Set-Based Hybrid Predictive Control for Autonomous Vehicles

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IPAM Workshop II:
Safe Operation of Connected and Autonomous Vehicle Fleets

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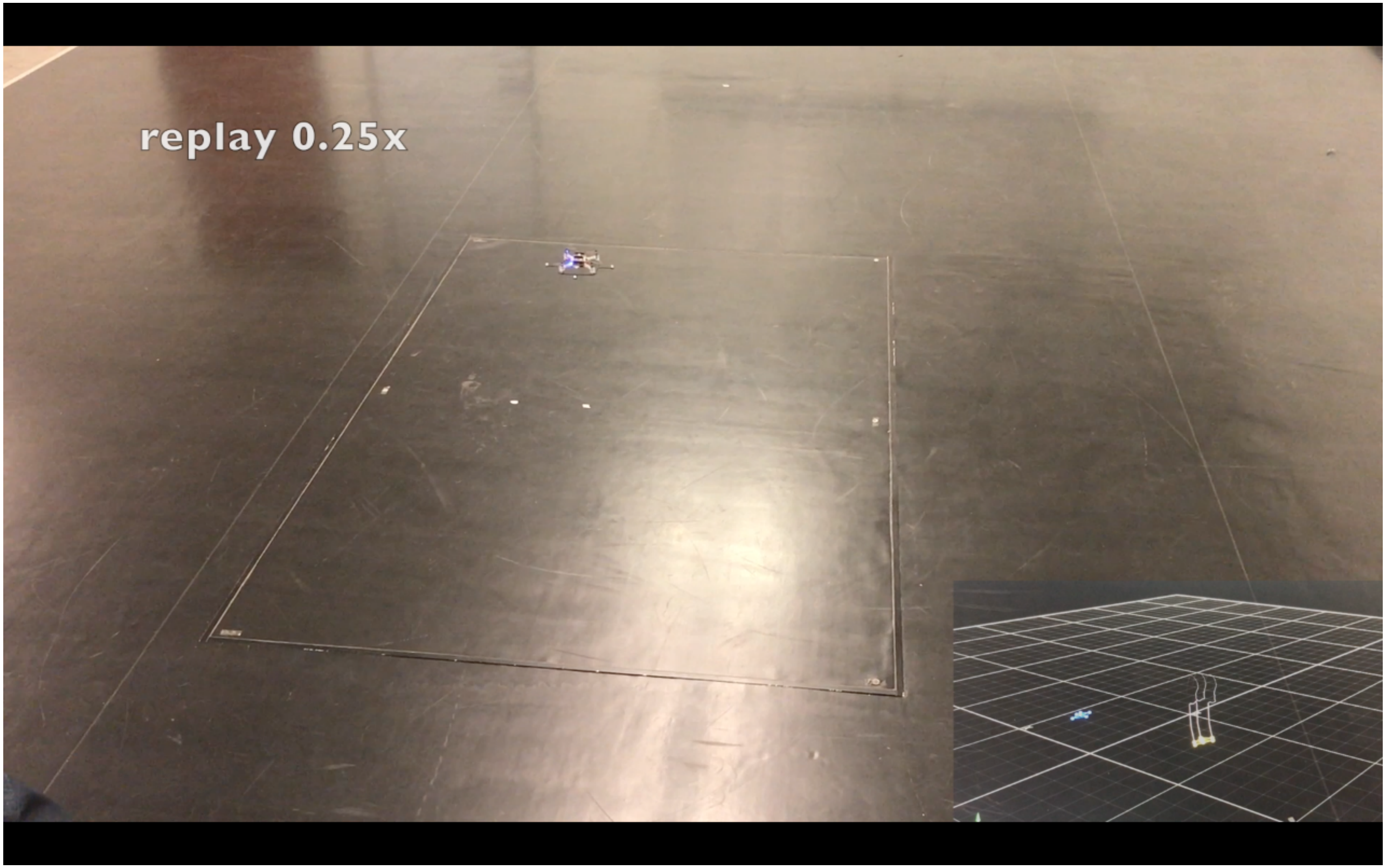
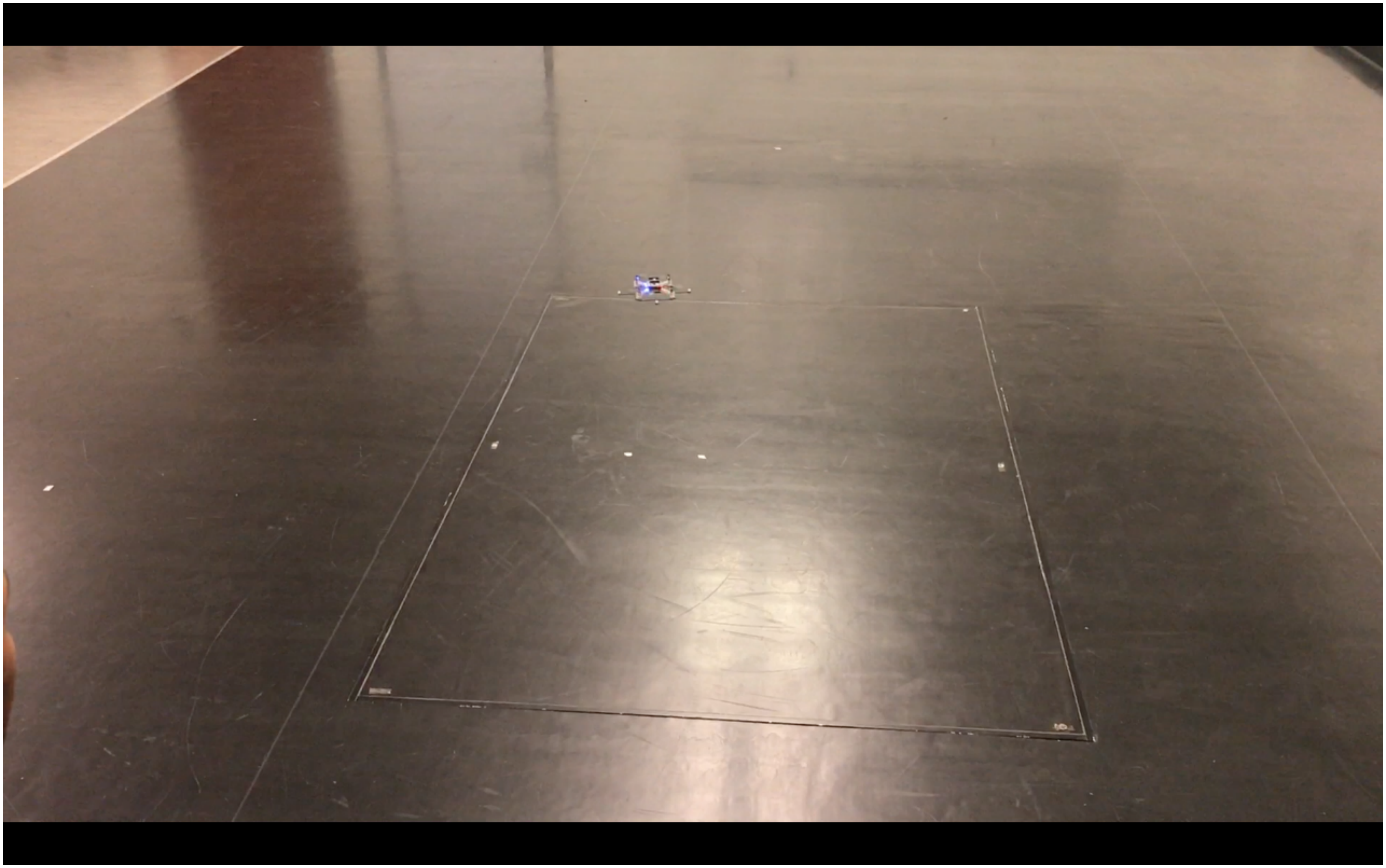
HYBRID SYSTEMS LABORATORY

Predictive Control of Evasion



esented at CASE 2019 – Best Paper Award Finalist

Predictive Control of Evasion



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Predictive Control of Evasion

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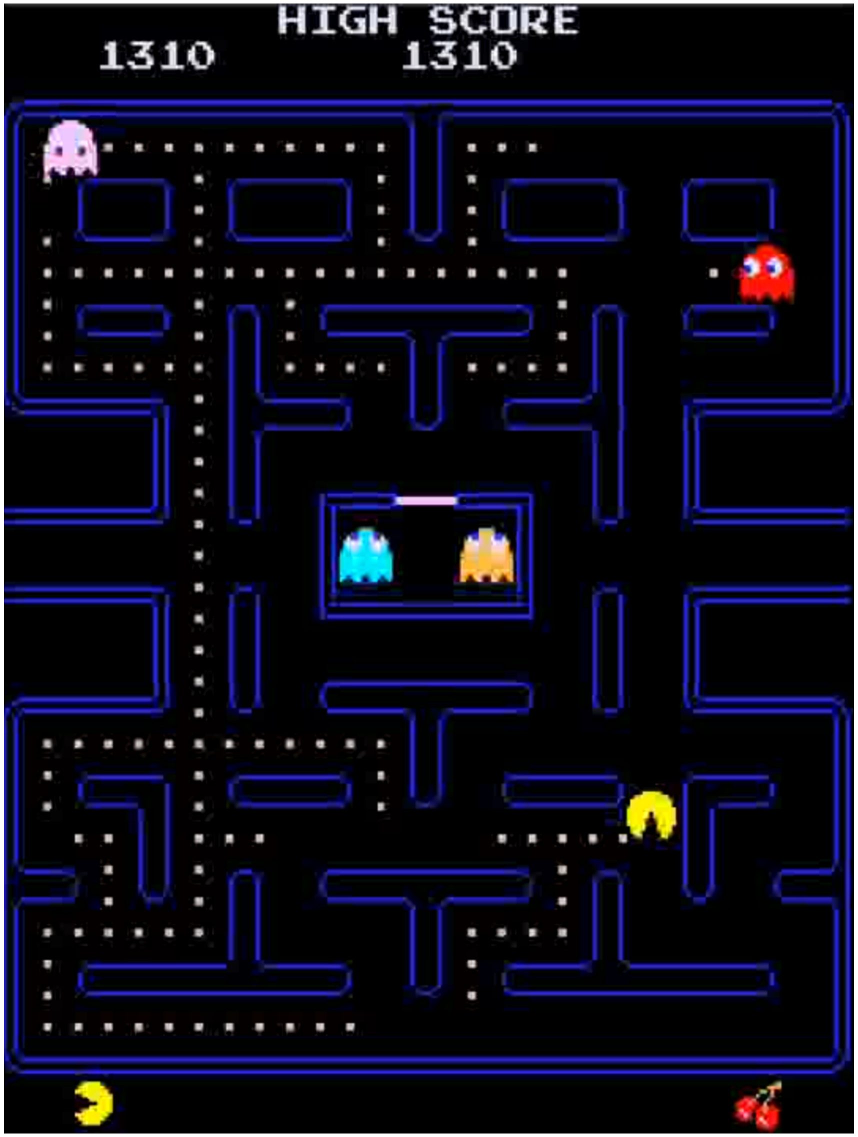
It is All About **Robustness**



Predict with Disturbances, and then Control

Model Predictive Control (MPC)

Predict system behavior, *select* best decision:

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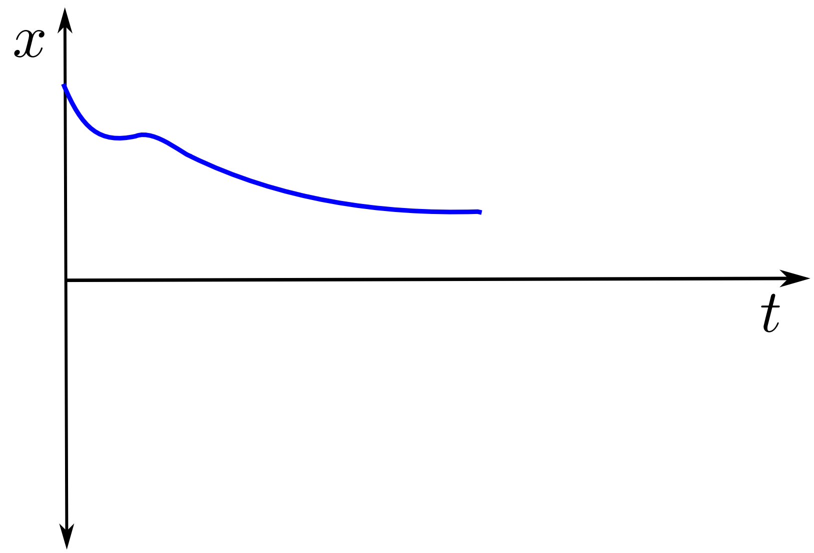

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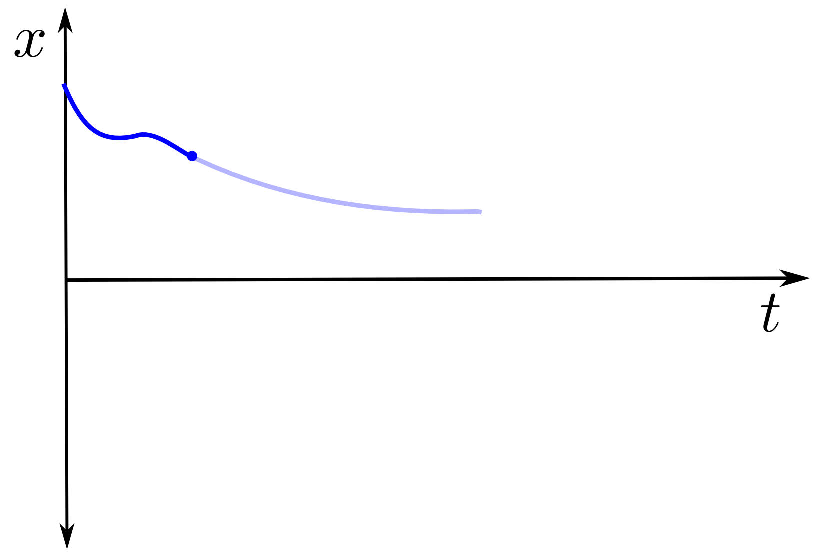

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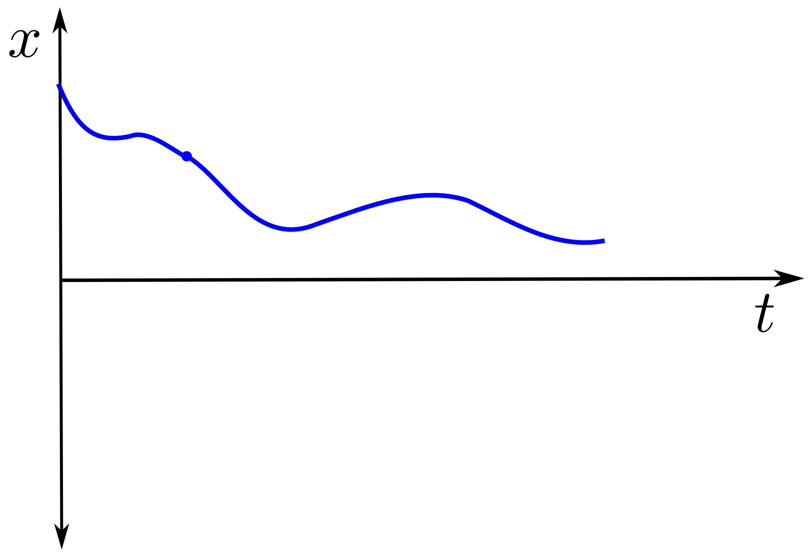

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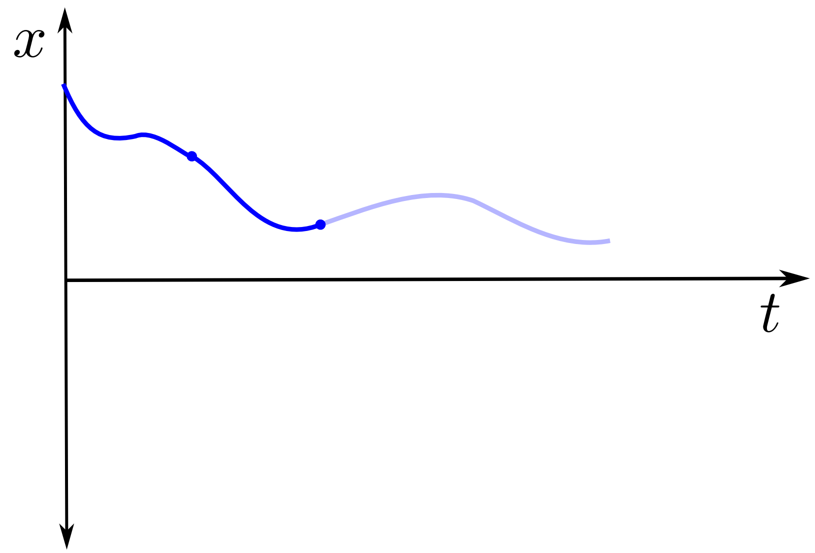

