\ell \in Out(J_k)} u_\ell^c(0) = u_\ell^c(L_l) & J_k \in \mathcal{J} \setminus \mathcal{T}^c,\ l \in Inc(J_k) \\ u_\ell^c(L_\ell) = 0\,, & \pi_\ell(L_\ell) \in \mathcal{T}^c \end{cases}$$

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$\longrightarrow$ **eikonal equation on network**

[Schieborn-Camilli 2013; Camilli-Festa-Schieborn 2013]

We set

$$W_k^c := \left\{ l \in Out(J_k) : \quad u_l^c(0) = \min_{j \in Out(J_k)} u_j^c(0) \right\}$$

and

$$A_k^c = \begin{cases} \alpha_{ji}^c = 1/|W_k^c|, & \text{if } j \in W_k^c \\ \alpha_{ji}^c = 0, & \text{otherwise} \end{cases}$$

System discretization

Conservation laws:

$$\rho_{\ell,1}^{c,\nu+1} = \rho_{\ell,1}^{c,\nu} - \frac{\Delta t}{\Delta x_{\ell,1}} \left(\frac{\rho_{\ell,1}^{c,\nu}}{\rho_{\ell,1}^{\nu}} F_{\ell,1}^{\nu} - \bar{\gamma}_{\ell,1}^{c,\nu} \right)$$

$$\rho_{\ell,h}^{c,\nu+1} = \rho_{\ell,h}^{c,\nu} - \frac{\Delta t}{\Delta x_{\ell,h}} \left(\frac{\rho_{\ell,h}^{c,\nu}}{\rho_{\ell,h}^{\nu}} F_{\ell,h}^{\nu} - \frac{\rho_{\ell,h-1}^{c,\nu}}{\rho_{\ell,h-1}^{\nu}} F_{\ell,h-1}^{\nu} \right)$$

$$\rho_{\ell,N_\ell}^{c,\nu+1} = \rho_{\ell,N_\ell}^{c,\nu} - \frac{\Delta t}{\Delta x_{\ell,N_\ell}} \left(\bar{\gamma}_{\ell,N_\ell}^{c,\nu} - \frac{\rho_{\ell,N_\ell-1}^{c,\nu}}{\rho_{\ell,N_\ell-1}^{\nu}} F_{\ell,N_\ell-1}^{\nu} \right)$$

where

$F_{\ell,h}^{\nu} = F_\ell(\rho_{\ell,h}^{\nu}, \rho_{\ell,h+1}^{\nu}) := \min\left\{ D_\ell(\rho_{\ell,h}^{\nu}), S_\ell(\rho_{\ell,h+1}^{\nu}) \right\}$ (Godunov scheme)

$\Delta t \leq \min_{\ell,h} \Delta x_{\ell,h}/V_\ell$ (CFL condition)

Eikonal equations:

$$\frac{u_{\ell,h+1}^{c,\nu} - u_{\ell,h}^{c,\nu}}{\Delta x_{\ell,h}} + \frac{1}{g^c(\rho_{\ell,h}^{\nu})} = 0$$

$$u_{\ell,N_\ell}^{c,\nu} = \min_{i \in Out(J_k)} u_{i,1}^{c,\nu}, \quad x_{\ell,N_\ell} = J_k \in \mathcal{J}$$

System discretization (cont'd)

Junction coupling conditions:

$$W_k^{c,\nu} = \left\{ l \in Out(J_k) \colon u_{l,1}^{c,\nu} = \min_{i \in Out(J_k)} u_{i,1}^{c,\nu} \right\},$$

$$A_k^{c,\nu} = \left\{ a_{ji}^{c,\nu} \right\}_{ji} : a_{ji}^{c,\nu} = \begin{cases} 1/|W_k^{c,\nu}|, & \text{if } j \in W_k^{c,\nu}, \\ 0, & \text{otherwise}, \end{cases}$$

$$A_k^{\nu} = \left\{ \sum_{c=1}^{N_c} a_{ji}^{c,\nu} \frac{\rho_{i,N_i}^{c,\nu}}{\rho_{i,N_i}^{\nu}} \right\}_{ji},$$

$$(\bar{\gamma}_{\ell_1}^{\nu}, \dots, \bar{\gamma}_{\ell_{n_k+m_k}}^{\nu}) = \mathcal{RS}_{A_k^{\nu}}(\rho_{\ell_1}^{\nu}, \dots, \rho_{\ell_{n_k+m_k}}^{\nu})$$

$$\bar{\gamma}_{i,N_i}^{c,\nu} = \frac{\rho_{i,N_i}^{c,\nu}}{\rho_{i,N_i}^{\nu}} \bar{\gamma}_i^{\nu}, \quad i \in Inc(J_k),$$

$$\bar{\gamma}_{j,1}^{c,\nu} = \sum_{i=\ell_1}^{\ell_{n_k}} a_{ji}^{c,\nu} \bar{\gamma}_i^{c,\nu}, \quad j \in Out(J_k),$$

