# On the influence of time delays in vehicular traffic

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## Outline

1. Framework and motivations
2. Delayed LWR model and its theoretical properties
   - Formal derivation
   - Conservation - Positivity - Boundedness
3. Numerical discretization and its properties
4. Numerical tests
   - Key example: Stop & Go waves
5. Second order traffic models with time delay

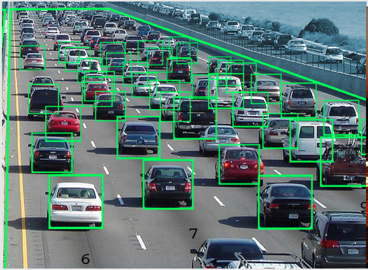
## Outline

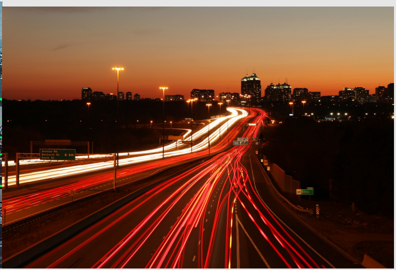
# Framework

## Framework

Mathematical modeling of traffic flow on a **single road**, by means of both

- microscopic (agent-based) follow-the-leader models based on a system of ODEs
- MACROSCOPIC (fluid-dynamic) models based on conservation laws



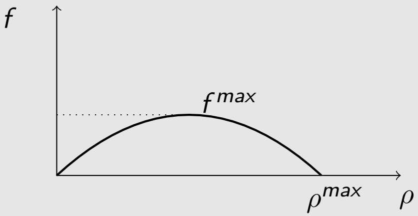
## Framework: Macroscopic Model

$\rho(x,t)$ density of cars at point $x$ and time $t$

$V(x,t)$ velocity of cars at point $x$ and time $t$

$f(x,t) = \rho(x,t)V(x,t)$ flux of cars at point $x$ and time $t$

The fundamental diagram establishes the relationship between the flux and the density of vehicles, i.e. $\{(\rho(x,t), f(x,t)) \ : \ x \in \mathbb{R},\ t > 0\}$



## Classical LWR Model

### Lighthill-Whitham-Richards (LWR) model (1955)

$$\partial_t \rho(x,t) + \partial_x(\rho(x,t)V(\rho(x,t))) = 0, \qquad x \in \mathbb{R},\ t > 0$$

Main features

- is a hyperbolic conservation law (the total mass has to be preserved)
- the velocity depends on the density and typically $V(\rho) = V_{\max}\left(1 - \frac{\rho}{\rho_{\max}}\right)$
- easy to implement and very cheap in terms of memory and time
- can be used also on large networks for traffic forecast
- **BUT** accelerations are considered to be instantaneous and traffic is described only at the equilibrium (no typical features of traffic dynamics as capacity drop or spontaneous congestions like Stop & Go waves)

## Outline

## Derivation of the delayed LWR model

**Question**

Is there a significant impact of time delays in (macroscopic) traffic models?

- Consider a delayed microscopic model by Newell[1]

$$\dot{x}_i(t) = W\Big(\frac{\Delta x_i(t-T)}{\Delta X}\Big), \quad i = 1, \ldots, N \tag{1}$$

where $W(\cdot)$ is a velocity function, $T > 0$ a reaction time, $\Delta x_i(t) = x_{i+1} - x_i$ the spacing between vehicle $i$ and $i+1$ and $\Delta X > 0$ a space scaling (i.e. average length of a vehicle)

- **Goal**: derive a LWR type model with explicit delay dependence

[1] Newell, Oper. Res., 1961

## Derivation of the delayed LWR model

- Define the traffic density $\rho$ as the inter-vehicle-spacing[2]

$$\rho_i(t) = \frac{\Delta X}{\Delta x_i(t-T)}$$

- Rewriting in terms of $V$ and inserting into (1)

$$\dot{x}_i(t) = W\Big(\frac{1}{\rho_i(t-T)}\Big) = V(\rho_i(t-T))$$

- To link the microscopic and macroscopic description, we consider

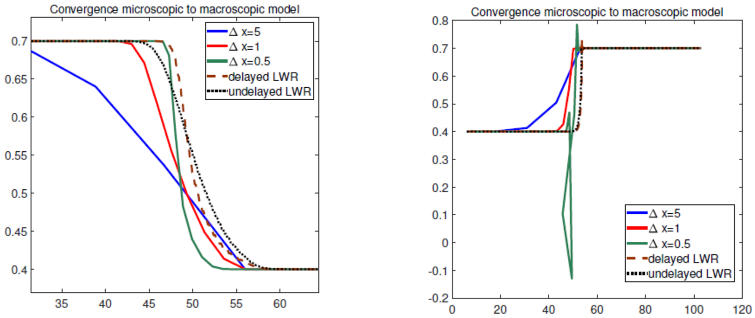
$$\partial_t \frac{1}{\rho_i(t)} = \partial_t \frac{\Delta x_i(t)}{\Delta X} = \frac{V(\rho_{i+1}(t-T)) - V(\rho_i(t-T))}{\Delta X}$$