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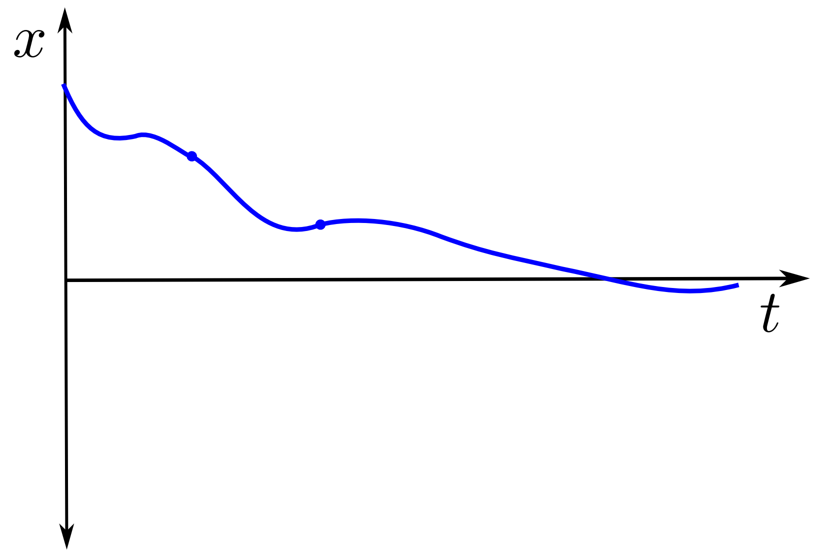

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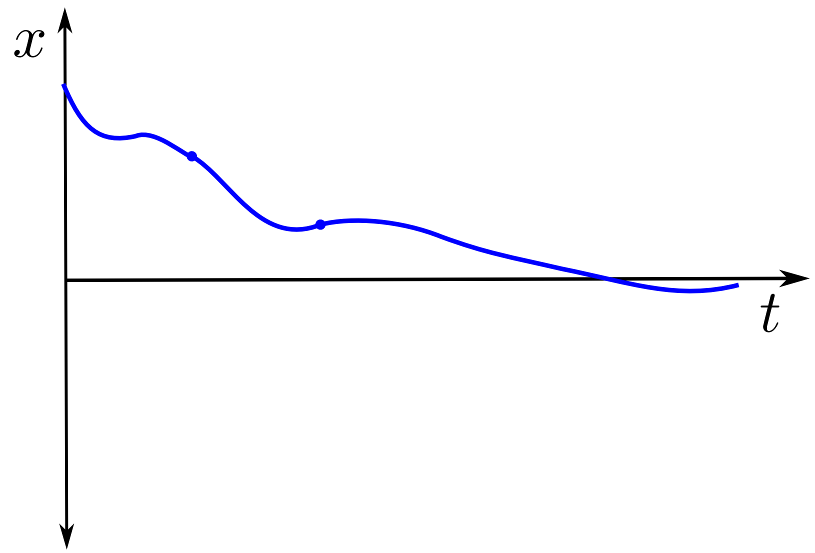

- Constraint satisfaction
- Optimal performance
- Closed-form controller not required

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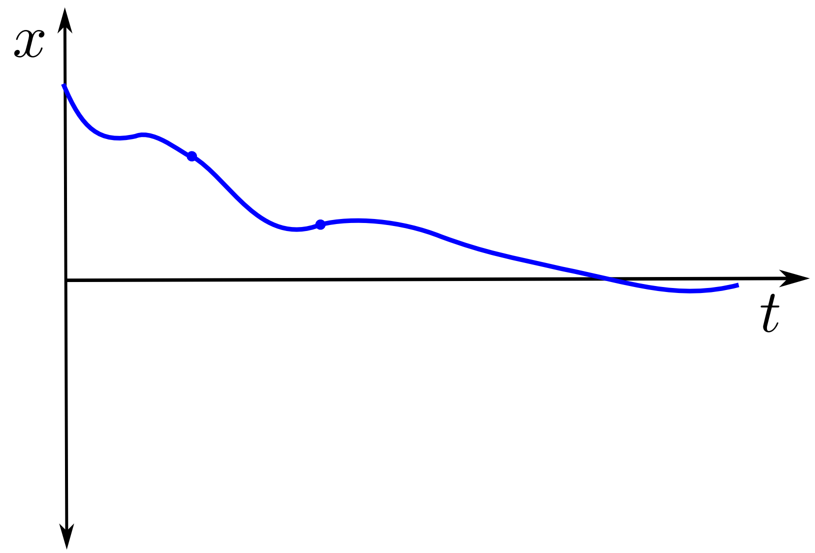

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Standard MPC exhibits a possible lack of robustness to perturbations and uncertainties. [Grimm et al. 2004] [Tuna et al. 2007]

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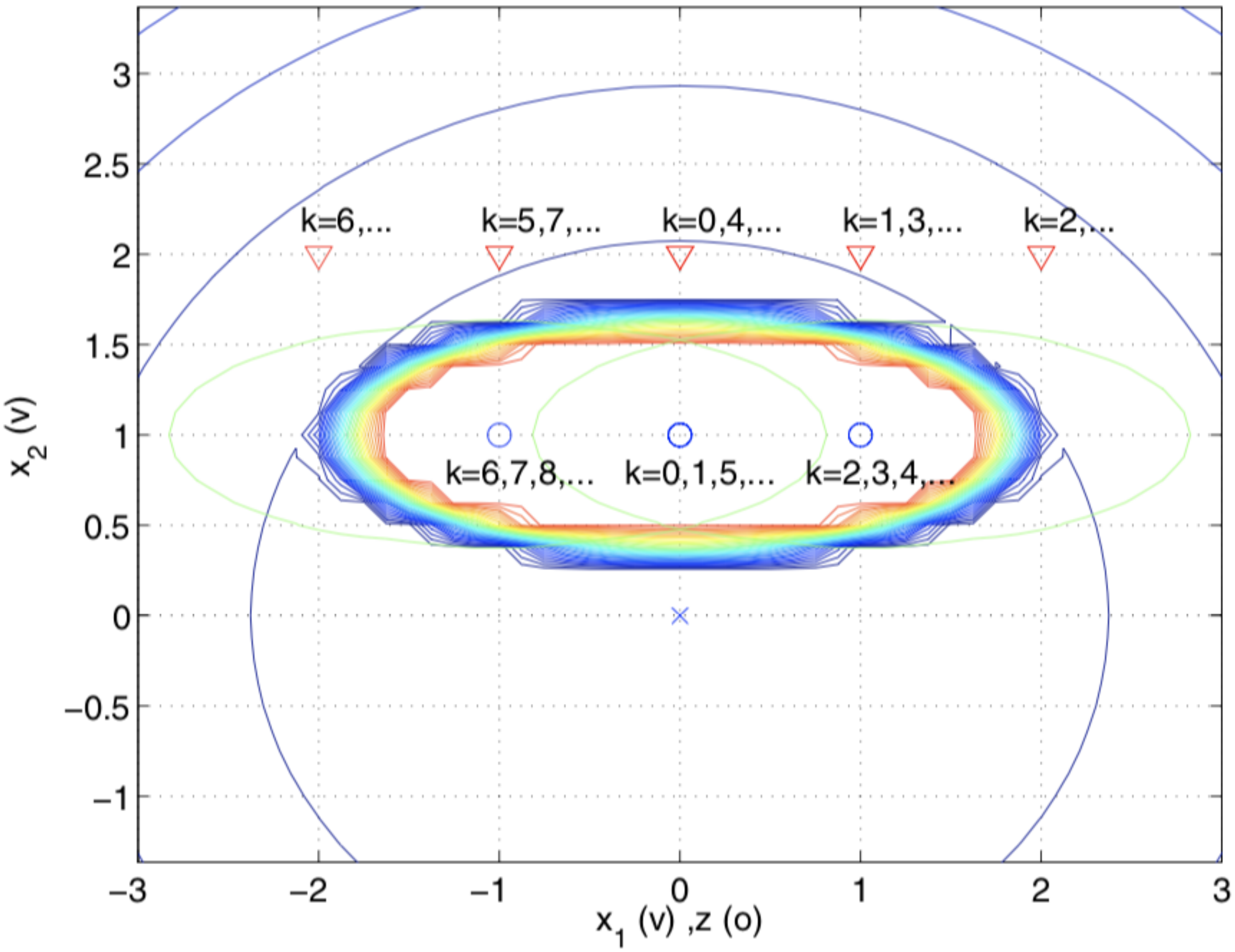
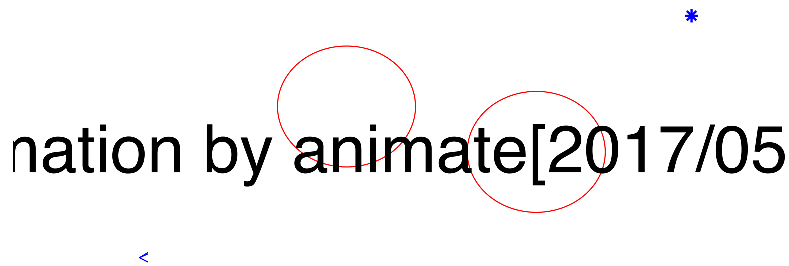


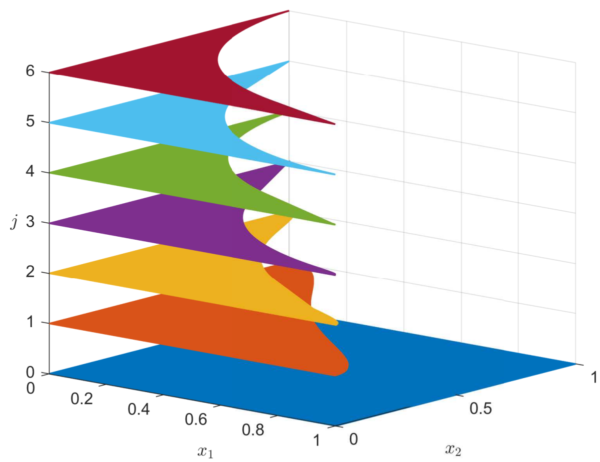
Background

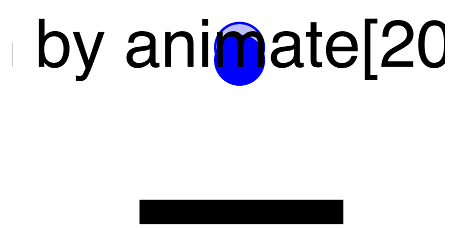
Related Fields:

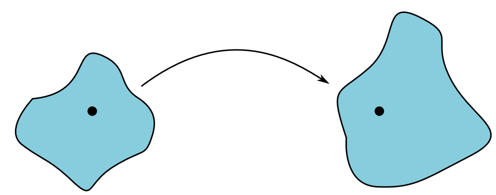
- Collision avoidance
- Set dynamical systems
- Uncertainty in MPC



[Chao et al. 2011], [Risso et al., 2016], [Raković et al., 2007.]







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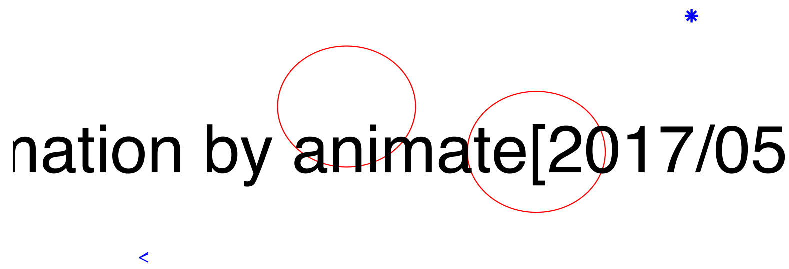

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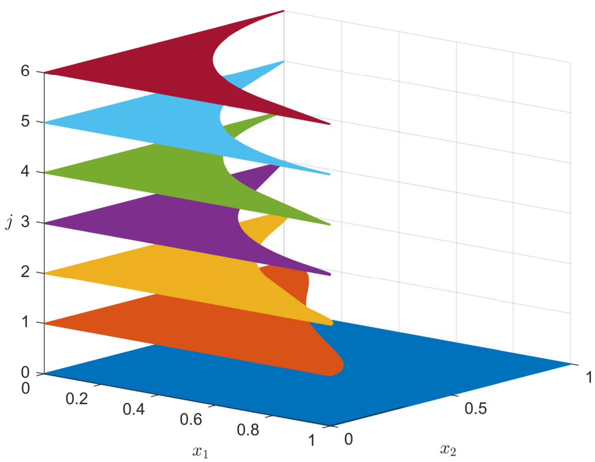

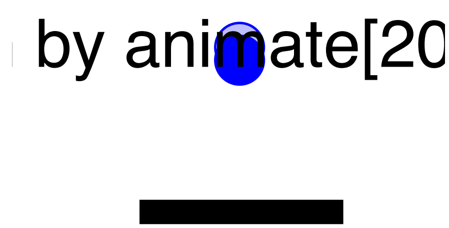

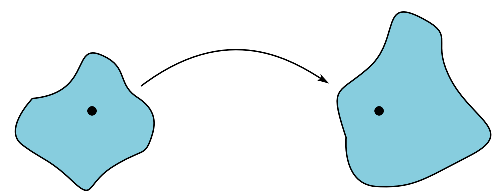

- Set-based consensus: [Gautam et al. 2011]
- Set-based adaptive control [Gonçalves et al. 2016]
- Set-based learning [Nguyen et al. 2018]

Background

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nation by animate[2017/05

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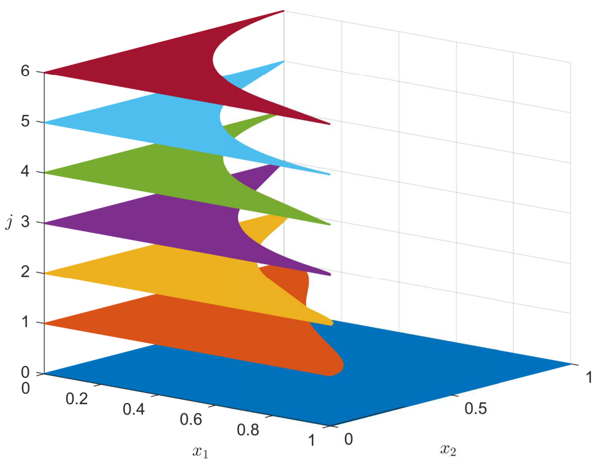

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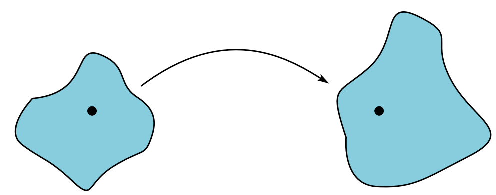

No set-based dynamics or optimization.