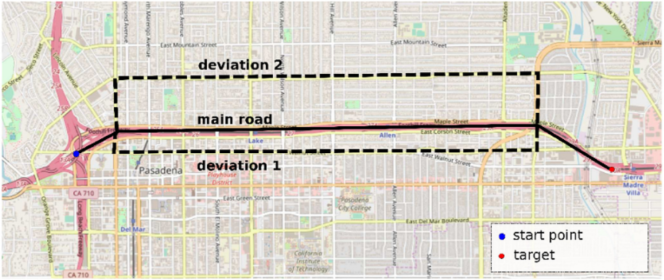
Initial and boundary conditions

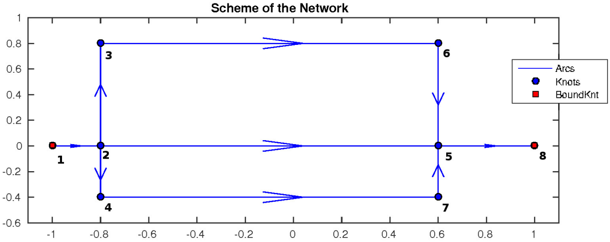
$$\rho_{\ell,h}^{c,0} = \frac{1}{\Delta x_{\ell,h}} \int_{x_{\ell,h}}^{x_{\ell,h+1}} \bar{\rho}_\ell^c(x)dx, \qquad u_{\ell,N_\ell}^{c,\nu} = \rho_{\ell,N_\ell}^{c,\nu} = 0, \quad x_{\ell,N_\ell} \in \mathcal{T}^c,$$

Outline of the talk

1. Conservation laws on networks
2. A multi-class model on networks
3. Examples

Example 1[6]: Pasadena





[6] [Thai-LaurentBrouty-Bayen, IEEE ITS 2016]

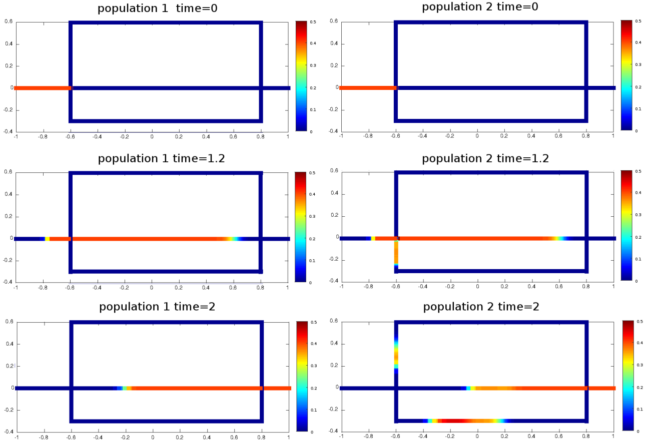
Example 1

$$v_\ell^1(\rho) = 1 - \rho, \quad g^1(\rho) = 1$$

$$v_\ell^2(\rho) = 1 - \rho, \quad g^2(\rho) = 1 - \rho$$

$$\rho_1^{1,0} = (1-P)\bar{\rho}, \quad \rho_1^{2,0} = P\bar{\rho}, \quad \bar{\rho} = 0.9$$

P=0.5



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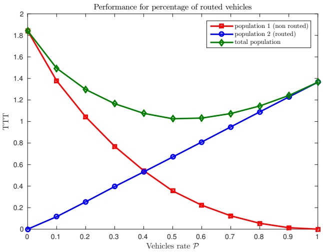
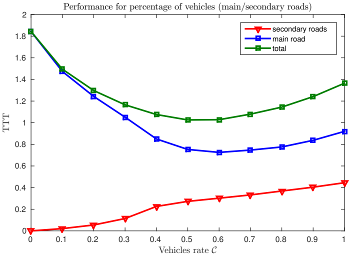


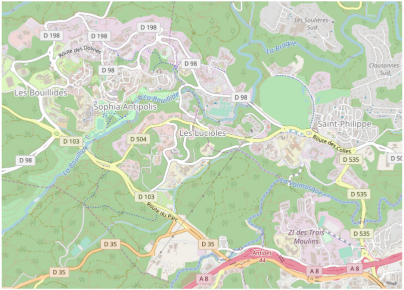
Total Travel Time in the whole network for each of the two populations and for the whole population depending on the penetration rate of routed vehicles $\mathcal{P}$