- Limit process $N \to \infty$ leads to $\partial_t \frac{1}{\rho(y,t)} = \partial_y V(\rho(y, t-T))$ or

$$\partial_t \rho(x,t) + \partial_x(\rho(x,t)V(\rho(x,t-T))) = 0$$

[2] Aw, Klar, Materne, Rascle, SIAP, 2002

## Microscopic vs. macroscopic approach

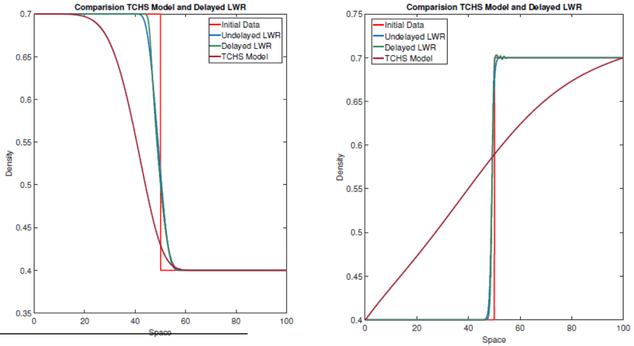



- Two special scenarios: rarefaction wave (left) and shock wave (right)
- Convergence of the microscopic model for $\Delta X$ small enough can be observed
- Since the microscopic model is NOT collision free, the model breaks down for $\Delta X = 0.5$

## Comments on the delayed LWR model

- A convection-diffusion equation[3] can be also derived from (1) by using a Taylor expansion in the argument of $W$:

$$\partial_t \rho + \partial_x(\rho V(\rho)) = -T\partial_x((\rho V'(\rho))^2 \partial_x \rho) \qquad \text{(TCHS)}$$




[3] Tordeux, Costeseque, Herty, Seyfried, SIAP, 2018

# Delayed LWR model

## Delayed LWR model

$$\begin{cases} \partial_t \rho(x,t) + \partial_x \left(\rho(x,t)\; V(\rho(x,t-T))\right) = 0 & x \in \mathbb{R},\ t \in [0, T_f] \\ \rho(x,t) = \rho^0(x) & x \in \mathbb{R},\ t \in [-T, 0] \end{cases}$$

- $T > 0$ is the time delay, which represents the reaction time of both drivers and vehicles
- Note: In order to guarantee the well-posedness of the problem, an initial history function as initial data has to be defined on $[-T, 0]$
- In the limit case of $T = 0$ the classical LWR model is recovered
- If the delay is not chosen appropriately, the maximum principle can be violated ($\rho$ can be greater than 1)

## Theoretical properties

*Lemma - Conservation of mass*

The delayed LWR model preserves the quantity $\rho(x, t)$.

[4] Keimer, Pflug, NoDEA, 2019

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*Lemma - Positivity*

Assume we have initial data with non-negative density $\rho(x, t) \geq 0$.
Then, for the delayed LWR model the density stays non-negative.

[4] Keimer, Pflug, NoDEA, 2019

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*Lemma - Unboundedness*

Assume $V$ is monotone decreasing and $V(\rho_{\max}) = 0$ for $\rho_{\max}$, the maximal density in the classical LWR model. The delayed first order model has no maximal density $\rho_{\max}$.