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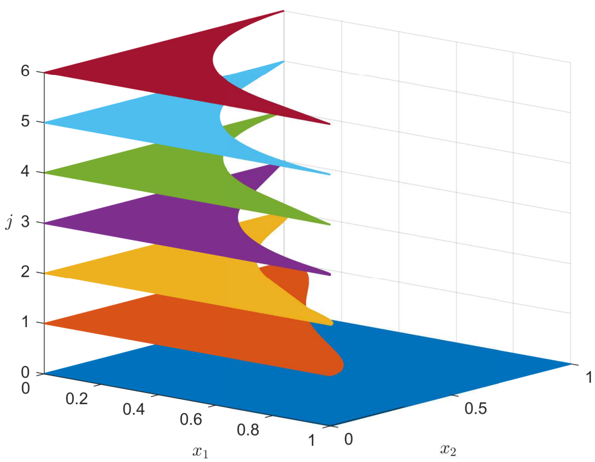

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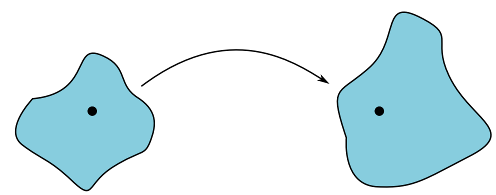

Tube MPC formulates constraints with isotropic sets around states and uses points to propagate dynamics and optimize trajectories

Our Contribution, with Multiple Applications

Robust optimization framework for dynamical systems that performs optimization over a sequence of sets and can be used for

- Uncertainty propagation

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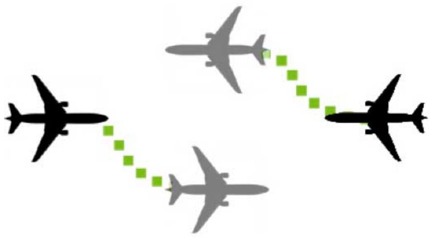
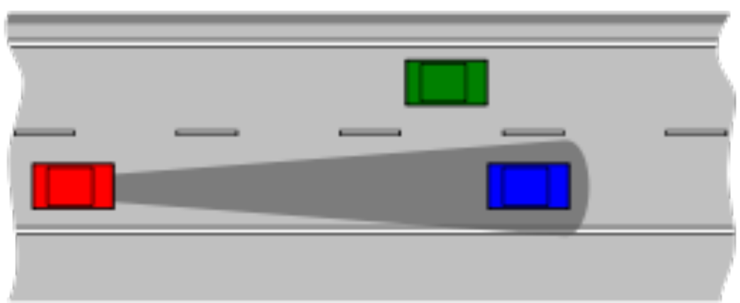


A collision detection and evasion application of set-based predictive control that is robust to uncertainties.



Towards Set Dynamical Systems

Discrete-time system with state x, disturbance w on the dynamics, and noise v on the state:

$$x_{j+1} = g(x_j, w_j)$$

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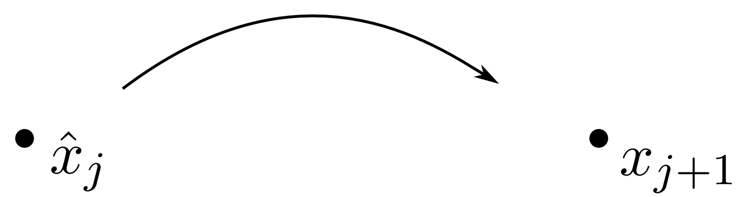
Discrete-time system with state x, disturbance w on the dynamics, and noise v on the state:

$$x_{j+1} = g(\hat{x}_j - v_j, w_j)$$

The measured state is $\hat{x}_j = x_j + v_j$.



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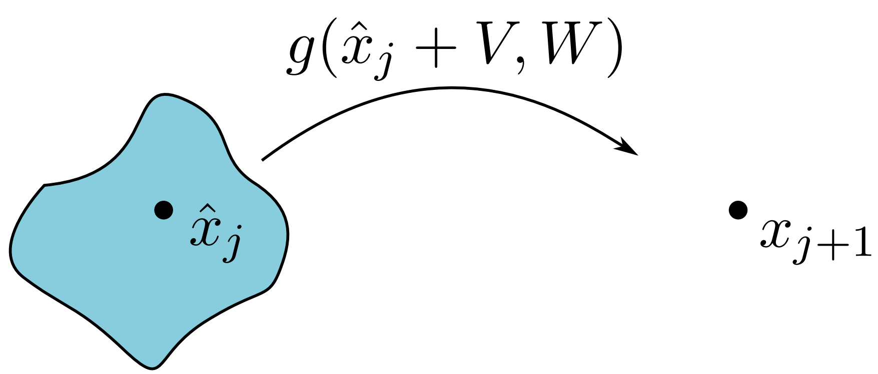

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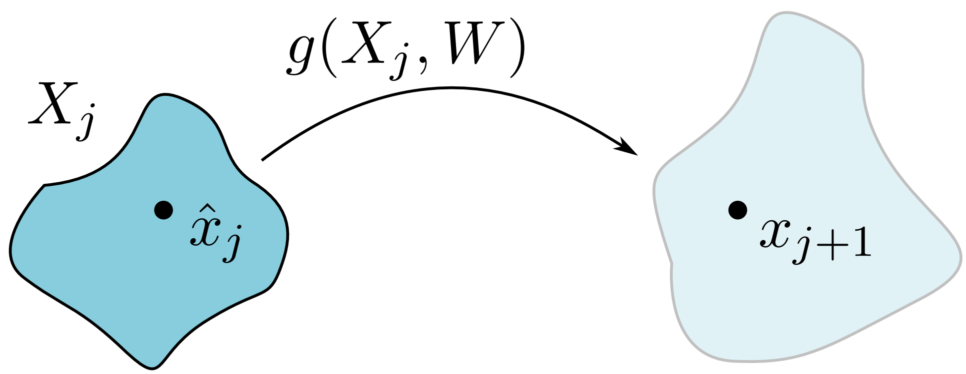

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Discrete-time system with state x, disturbance w on the dynamics, and noise v on the state:

$$x_{j+1} \in g(X_j, W) =: G(X_j)$$

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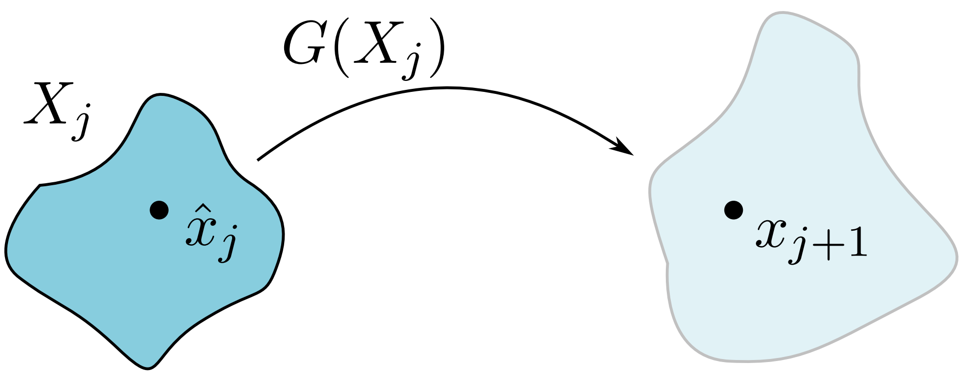

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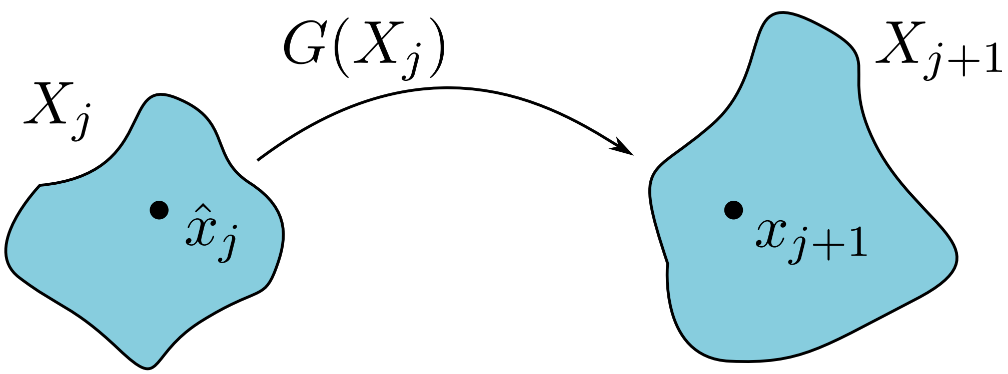
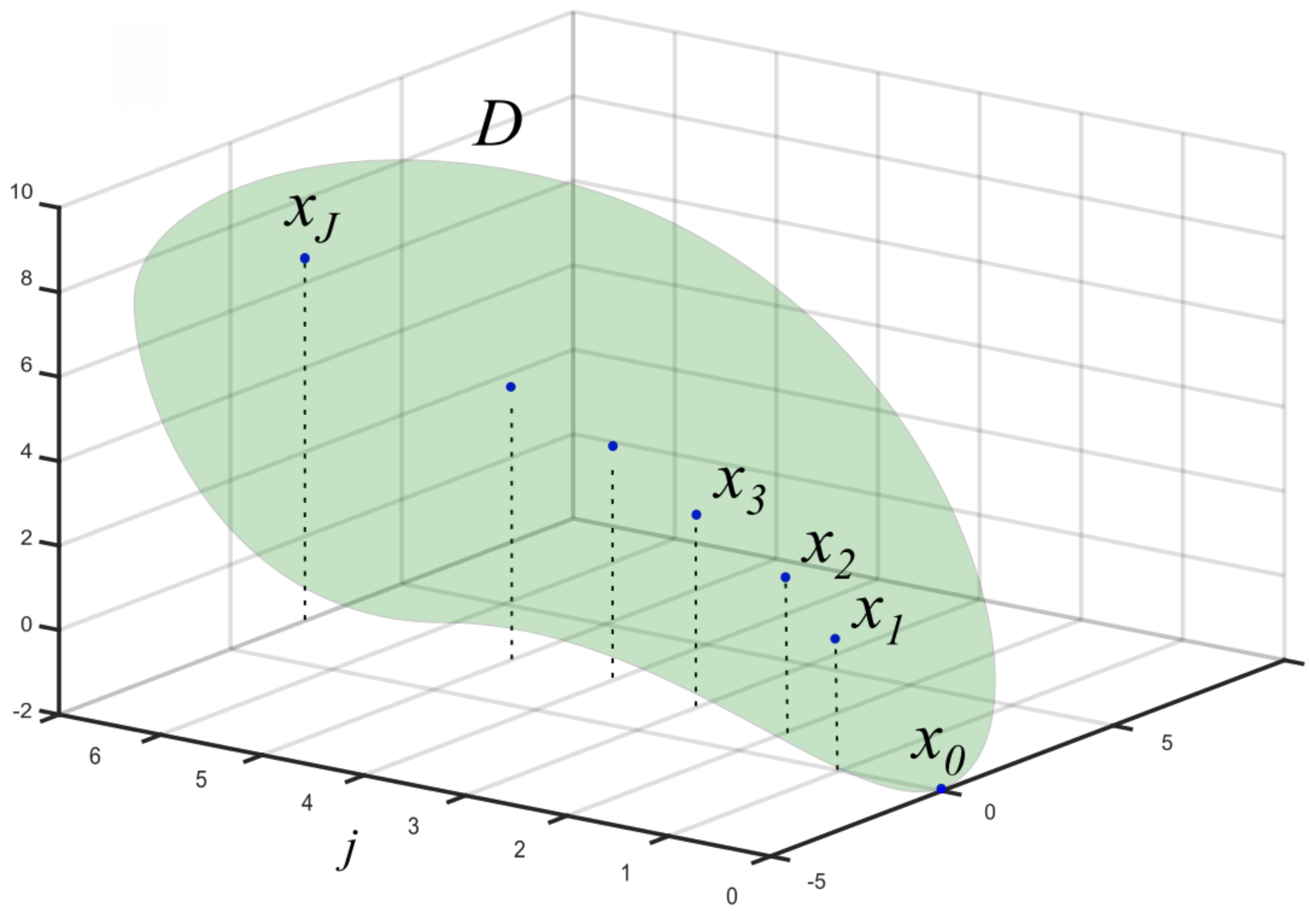


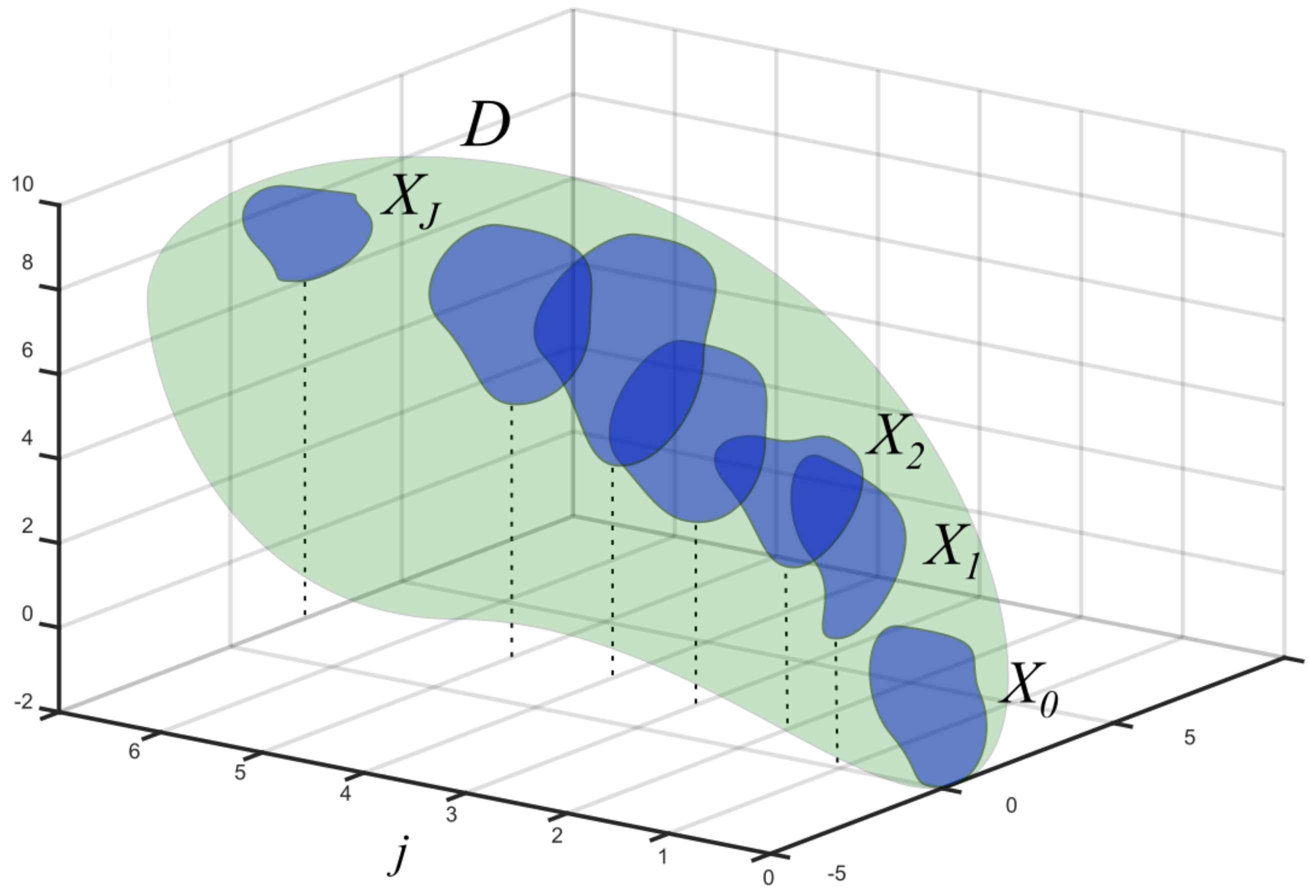
Set Dynamical Systems

Discrete-time system state evolution



Set Dynamical Systems

Discrete-time set-valued state evolution



Set Dynamical Systems

We consider dynamical systems of the form

$$
\begin{aligned}
X^+ &= G(X) \\
Y &= H(X) \\
X &\subset D
\end{aligned}
$$

where

- $X \subset \mathbb{R}^n$ is the set-valued state
- $Y \subset \mathbb{R}^m$ is the system's output
- $G\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ and $H\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ are set-valued maps
- $D \subset \mathbb{R}^n$ defines a constraint that solutions to the system must satisfy

Due to the set-valuedness of its solutions, we refer to this class of systems as **set dynamical systems**.

Set Dynamical Systems

Solution to a set dynamical system

A sequence of nonempty sets $\{X_j\}_{j=0}^{J}$, $J \in \operatorname{dom} X$, is a *solution* to $X^+ = G(X)$ if

$$\begin{array}{rclr} X_{j+1} & = & G(X_j) & \forall j \in \{0, 1, \ldots, J-1\} \cap \mathbb{N} \\ X_j & \subset & D & \forall j \in \operatorname{dom} X \end{array}$$

where X_0, is the initial set and $\operatorname{dom} X = \{0, 1, 2, \ldots, J\} \cap \mathbb{N}$ is the domain of definition of the solution. We say the solution is:

- trivial, if it has $J = 0$
- nontrivial, if it has $J > 0$
- maximal if it cannot be further extended.