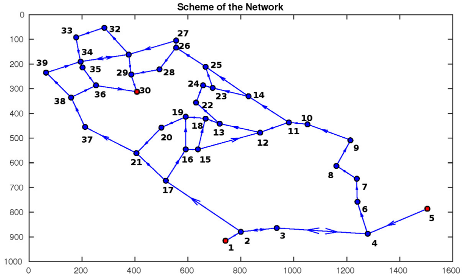


Total Travel Time in the main road and in the two detours to reach destination depending on the penetration rate of routed vehicles $\mathcal{P}$

$$TTT(\rho) = \Delta t \Delta x \sum_{\nu=0}^{N_f} \sum_{\ell \in \mathcal{L}} \sum_{h=1}^{N_\ell} \rho_{\ell,h}^{\nu}$$

Example 2: Sophia Antipolis

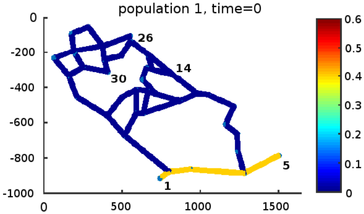




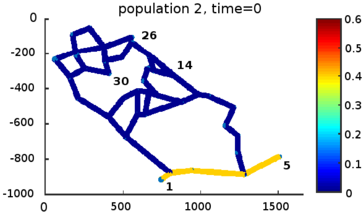
Example 2

initial condition $\bar{\rho} = 0.8$

P=0.5



non informed



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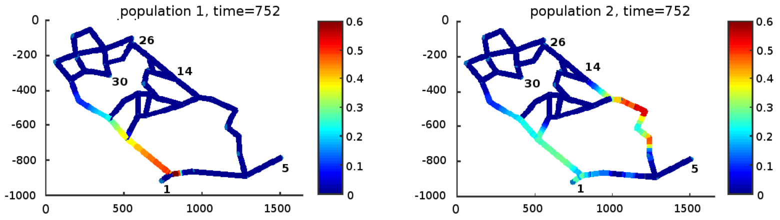

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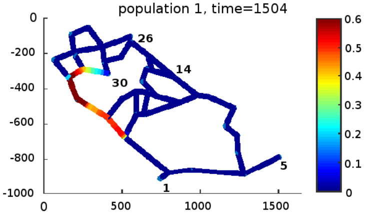
P=0.5



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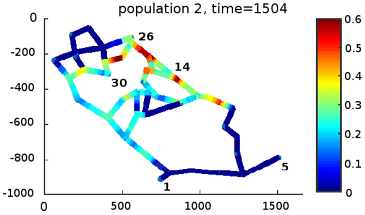

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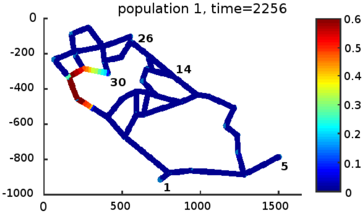

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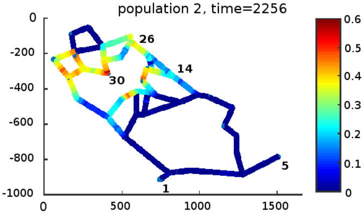

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Conclusion

[A. Festa and P. Goatin, Modeling the impact of on-line navigation devices in traffic flows, 2019 IEEE 58th Conference on Decision and Control (CDC), Nice, France (2019), 323-328.]

Multi-population model accounting for routing choices:

- Can be applied to any Riemann Solver at junction
- Solves eikonal equations on networks
- Reproduces expected behaviour
- Can be extended to route choice based on traffic forecast
- Convergence?

Related reference

[N. Laurent-Brouty, A. Keimer, P. Goatin and A. Bayen, A macroscopic traffic flow model with finite buffers on networks: Well-posedness by means of Hamilton-Jacobi equations, Comm. Math. Sci., 18(6) (2020), 1569-1604.]

Multi-buffer junction model accounting for variable routing ratios:

- Hamilton-Jacobi formulation of LWR
- Well-posedness by fixed-point theorem
- Suitable for solving optimal control problems as DTA
- Can (in principle) be extended to multi-commodity

Related reference

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Dr. Paola Goatin
Inria Sophia Antipolis - Méditerranée
France
paola.goatin@inria.fr
http://www-sop.inria.fr/members/Paola.Goatin/