**Note**: In general $\rho > \rho_{\max}$ is allowed. Moreover, existence and uniqueness of the solution needs to be considered carefully[4]

[4] Keimer, Pflug, NoDEA, 2019

# Outline

## Numerical discretization

**Idea:** Investigate properties from a numerical viewpoint

- $\Delta x$, $\Delta t > 0$ define a grid in space $\{x_i = i\Delta x, i \in \mathbb{Z}\}$ and time $\{t^n = n\Delta t, n \in \mathbb{N}\}$.
- $\rho_i^n = \rho(x_i, t^n)$ is the discretized variable
- classical CFL condition: $\Delta t \leq \frac{\Delta x}{\max_k(\lambda_k)}$, where $\lambda_k$ are the eigenvalues of the Jacobian matrix of $f$

**Lax-Friedrichs Method**

$$\rho_i^{n+1} = \frac{1}{2}(\rho_{i+1}^n + \rho_{i-1}^n) - \frac{\Delta t}{2\Delta x}(f(\rho_{i+1}^n) - f(\rho_{i-1}^n))$$

**BUT** we are dealing with the delayed model...

## Numerical scheme

### Altered Lax-Friedrichs method

$$\rho_i^{n+1} = \frac{1}{2}(\rho_{i+1}^n + \rho_{i-1}^n) - \frac{\Delta t}{2\Delta x}(f(\rho_{i+1}^{n-T_\Delta}, \rho_{i+1}^n) - f(\rho_{i-1}^{n-T_\Delta}, \rho_{i-1}^n))$$

- $T \geq \Delta t$ to be able to treat with the delay
- $T_\Delta$ is the number of steps that make up the time delay $T$
- Identify flux as $f(\rho_{i+1}^{n-T_\Delta}, \rho_{i+1}^n) = \rho_{i+1}^n V(\rho_{i+1}^{n-T_\Delta})$
- For the well-posedness of the discrete problem, we have to provide an initial history function as initial data defined on $[-T, 0]$, i.e., $\rho^0(x, 0)$ is assumed to be constant in $t$ for $t \in [-T, 0]$

## Properties of the discretization

### Conservation

At the discrete level, we have:

$$\Delta x \sum_i \rho_i^{n+1} = \Delta x \sum_i \left( \frac{1}{2}(\rho_{i+1}^n + \rho_{i-1}^n) - \frac{\Delta t}{2\Delta x}(V(\rho_{i+1}^{n-T_\Delta})\rho_{i+1}^n - V(\rho_{i-1}^{n-T_\Delta})\rho_{i-1}^n) \right)$$

This gives us

$$\Delta x \sum_i \rho_i^{n+1} = \Delta x \sum_i \rho_i^n$$

and therefore conservation of mass.

## Properties of the discretization

### Positivity

To ensure positivity, we need to guarantee

$$\frac{\Delta t}{2\Delta x}\left(V(\rho_{i+1}^{n-T_\Delta})\rho_{i+1}^{n} - V(\rho_{i-1}^{n-T_\Delta})\rho_{i-1}^{n}\right) \leq \frac{1}{2}(\rho_{i+1}^{n} + \rho_{i-1}^{n}).$$

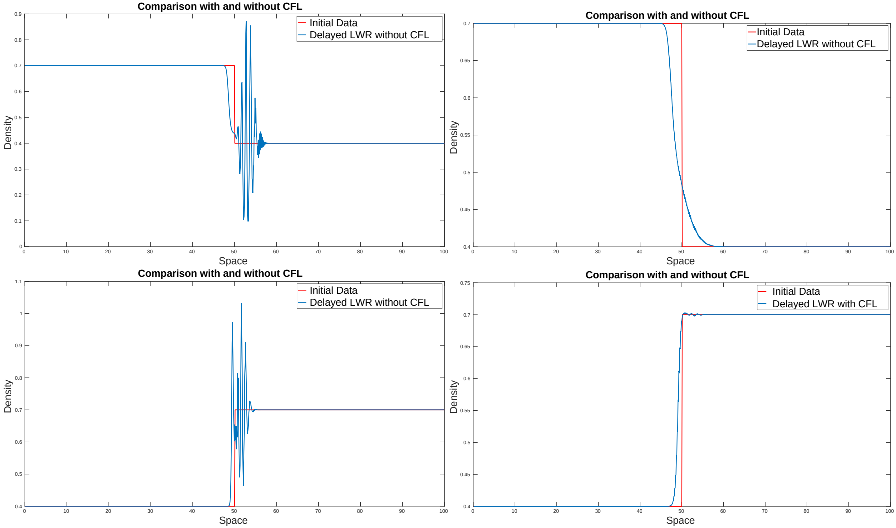
We achieve this result by

- introducing an altered **CFL** condition:

$$\Delta t \leq \frac{\Delta x}{\max\{|\rho^n|, |\rho^{n-T_\Delta}|\}}$$

- assuming $|V(\rho)| \leq \max\{|\rho^n|, |\rho^{n-T_\Delta}|\}$ with $V_{\max} = 1$