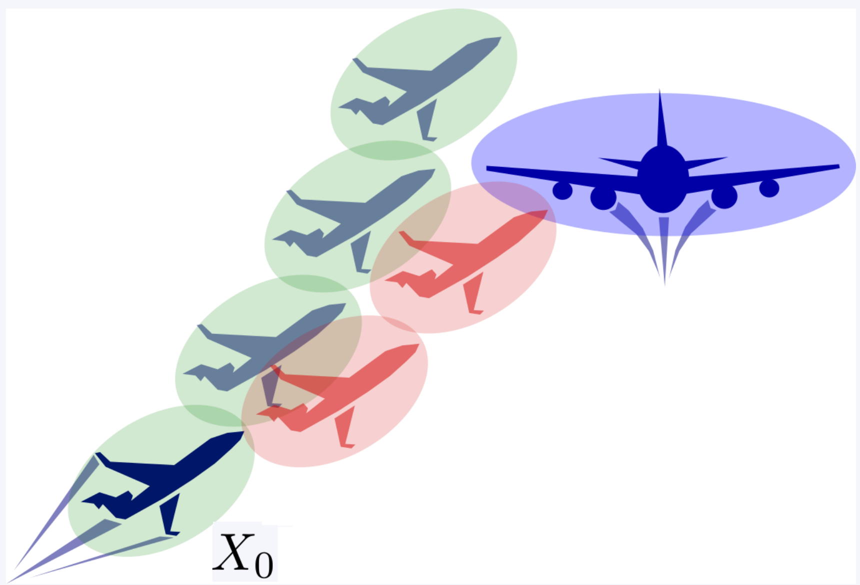
$X = \mathcal{S}(X_0)$: A solution that starts from the initial set $X_0 \subset \mathbb{R}^n$
We refer to the solution $\{X_j\}_{j=0}^{J}$ as X along this presentation.

Set Dynamical Systems

Reachability and Safety

Given the discrete time system $x^+ = g(x)$, for a set of initial conditions, determine where can the system's states evolve. Are there any potentially reachable unsafe configurations, given an initial set?

$$\text{reach}_{j \leq J}(X_0) := \bigcup_{j \in \operatorname{dom} X} g(X_j)$$

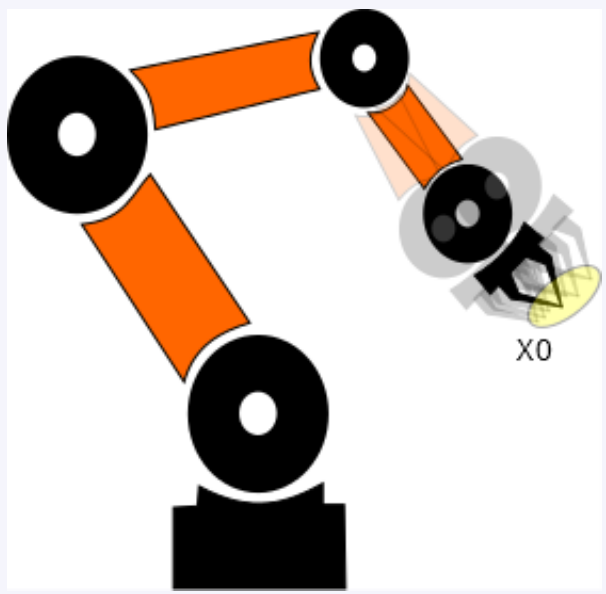


Set Dynamical Systems

Uncertainty Propagation

Given the discrete time system $x^+ = g(x)$, initial conditions $X_0 \subset \mathbb{R}^n$ and the state-dependent bounded disturbances d_1 and d_2, determine the worst case effect of the uncertainty over the system's behavior.

$$G(X) = \bigcup_{x \in X} \{g(x, d_1(x)\mathbb{B}) + d_2(x)\mathbb{B}\}$$



Related Contributions

- Generalized pseudo-dynamical systems and stability results

  Pelczar, Prace Matematyczne, 77

  Sobanski, Zeszyty Naukowe III, Prace Mat, 78
- Set-dynamics framework for the invariance of sets under output feedback

  Artstein and Rakovic, International Journal of Systems Science, 11
- Dynamical properties of continuous-time systems with set-valued solutions
  - Calculus for set-valued maps and set-valued evolution equations

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  - Dynamics of sets defined by differential inclusions

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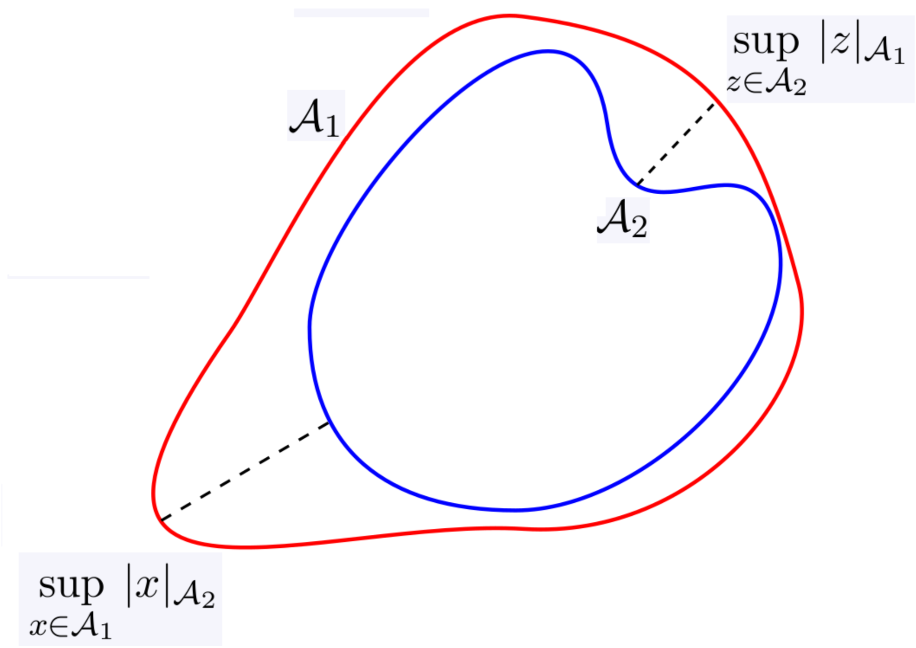
We characterize the limiting properties of solutions to these systems via tools from variational analysis, using
Rockafellar and Wets, **Variational Analysis**, Springer, 1998

Distance Between Sets

Definition: Distance between Sets

The Hausdorff distance between two closed sets $\mathcal{A}_1$, $\mathcal{A}_2 \subset \mathbb{R}^n$ is given by

$$d(\mathcal{A}_1, \mathcal{A}_2) = \max\left\{ \sup_{x \in \mathcal{A}_1} |x|_{\mathcal{A}_2}, \sup_{z \in \mathcal{A}_2} |z|_{\mathcal{A}_1} \right\}$$

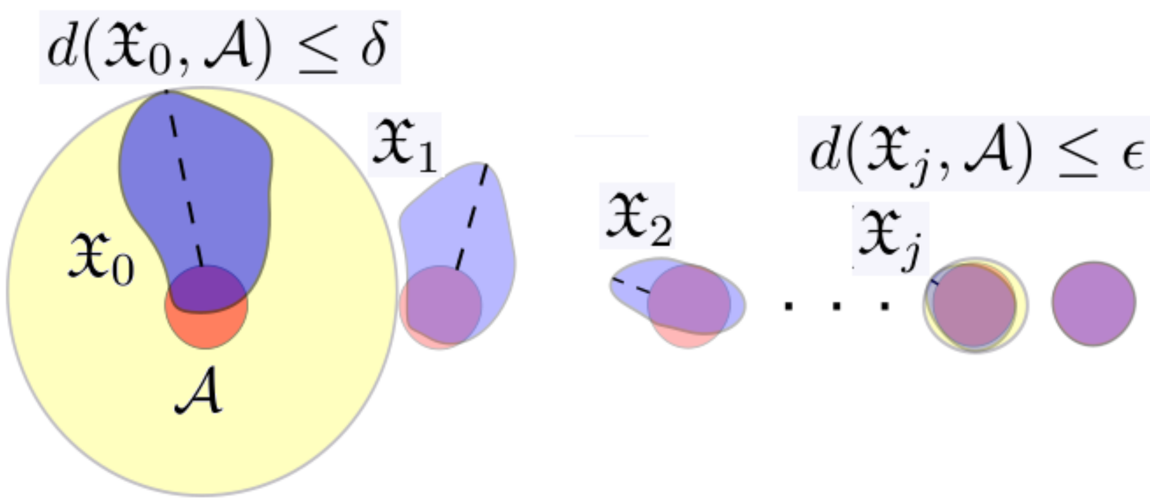


Asymptotic Stability

Definition: Asymptotic Stability of a Set

The closed set $\mathcal{A} \subset \mathbb{R}^n$ is

- stable if for each $\epsilon > 0$ there exists $\delta > 0$ such that each solution X with $d(X_0, \mathcal{A}) \leq \delta$ satisfies $d(X_j, \mathcal{A}) \leq \epsilon$ for all $j \in \text{dom } X$
- asymptotically stable if is stable and if there is $\rho > 0$ such that for any compact set $X_0 \subset \mathcal{A} + \rho\mathbb{B}$, $X \in S(X_0)$ is complete and satisfies $\lim_{j\to\infty} d(X_j, \mathcal{A}) = 0$



Regularity of the Data

Standing Assumption

The data (G, D) of the set dynamical system

$$X^+ = G(X) \qquad X \subset D$$

satisfies the following properties:

(A0) The set-valued map $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is outer semicontinuous, locally bounded, and, for each $x \in D$, $G(x)$ is a nonempty subset of $\mathbb{R}^n$.

(A1) The set $D \subset \mathbb{R}^n$ is closed.

Semi-group and Sequential Compactness

Theorem

The following properties hold for

$$X^+ = G(X) \qquad X \subset D$$

(B1) For any solution X and any $\bar{j} \in \operatorname{dom} X$, we have that $\overline{X}$ given by $\overline{X}_j = X_{j+\bar{j}}$ for each $j \in \operatorname{dom} \overline{X} = \{j \ : \ j + \bar{j} \in \operatorname{dom} X \ \}$ is a solution to the set dynamical system.

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(B2) Let $\{X_0^i\}_{i=0}^\infty$ be an eventually bounded sequence of sets converging to a bounded set X_0 and suppose $\{X^i\}_{i=0}^\infty$ is such that $X^i \in \mathcal{S}(X_0^i)$.

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(B2) Let $\{X_0^i\}_{i=0}^{\infty}$ be an eventually bounded sequence of sets converging to a bounded set X_0 and suppose $\{X^i\}_{i=0}^{\infty}$ is such that $X^i \in \mathcal{S}(X_0^i)$. Then, there exists a subsequence of

$$\{X^i\}_{i=0}^{\infty}$$

converging to

$$X \in \mathcal{S}(X_0)$$

Semi-group and Sequential Compactness

Proof sketch:

Semi-group and Sequential Compactness

Proof sketch: Since $\{X_0^i\}_{i=0}^{\infty}$ is eventually bounded, then

$$\bigcup_{i \geq i^*} X_0^i$$

for some $i^* \in \mathbb{N}$, is bounded.

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$X_0^i \subset X_0 + \varepsilon\mathbb{B} \qquad \forall i \geq i^{**}$

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$$X_0^i \subset X_0 + \varepsilon\mathbb{B} \qquad \forall i \geq i^{**} \;\Rightarrow\; G(X_0^i) \subset G(X_0 + \varepsilon\mathbb{B}) \qquad \forall i \geq i^{**}$$

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Note that $X_1^i = G(X_0^i) \subset D$.

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$$X_0^i \subset X_0 + \varepsilon\mathbb{B} \qquad \forall i \geq i^{**} \;\Rightarrow\; G(X_0^i) \subset G(X_0 + \varepsilon\mathbb{B}) \qquad \forall i \geq i^{**}$$

Note that $X_1^i = G(X_0^i) \subset D$. Since G is locally bounded and X_0 is bounded, $G(\overline{X_0 + \varepsilon\mathbb{B}})$ is bounded and

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Semi-group and Sequential Compactness

Proof sketch: Since $\{X_0^i\}_{i=0}^\infty$ is eventually bounded, then

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Then, $\{X_1^i\}_{i=0}^\infty$ has a subsequence $\{X_1^{i_k}\}_{k=0}^\infty$ that converges to a closed set satisfying

$$X_1 := \lim_{k\to\infty} X_1^{i_k} = G(X_0)$$

Semi-group and Sequential Compactness

Proof sketch: Since $\{X_0^i\}_{i=0}^{\infty}$ is eventually bounded, then

Definition: Inner and Outer Limit

Let $\{T_i\}_{i=0}^{\infty}$ be a sequence of sets in $\mathbb{R}^n$

- Inner limit of the sequence $\{T_i\}_{i=0}^{\infty}$: $\liminf_{i\to\infty} T_i$, is the set of all $x \in \mathbb{R}^n$ for which there exist points $x_i \in T_i$, $i \in \mathbb{N}$, such that $\lim_{i\to\infty} x_i = x$
- Outer limit of the sequence $\{T_i\}_{i=0}^{\infty}$, denoted $\limsup_{i\to\infty} T_i$, is the set of all $x \in \mathbb{R}^n$ for which there exist a subsequence $\{T_{i_k}\}_{k=0}^{\infty}$ of $\{T_i\}_{i=0}^{\infty}$ and points $x_k \in T_{i_k}$, $k \in \mathbb{N}$, such that $\lim_{k\to\infty} x_k = x$

If the inner and the outer limit coincide, the sequence is said to be convergent, and its limit is given by

$$\lim_{i\to\infty} T_i = \liminf_{i\to\infty} T_i = \limsup_{i\to\infty} T_i$$

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Theorem (Rockafellar and Wets, 98)

Every sequence of nonempty sets $\{T_i\}_{i=0}^\infty$ in $\mathbb{R}^n$ either escapes to the horizon or has a subsequence converging to a nonempty set $T \subset \mathbb{R}^n$, i.e., there exists a subsequence $\{T_{i_k}\}_{k=0}^\infty$ of $\{T_i\}_{i=0}^\infty$ such that $\lim_{k\to\infty} T_{i_k} = T$.

Limit for Decreasing V

Theorem (Convergence)

Let $\mathcal{M} \subset D$ be closed. Suppose there exist a continuous function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ and functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that

$$\begin{array}{rcll} \alpha_1(|x|_{\mathcal{M}}) \leq V(x) \leq \alpha_2(|x|_{\mathcal{M}}) & & & \forall x \in D \cup G(D) \\ V(\eta) - V(x) & \leq & 0 & \forall x \in \mathcal{M}, \eta \in G(x) \\ V(\eta) & \leq & \gamma V(x) & \forall x \in D \setminus \mathcal{M}, \eta \in G(x) \end{array}$$

for some $\gamma \in [0, 1)$.

Limit for Decreasing V

Theorem (Convergence) (cont'd)

Let $X_0 \subset D$ be compact. Then, for each solution $\{X_j\}_{j=0}^{J}$, $J \in \mathbb{N} \cup \{\infty\}$ from $X_0 \subset D$ there exists a sequence of positive numbers $\{\epsilon_j\}_{j=0}^{J}$ such that

$$L_V(c_{j+1}) + \epsilon_j \mathbb{B} \subset L_V(c_j)$$

for all $j \in \{0, 1, \ldots, J-1\} \cap \mathbb{N}$, where

$$c_j = \max_{x \in X_j} V(x)$$

Limit for Decreasing V

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for all $j \in \{0, 1, \ldots, J-1\} \cap \mathbb{N}$, where

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Moreover, if $J = \infty$ then

$$\lim_{j \to \infty} L_V(c_j) = \mathcal{M}$$

Towards Set Dynamical Systems with Inputs

disturbance w on the dynamics, and noise v on the state:

$$x_{j+1} = g(x_j, u_j, w_j)$$

The measured state is $\hat{x}_j = x_j + v_j$.

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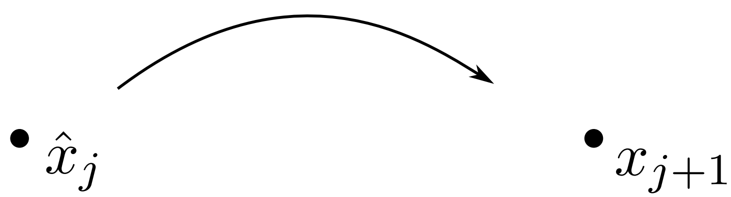

Towards Set Dynamical Systems with Inputs

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$$x_{j+1} \in g(\hat{x}_j + V, u_j, W)$$

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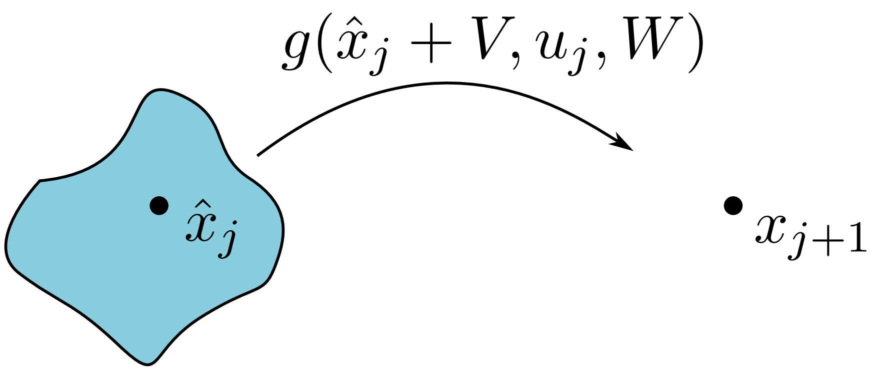

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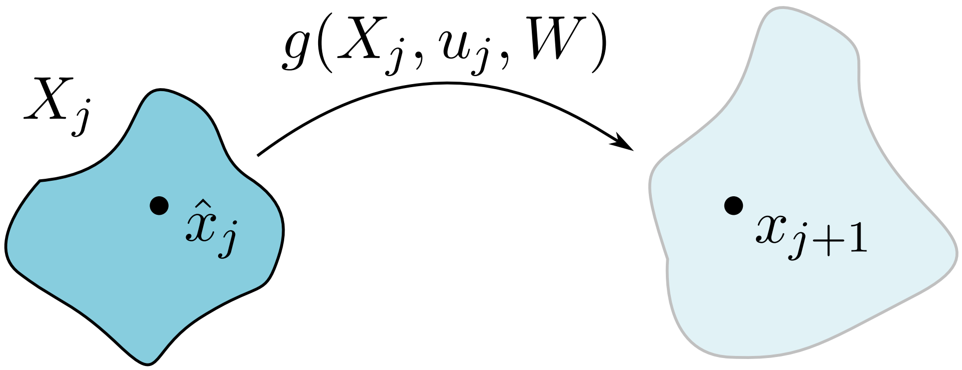

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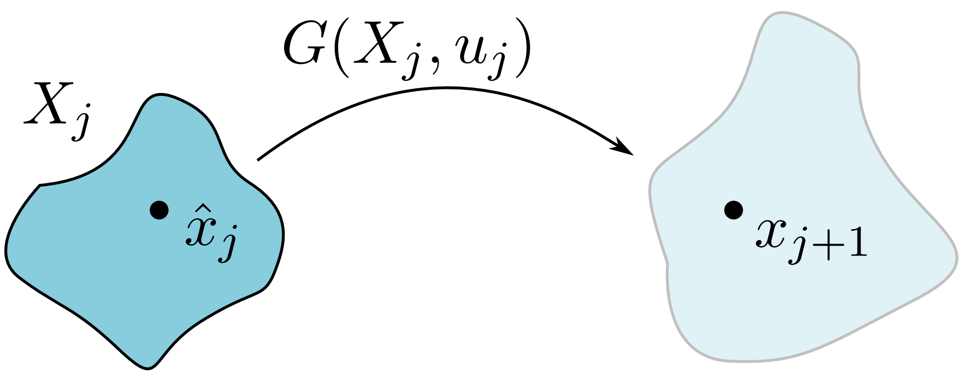

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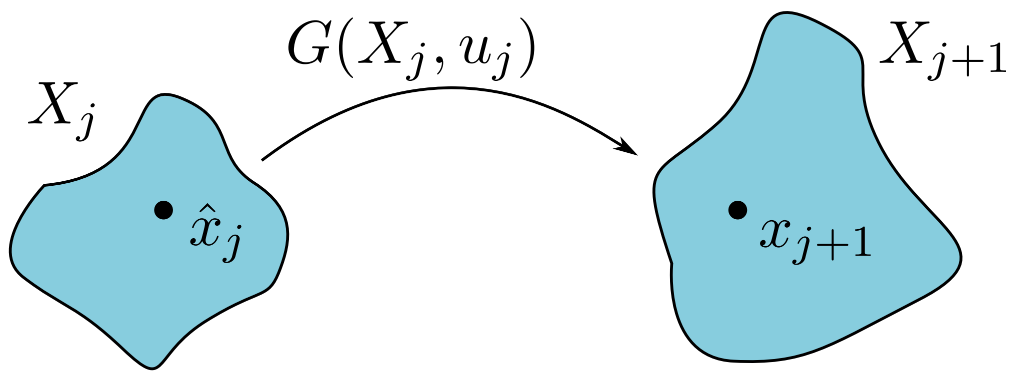

Point-based vs. (Proposed) Set-based MPC

- Point-based MPC optimizes over a sequence of points
- Set-based MPC optimizes over a sequence of sets.

Point-based

- Point-based dynamics
  - $x^+ = g(x, u)$
- Point-based cost functional
  - $J(x, u) = \sum_{j=0}^{N-1} l(x_j) + v(x_N)$

Set-Based

- Set-based dynamics
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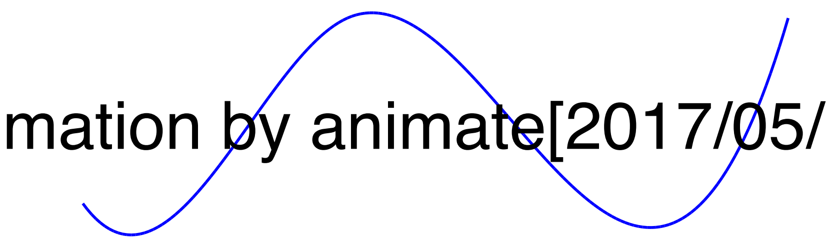
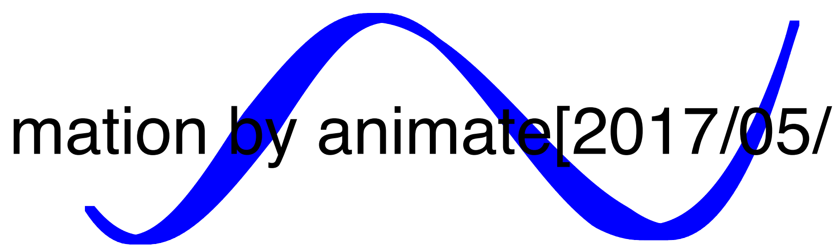




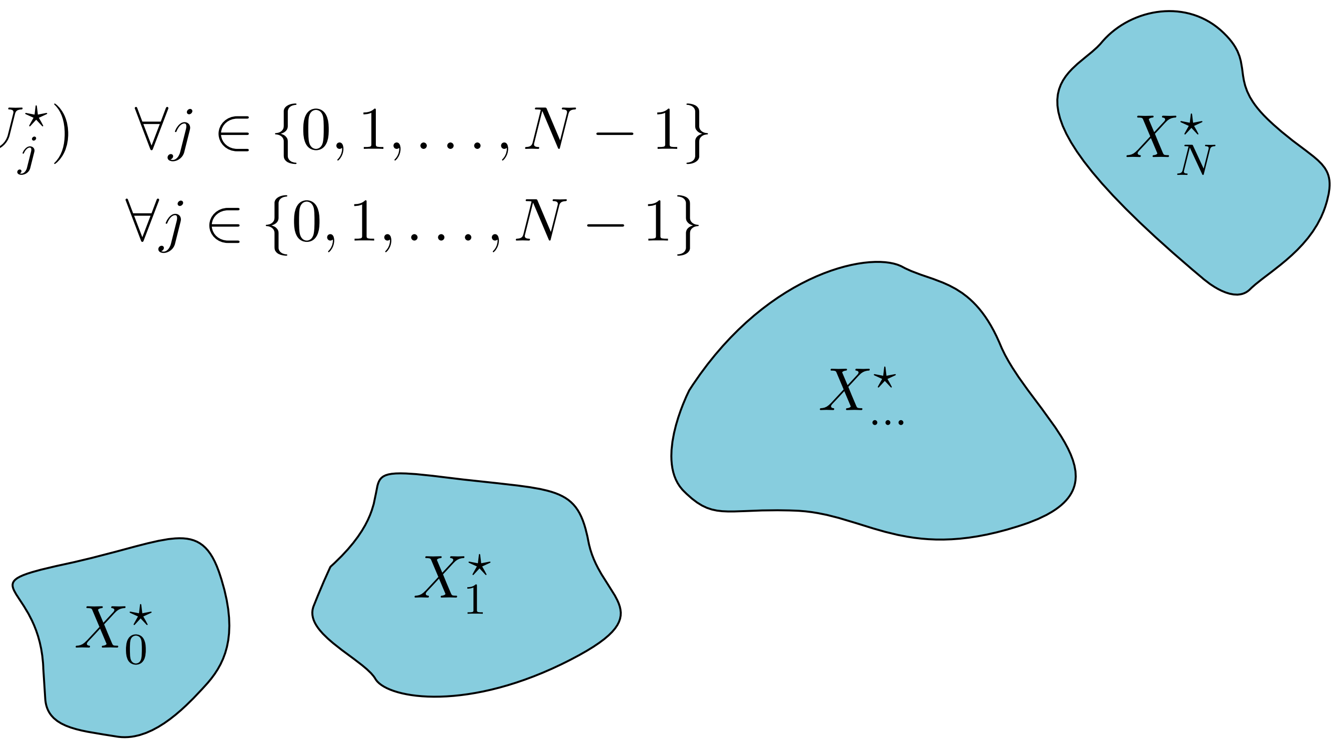
Optimization Problem in Set Based MPC

Given an initial condition X_0, minimize the cost functional

$$\mathcal{J}(X^\star, U^\star) := \left(\sum_{j=0}^{N-1} L(X_j^\star, U_j^\star) \right) + V(X_N^\star)$$

subject to

- $X_0^\star = X_0$
- $X_{j+1}^\star = G(X_j^\star, U_j^\star) \quad \forall j \in \{0, 1, \ldots, N-1\}$
- $(X_j^\star, U_j^\star) \subset D \qquad \forall j \in \{0, 1, \ldots, N-1\}$
- $X_N^\star \subset X_V$



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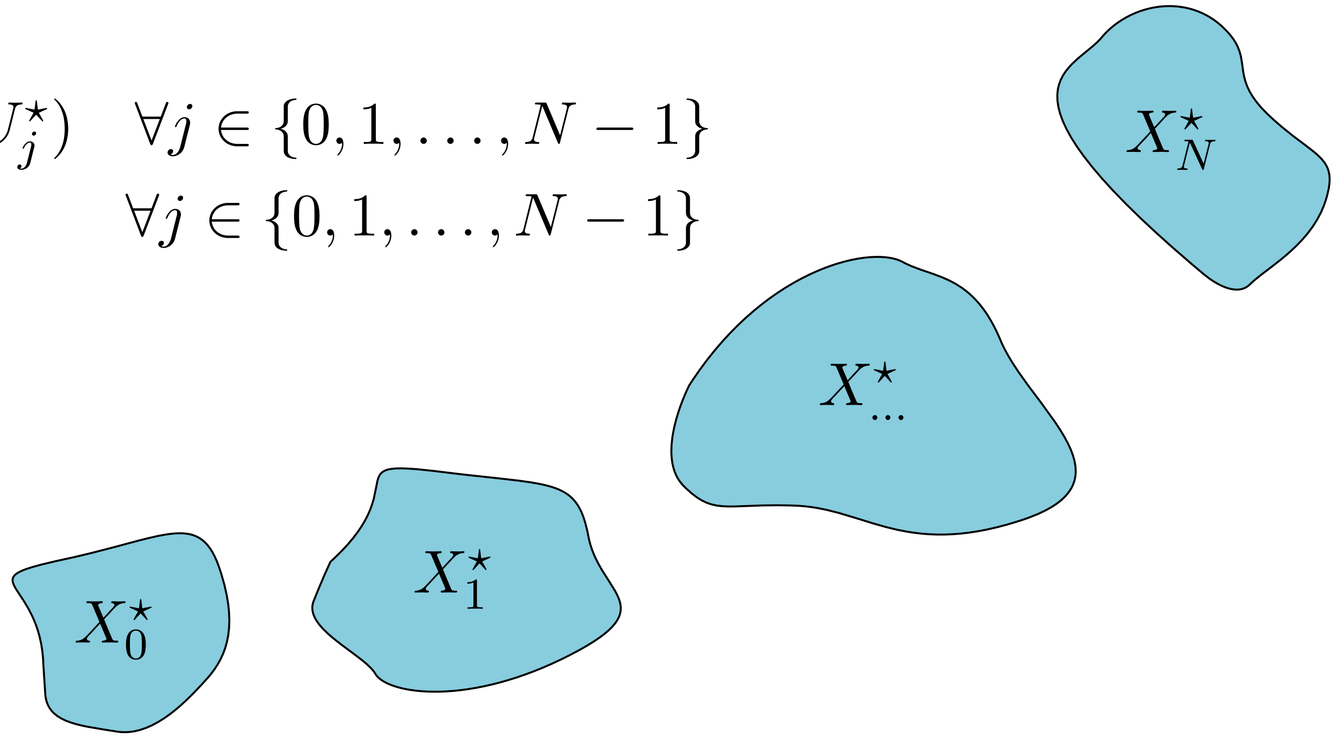

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* Real-valued functions

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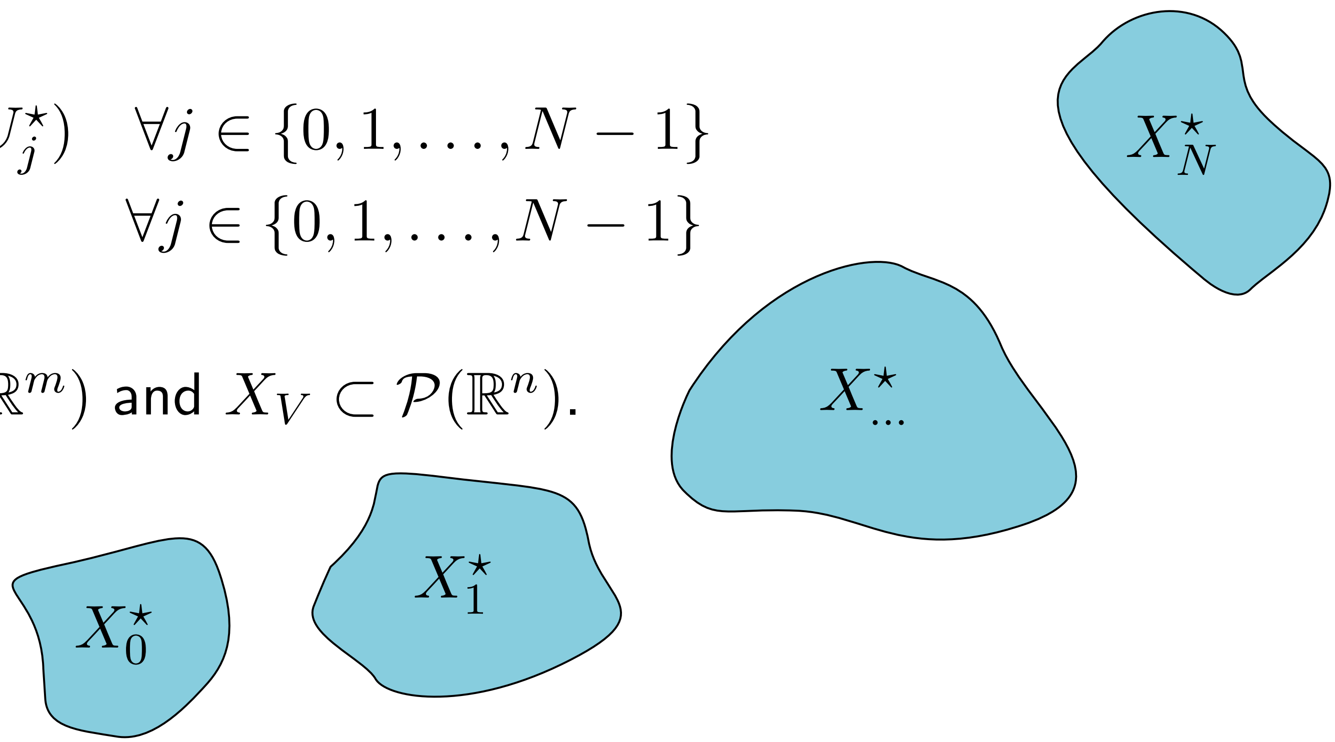
where $D \subset \mathcal{P}(\mathbb{R}^n \times \mathbb{R}^m)$ and $X_V \subset \mathcal{P}(\mathbb{R}^n)$.