# Rarefaction (1st row) vs. shock wave (2nd row)



## Properties of the discretization

### $L^\infty$ bound

Under the altered CFL condition, it holds

$$||\rho^{n+1}||_{L^\infty} \leq 2\max\{|\rho^n|, |\rho^{n-T_\Delta}|\}$$

### TV bound

Under the altered CFL condition, it holds

- in space, assuming $TV(\rho^n_\Delta) = \sum_j |\rho^n_{j+1} - \rho^n_j|$:

$$\sum_j |\rho^{n+1}_{j+1} - \rho^{n+1}_j| \leq 2\left(5 + \frac{1}{\max\{|\rho_j^{n-T_\Delta}|, |\rho_j^n|\}}\right)\max\{TV(\rho^n_\Delta), TV(\rho_\Delta^{n-T_\Delta})\}$$

- in time:

$$\sum_j |\rho_j^{n+1} - \rho_j^n| \leq \sum_j 4\max\{|\rho_j^{n-T_\Delta}|, |\rho_j^n|\} + 2.$$

## Outline

## Numerical tests

Starting from empirical observations[5], we assume the velocity function as

$$V(\rho) = \begin{cases} V_{max} & \rho \leq \rho_f \\ \alpha\left(\frac{1}{\rho} - \frac{1}{\rho_c}\right) & \rho_f < \rho < \rho_c \\ 0 & \rho \geq \rho_c \end{cases}$$

- $\alpha > 0$ (parameter)
- $\rho_c \in (0, \rho_{max}]$ represents the so-called *safe distance* at the macroscopic level
- $\rho_f \in [0, \rho_{max})$ represents the distance until vehicles influence each other

**Note** that $V(\rho)$ **respects** the hypothesis $|V(\rho)| \leq |\rho_{max}|$ as $V_{max} = 1$

[5] Cristiani, Iacomini, DCDS-B, 2019

## Discretization parameters

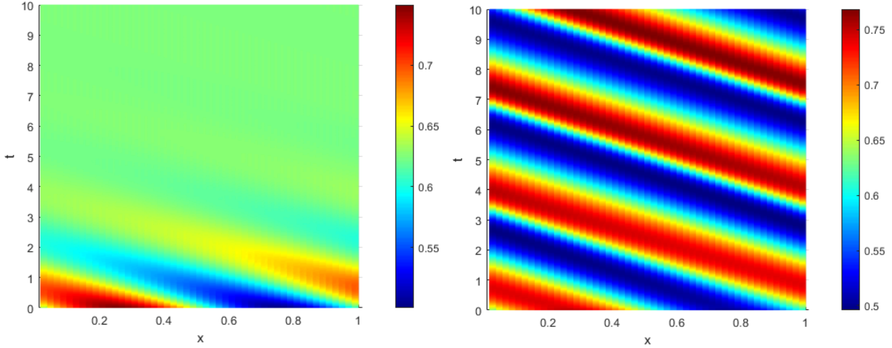
Let us fix the discretization parameters as follows:

- space interval $[a, b] = [0, 1]$, $\Delta x = 0.02$
- periodic boundary conditions
- $\Delta t$ is chosen such that the CFL condition is satisfied
- $\rho_c = 0.75$ and $\rho_f = 0.2$ as real data suggests[6]
- for simplicity the delay is a multiple of $\Delta t$: $T_\Delta \approx (10\Delta t, 20\Delta t)$.

[6] Balzotti, Iacomini, SEMA SIMAI Springer Series, 2020

# LWR vs. delayed LWR

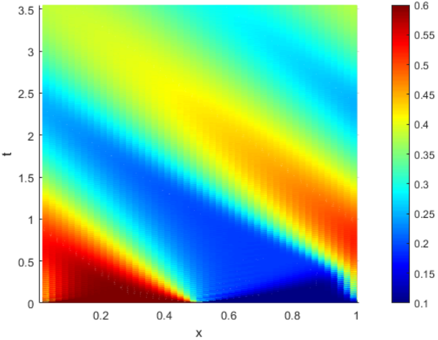
$$\rho^0(x) = \frac{5}{8} + \frac{1}{8}\sin(2\pi x) \qquad T_\Delta = 15\Delta t$$

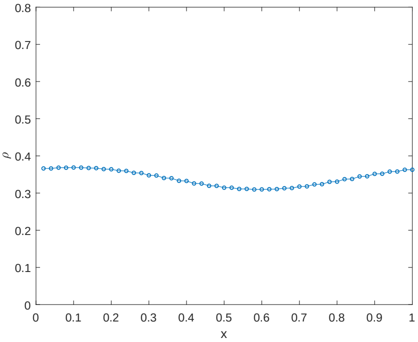



⇝ Density evolution shows that the LWR model smears out the perturbations while the delayed LWR model preserves them