Mixed Constraint Set for Collision Avoidance

The mixed constraint set D is used to encode safety constraints.

$$D = \{X \times U \subset \mathcal{X} \times \mathcal{U} : \sigma(X, X_{\mathsf{obst}}) \geq \varsigma\}$$

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- $\mathcal{X}$, state constraint set
- $\mathcal{U}$, input constraint set
- $\sigma(X, X_{\mathsf{obst}})$, minimum distance between X and X_{obst}
- $\varsigma > 0$, safety parameter

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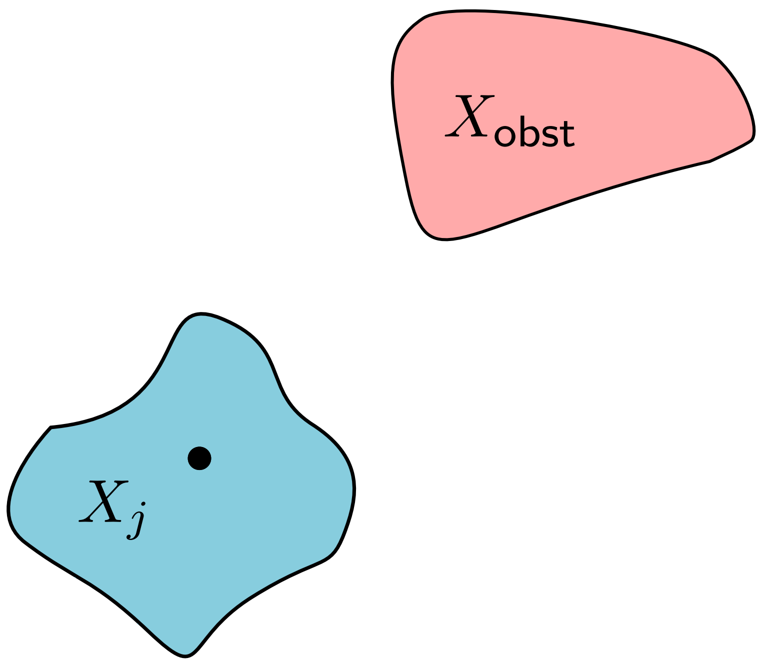

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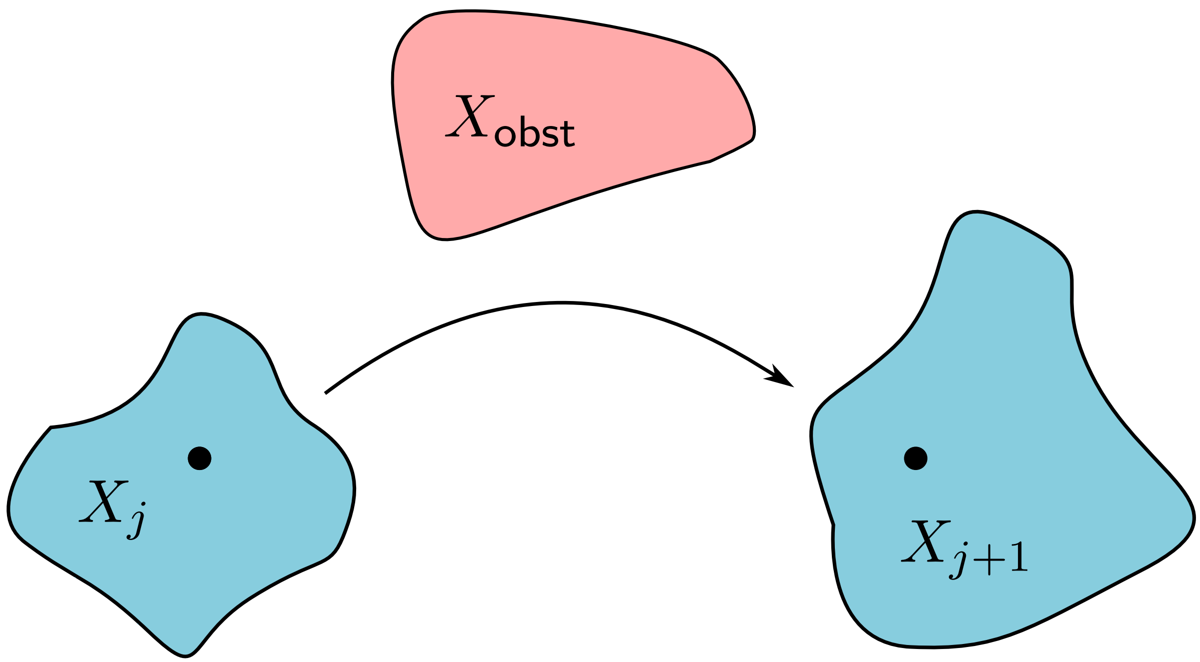

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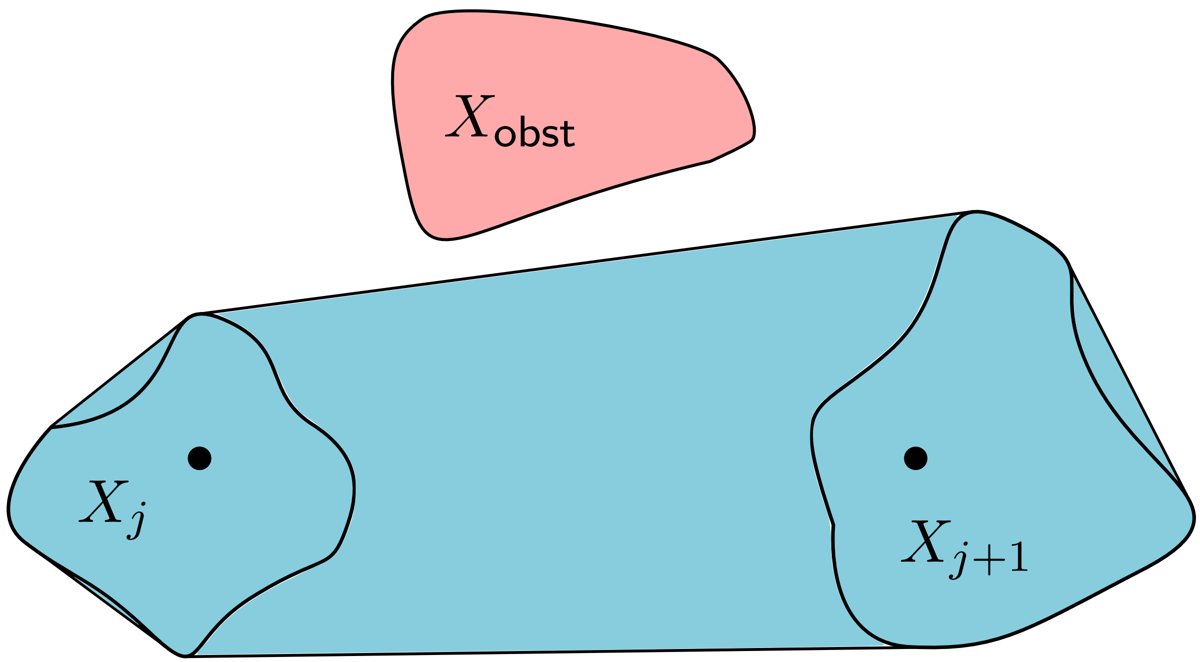

$\mathsf{con}(X_j \cup G(X_j, U_j))$

Computationally Tractable Set Dynamics

Special case: sequences of compact convex polytopic sets

$$x^+ = g(x,u) = Ax + Bu + w$$

Computationally Tractable Set Dynamics

Special case: sequences of compact convex polytopic sets

If $x \in X$, $u \in U$, $w \in W$ all compact convex polytopes, then

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$$\begin{bmatrix} (x^1)^+ \\ (x^2)^+ \\ \vdots \\ (x^p)^+ \end{bmatrix} = \begin{bmatrix} Ax^1 + Bu + w \\ Ax^2 + Bu + w \\ \vdots \\ Ax^p + Bu + w \end{bmatrix}$$

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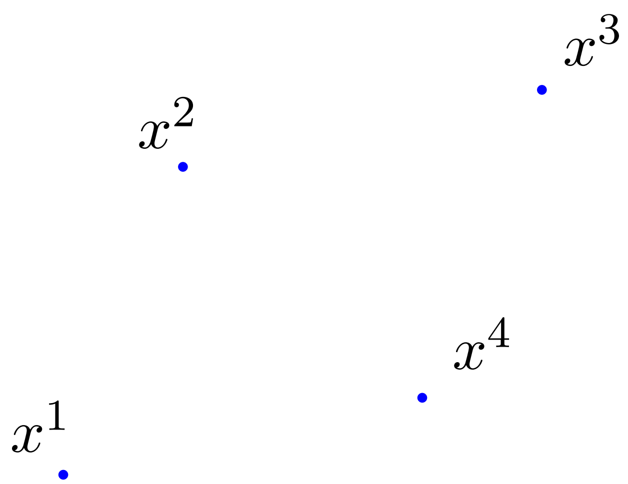

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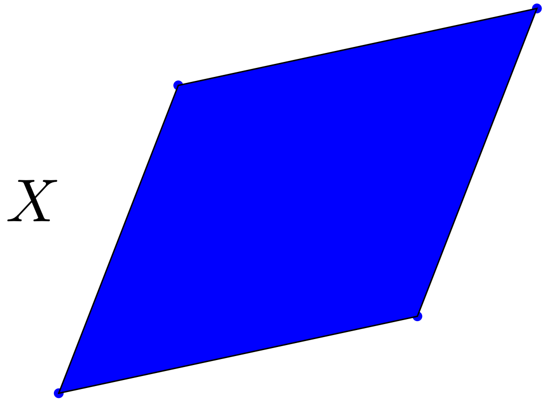

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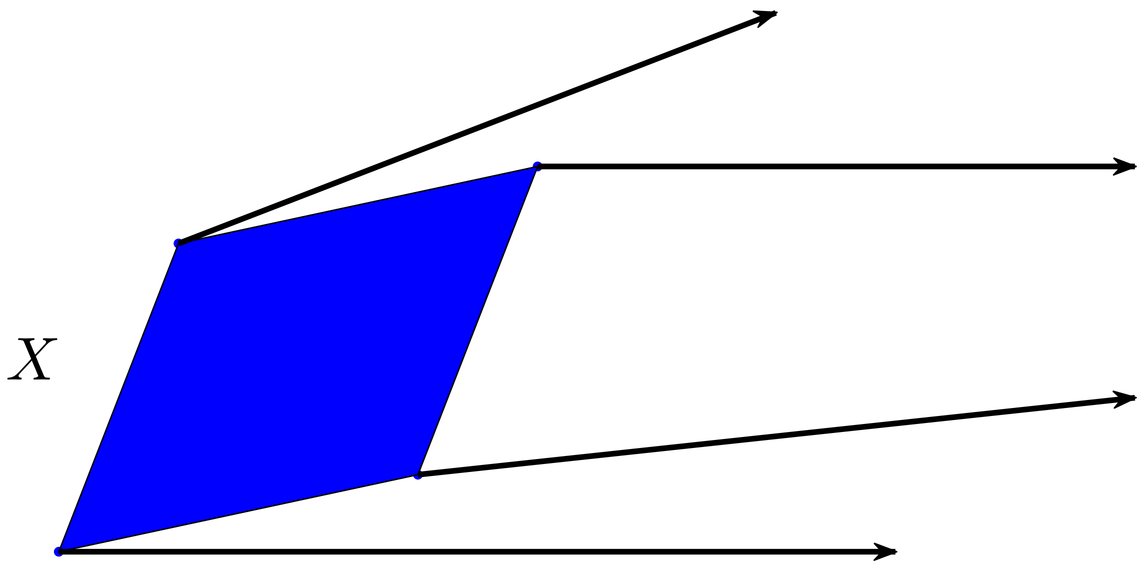

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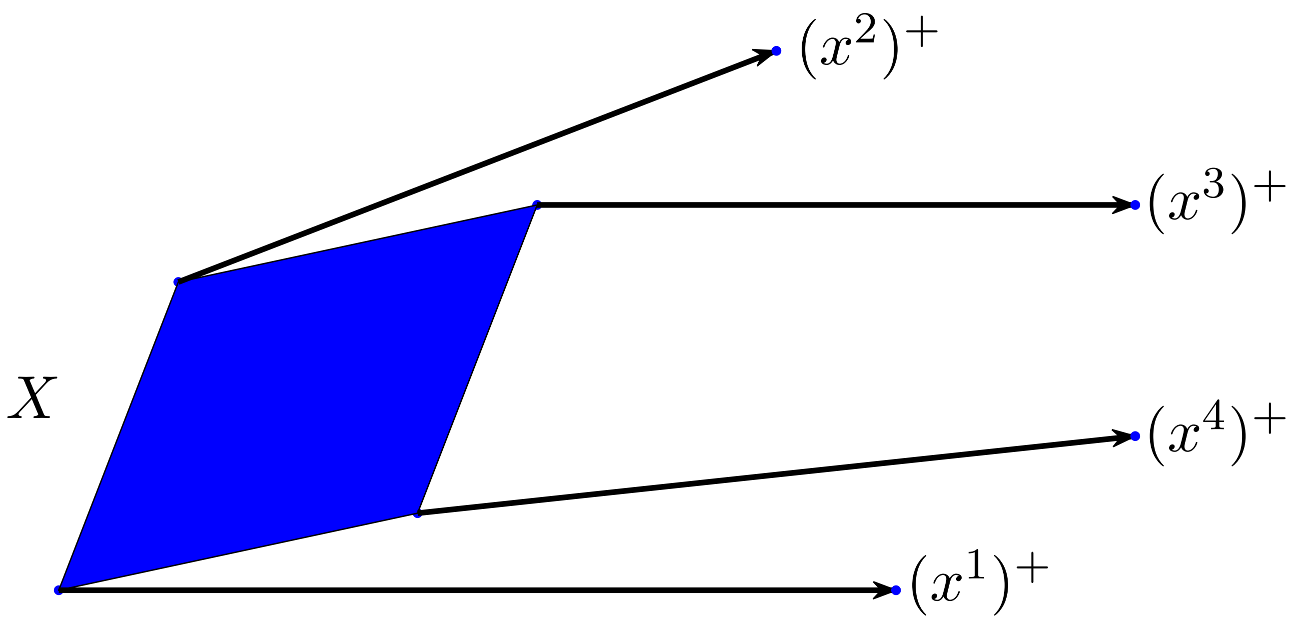

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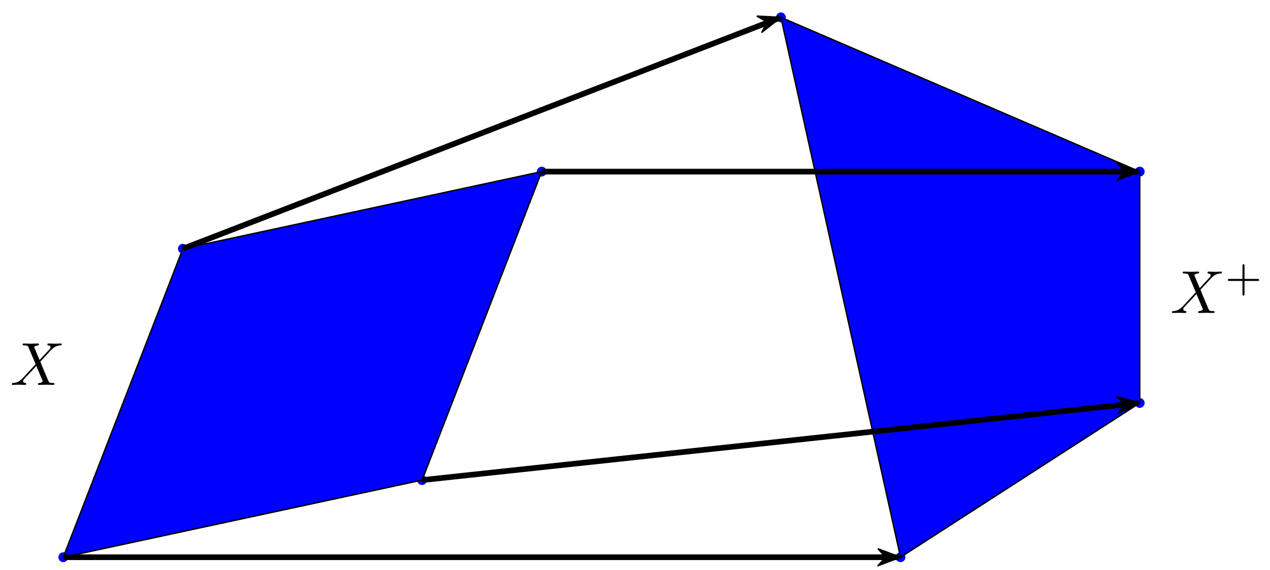