## The crucial role of the delay: too small

$$\rho^0(x) = \frac{5}{8} + \frac{1}{8}\sin(2\pi x) \qquad T_\Delta = 4\Delta t$$

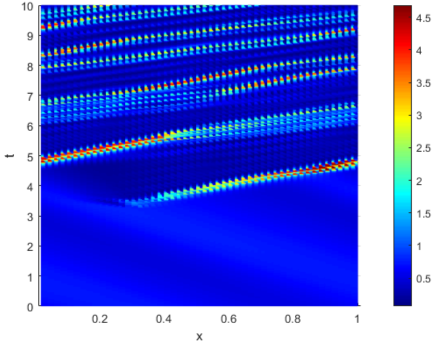


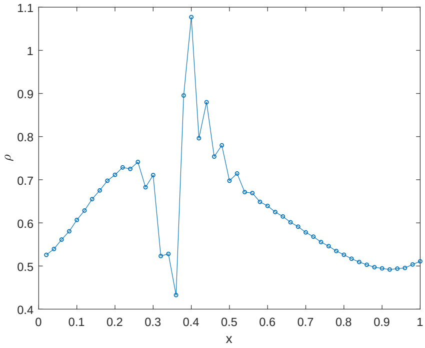




⇝ If the delay is too small, we recover a situation similar to the LWR model

## The crucial role of the delay: too large

$$\rho^0(x) = \frac{5}{8} + \frac{1}{8}\sin(2\pi x) \qquad T = 18\Delta t$$




⇝ The hypothesis are **no longer** satisfied and the density starts to oscillate

# Stop & Go waves

## Definition

Stop-and-Go waves are a typical feature of congested traffic. A S&G wave is detected when vehicles stop and restart without any apparent reason generating a wave that travels backward with respect to the cars' trajectories.

## Relevance

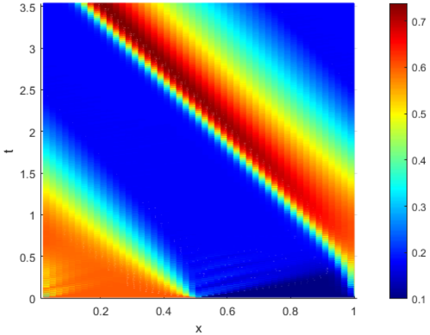
S&G waves are one of the main reasons for

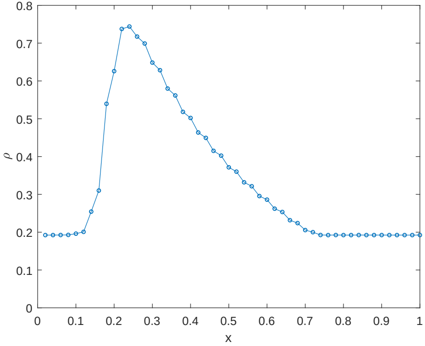
- accidents,
- longer travel times,
- high fuel consumption,
- pollution

$\rightsquigarrow$ VIDEO by Stern et al.: `https://youtu.be/2mBjYZTeaTc`

## Backward propagation of Stop & Go waves

$$\rho^0(x) = \begin{cases} 0.6 & x < 0.5 \\ 0.1 & x \geq 0.5. \end{cases} \qquad T_\Delta = 10\Delta t$$



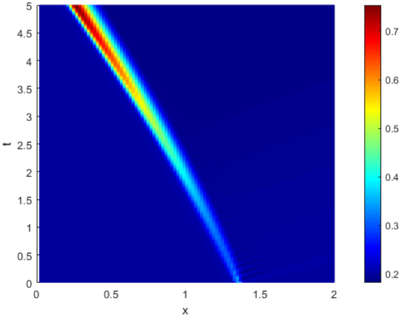




⇝ Density profile at $T_{\text{final}} = 3.5$ yields a well-defined S&G wave[7]

[7] Flynn et al., Phys. Rev. E, 2009

# Triggering of Stop & Go waves

$$\rho^0(x) = \begin{cases} 0.35 & 1.34 \leq x \leq 1.342 \\ 0.2 & \textit{elsewhere}, \end{cases} \qquad T = 21\Delta t$$




⇝ Initial perturbation increases and moves backward as time increases

# Outline

## A note on delayed second order models

**Follow-up question**

What do we observe in second order delayed models?