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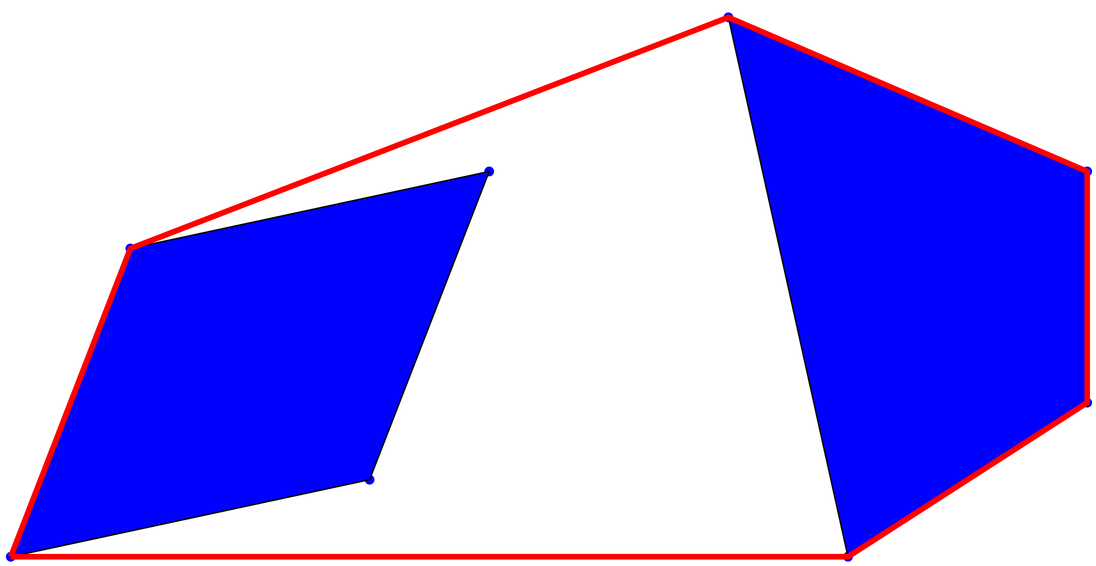

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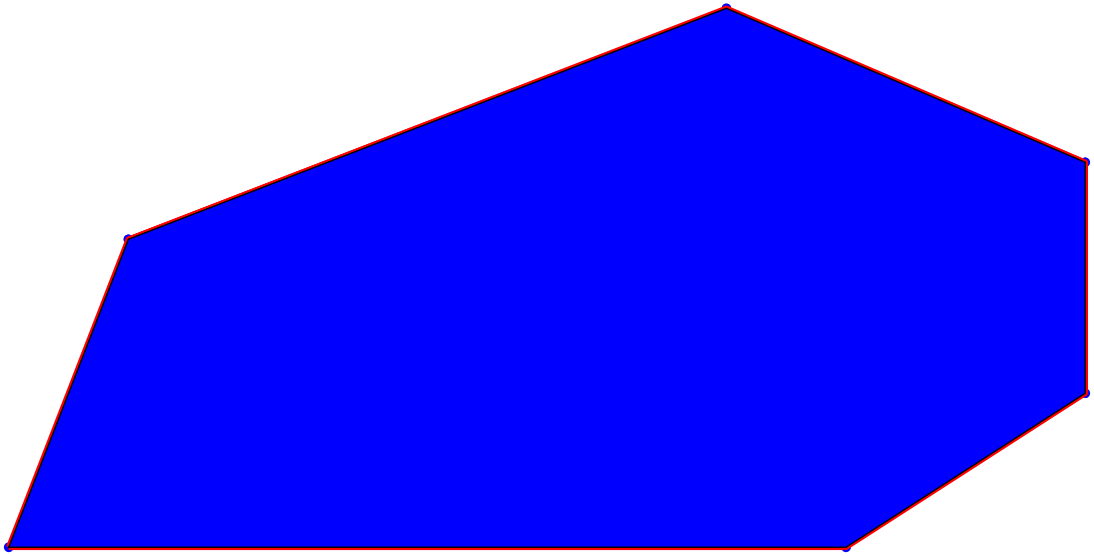

Determining Set Based Cost

Define cost function, constraint sets, and dynamics appropriately

- $\mathcal{J}(X, U)$ generates a cost associated with a target set
  - "LQR" style cost with weighted state error

animation by animate[2017/05/18]

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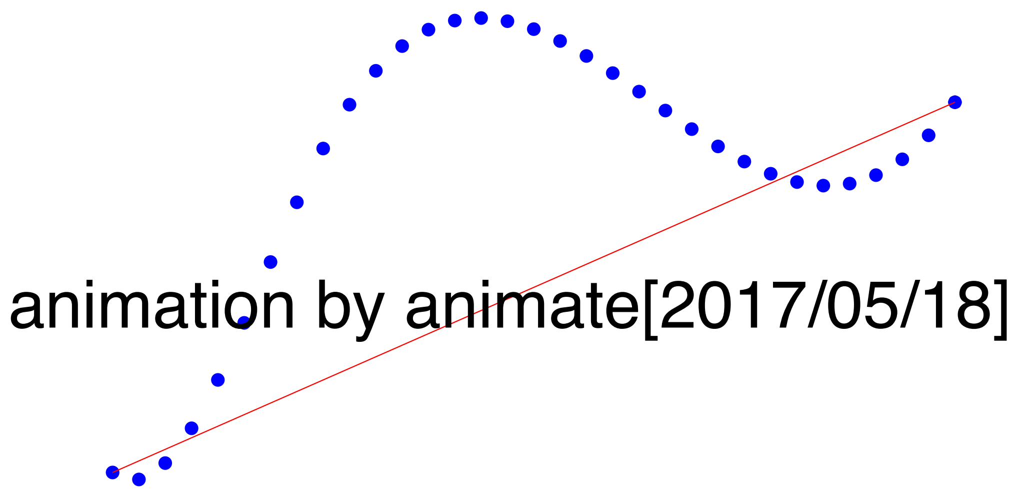

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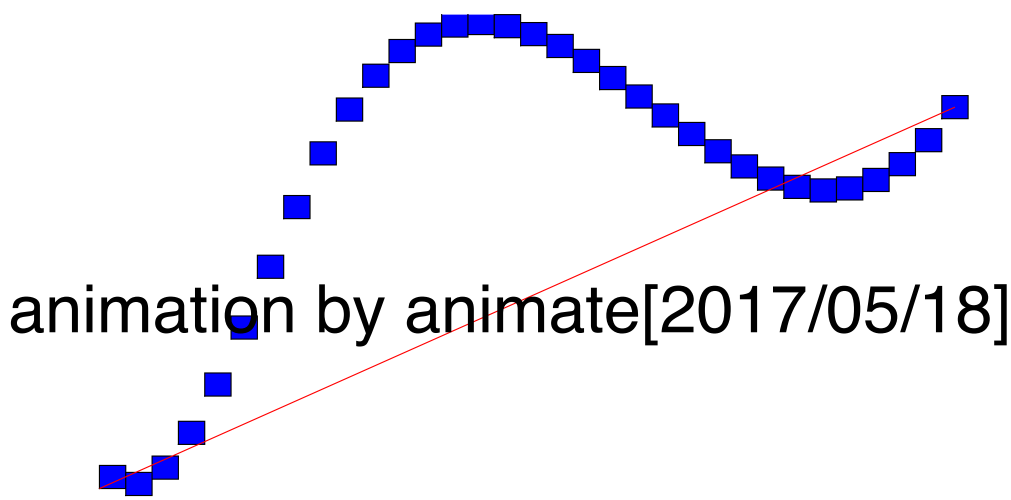
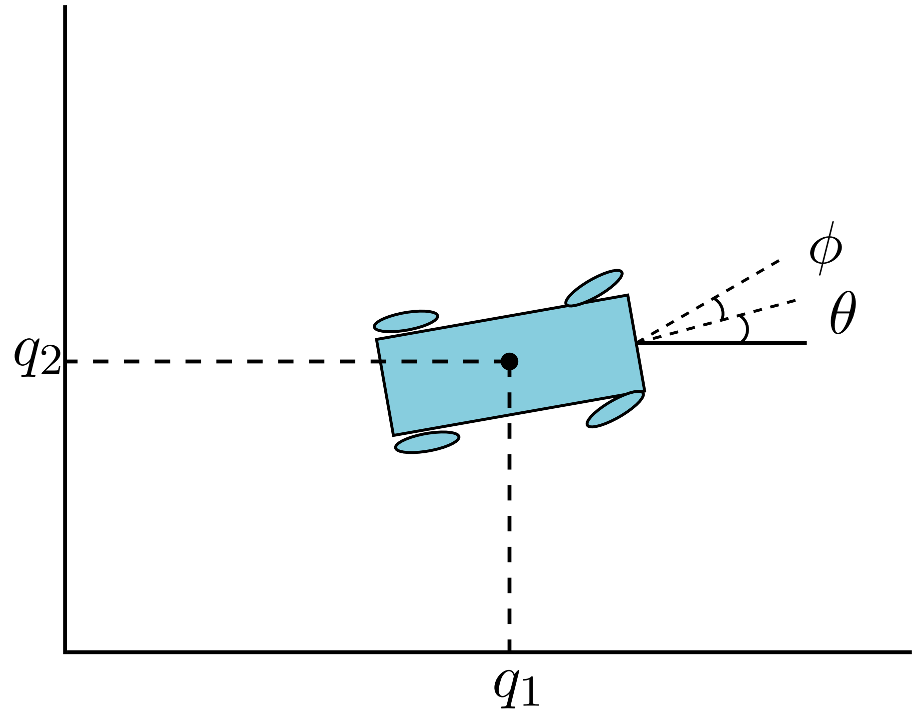


Case Study: Vehicle Dynamics

Using **Dubins model**

$$\dot{q}_1 = v\cos(\theta)$$
$$\dot{q}_2 = v\sin(\theta)$$
$$\dot{\theta} = (v/L)\tan(\phi) := \omega$$



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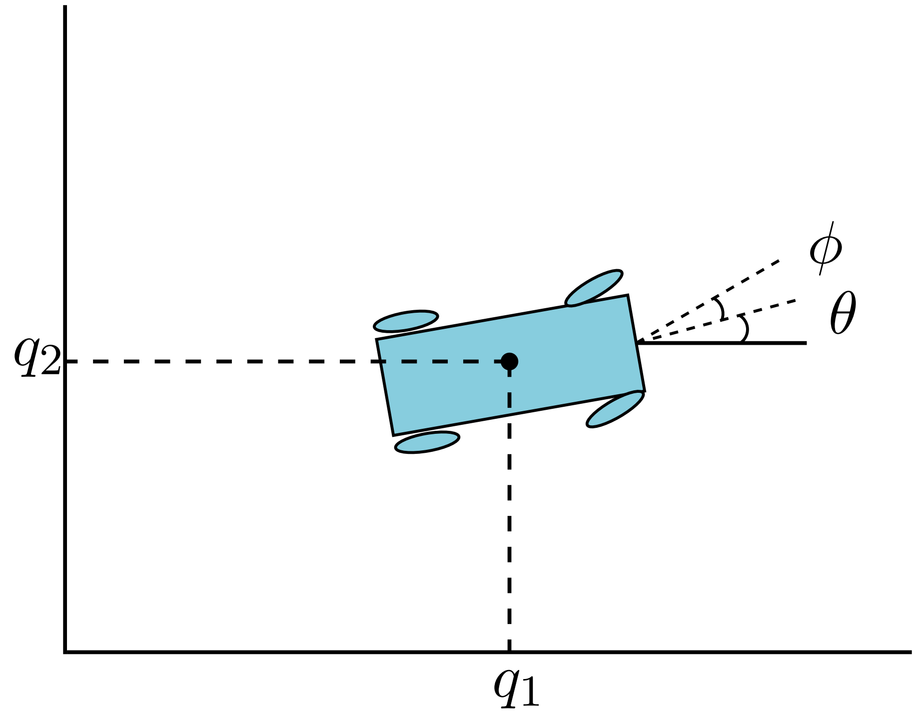


State $x := (q_1, q_2, \theta)$ and input $u := (v, T\omega/2)$

$$x^+ = g(x,u,w) = \begin{bmatrix} q_1 + u_1 \frac{2\cos(\theta+u_2)\sin(u_2)}{\omega} \\ q_2 + u_1 \frac{2\sin(\theta+u_2)\sin(u_2)}{\omega} \\ \theta + 2u_2 \end{bmatrix}$$

Case Study: Set-Based Vehicle Dynamics

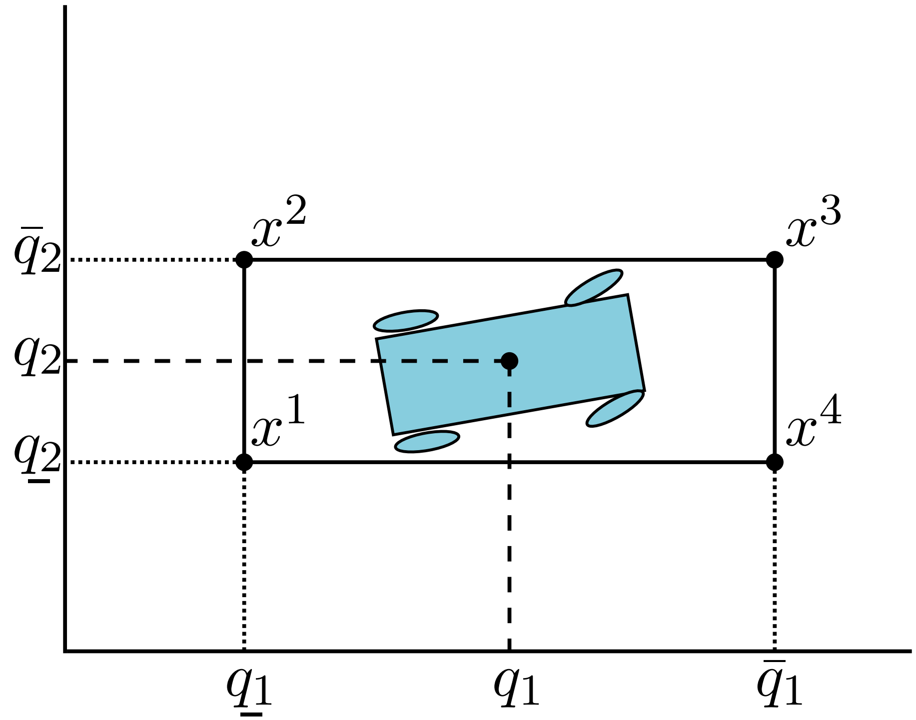
To formalize the set dynamical model, we assume

$$x \in X := [\underline{q}_1, \bar{q}_1] \times [\underline{q}_2, \bar{q}_2] \times \{\theta\}.$$

Polytope with vertices:

$$x^1 = (\underline{q}_1, \underline{q}_2, \theta) \quad x^3 = (\bar{q}_1, \bar{q}_2, \theta)$$

$$x^2 = (\underline{q}_1, \bar{q}_2, \theta) \quad x^4 = (\bar{q}_1, \underline{q}_2, \theta)$$