Starting from the model by Gazis, Herman, Rothery (1961)

$$\dot{x}_i(t) = v_i(t)$$
$$\dot{v}_i(t) = C\frac{(v_{i+1}(t-T) - v_i(t-T))}{(x_{i+1}(t-T) - x_i(t-T))^{\gamma+1}}, \qquad i = 1, \dots, N,$$

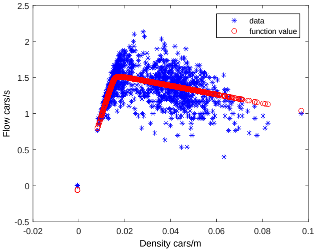
with delay $T > 0$ and model constants $C = v_{\text{ref}}\Delta X^{\gamma} > 0$ and $\gamma \geq 0$, we derive the delayed Aw-Rascle-Zhang (ARZ) model

$$\partial_t \rho(x,t) + \partial_x(\rho(x,t)v(x,t)) = 0 \qquad \text{(RSD)}$$
$$\partial_t(\rho(x,t)w(x,t)) + \partial_x(\rho(x,t)v(x,t)w(x,t)) = Q(\partial_x v, \rho)$$
$$\rightsquigarrow \; Q(\partial_x v, \rho) = v_{\text{ref}}(\partial_x v(x,t-T)\rho(x,t-T)^{\gamma} - \partial_x v(x,t)\rho(x,t)^{\gamma}),$$

where $w = v + P(\rho)$ and $P(\rho)$ is a known pressure function

## Data-fitting

- Take real data from the Minnesota Department of Transportation[8]



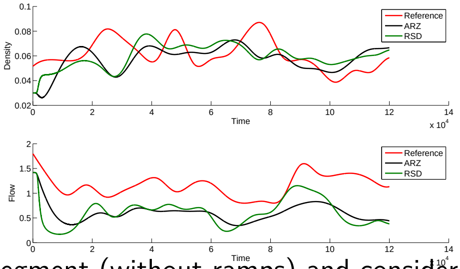

- $Q(\rho)$ fits the fundamental diagram data in a least squares sense[9]:

$$Q(\rho) = \alpha\Big[\sqrt{1+(\lambda p)^2} + \big(\sqrt{1+(\lambda(1-p))^2} - \sqrt{1+(\lambda p)^2}\big)\frac{\rho}{\rho_{max}} - \sqrt{1+\lambda^2(\frac{\rho}{\rho_{max}} - p)^2}\Big]$$

[8] http://data.dot.state.mn.us/datatools/

[9] Fan, Herty, Seibold, NHM, 2014

## Data-fitting: Comparison of the ARZ to the delayed ARZ model




- Take a road segment (without ramps) and consider the time interval 16:00-16:10 (workday)
- The classical ARZ model tends to avoid deviations
- Error between real data and simulated models:

$$E(x,t) = \frac{|\rho_{\text{model}}(x,t) - \rho_{\text{data}}(x,t)|}{\Delta\rho} + \frac{|v_{\text{model}}(x,t) - v_{\text{data}}(x,t)|}{\Delta v},$$

where $E_{\text{ARZ}} = 0.6404$ and $E_{\text{RSD}} = 0.6034$

# Conclusion and future perspective

## Conclusions

- introduced new delayed macroscopic traffic models, pointing out similarities and differences to the undelayed versions
- proposed an altered Lax-Friedrichs method to compute the numerical solution
- showed that the LWR delayed model is able to reproduce Stop & Go waves while the delayed ARZ model performs well regarding real data

## Future Perspective

- advanced parameter estimation techniques
- inclusion of stochastic events for route planning

## This talk is based on the following references:

- M. Burger, S. Göttlich and T. Jung, *Derivation of a first order traffic flow model of Lighthill-Whitham-Richards type*, IFAC PapersOnLine, 51, pp. 49–54, 2018.
- M. Burger, S. Göttlich and T. Jung, *Derivation of second order traffic flow models with time delays*, Netw. Heterog. Media, Vol. 14(2), pp. 265-288, 2019.
- S. Göttlich, E. Iacomini and T. Jung, *Properties of the LWR model with time delay*, to appear in Netw. Heterog. Media, 2020.
- T. Jung, *Delayed Traffic Models in Multiple Scales: New Macroscopic Models and Their Numerics*, PhD Thesis, University of Mannheim, to appear 2020.
- ... and was supported by the German Research Foundation (DFG).

## Thank you for your attention!