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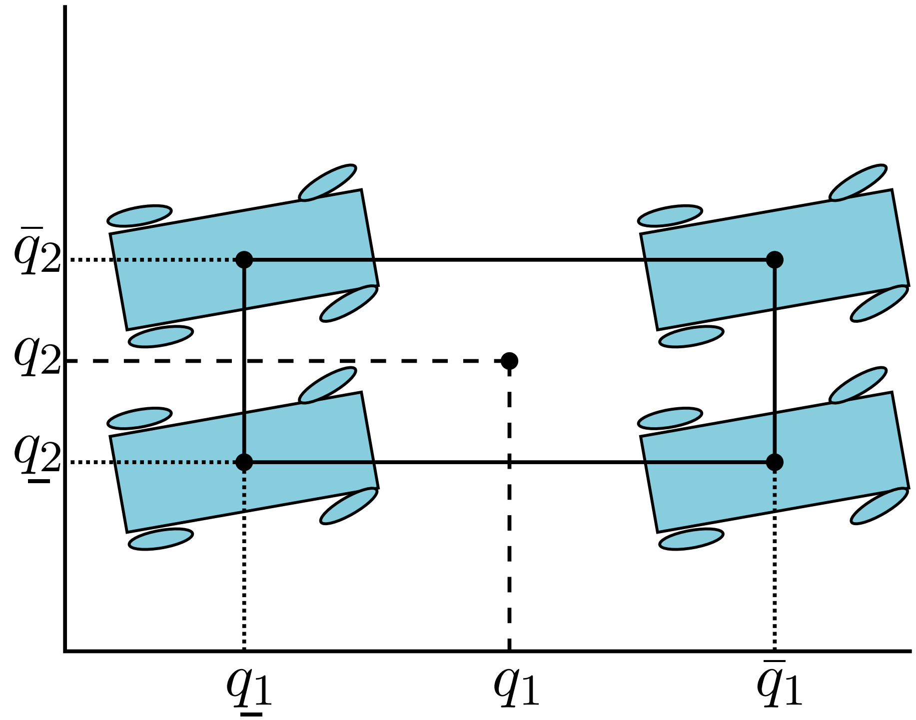

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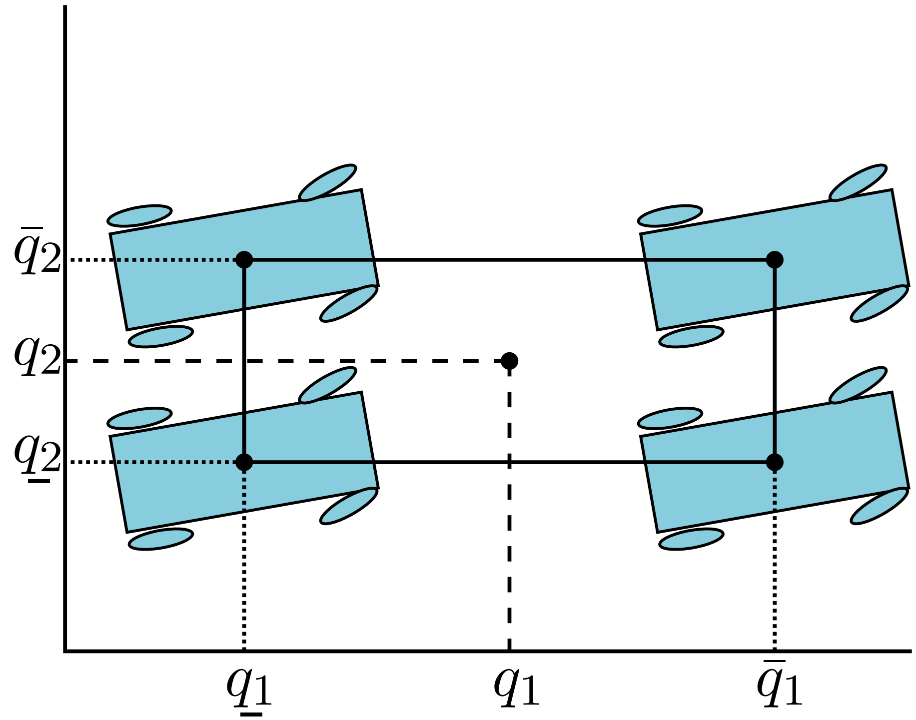
$$x \in X := [\underline{q}_1, \bar{q}_1] \times [\underline{q}_2, \bar{q}_2] \times \{\theta\}.$$

Polytope with vertices:

$$x^1 = (\underline{q}_1, \underline{q}_2, \theta) \quad x^3 = (\bar{q}_1, \bar{q}_2, \theta)$$

$$x^2 = (\underline{q}_1, \bar{q}_2, \theta) \quad x^4 = (\bar{q}_1, \underline{q}_2, \theta)$$



The dynamics of $\mathbf{q_1}$ and $\mathbf{q_2}$ are decoupled, so we can remove redundant states to get a 5 dimensional system.

Selecting the Cost Function and Constraints

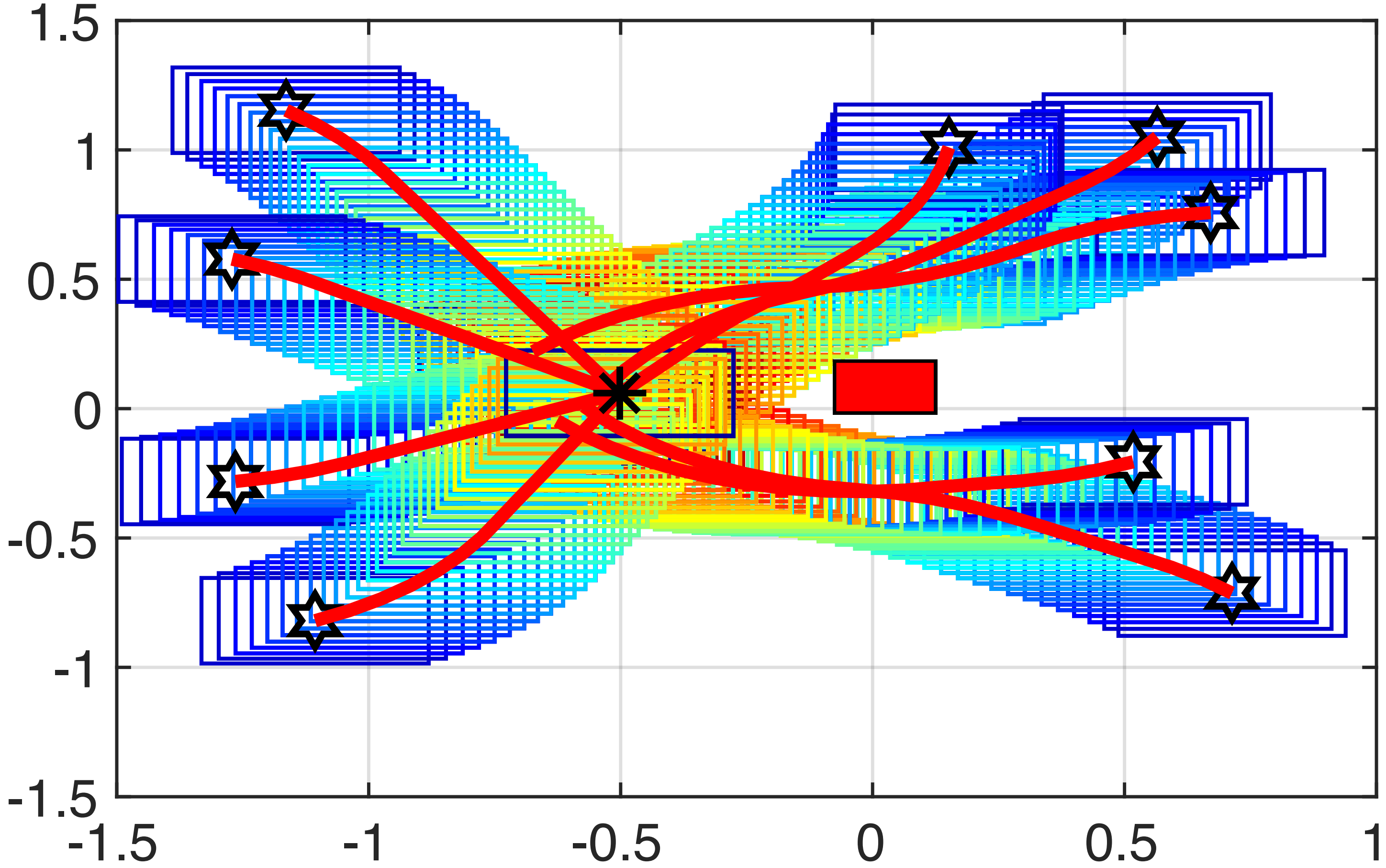
Cost Function

- $L(X, U) = \sum_{i=1}^{p} |x^i|_{X_{\text{target}}}$
- $V(X) = c \sum_{i=1}^{p} |x^i|_{X_{\text{target}}}$ where $c > 0$

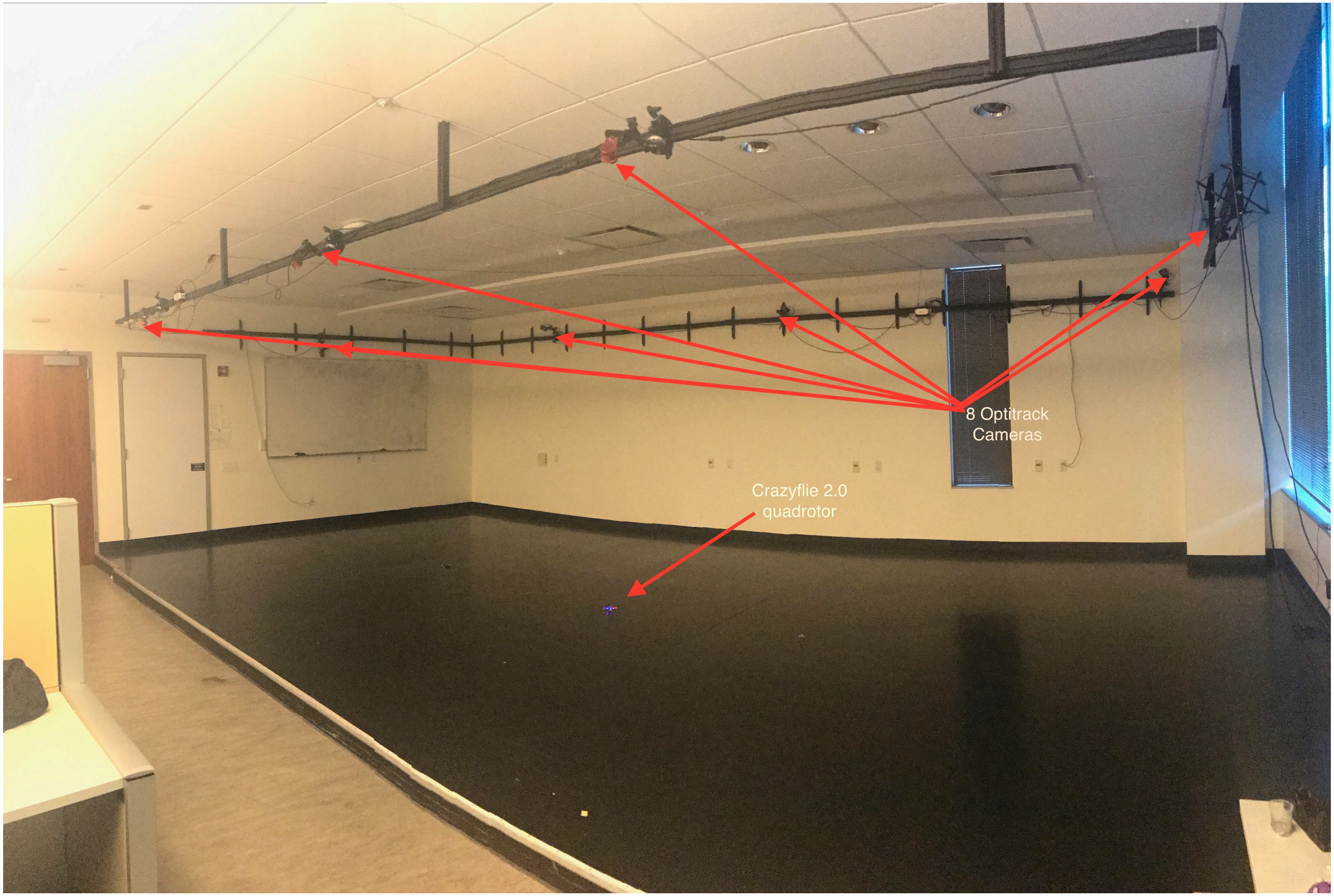
Constraints

- $D = \{X \times U \subset \mathcal{X} \times \mathcal{U} : \sigma(\text{con}(X \cup G(X, U)), X_{\text{obst}}) \geq \varsigma\}$
- $X_V = \mathcal{P}(\mathbb{R}^3)$

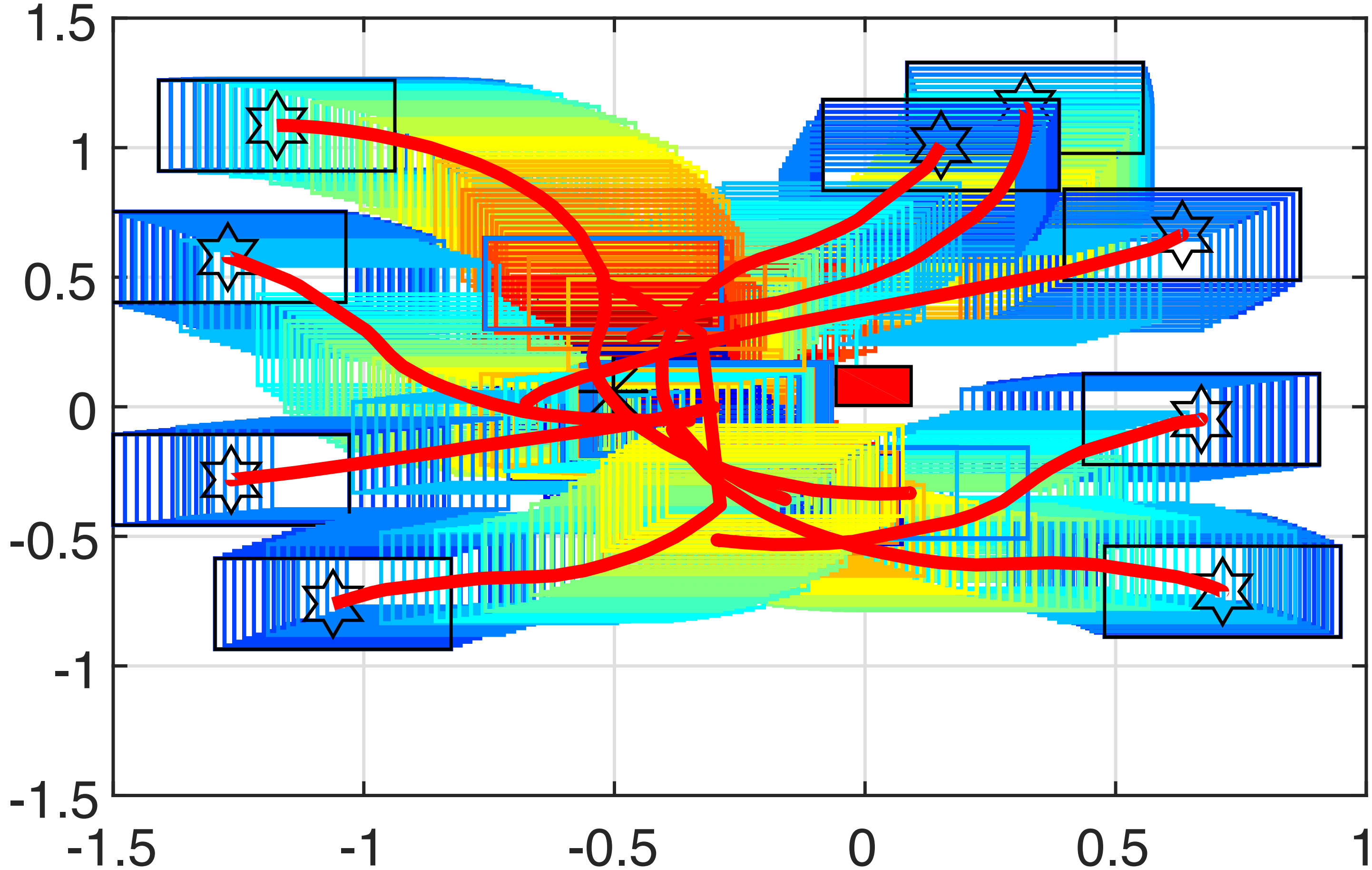
Static Obstacle Simulations



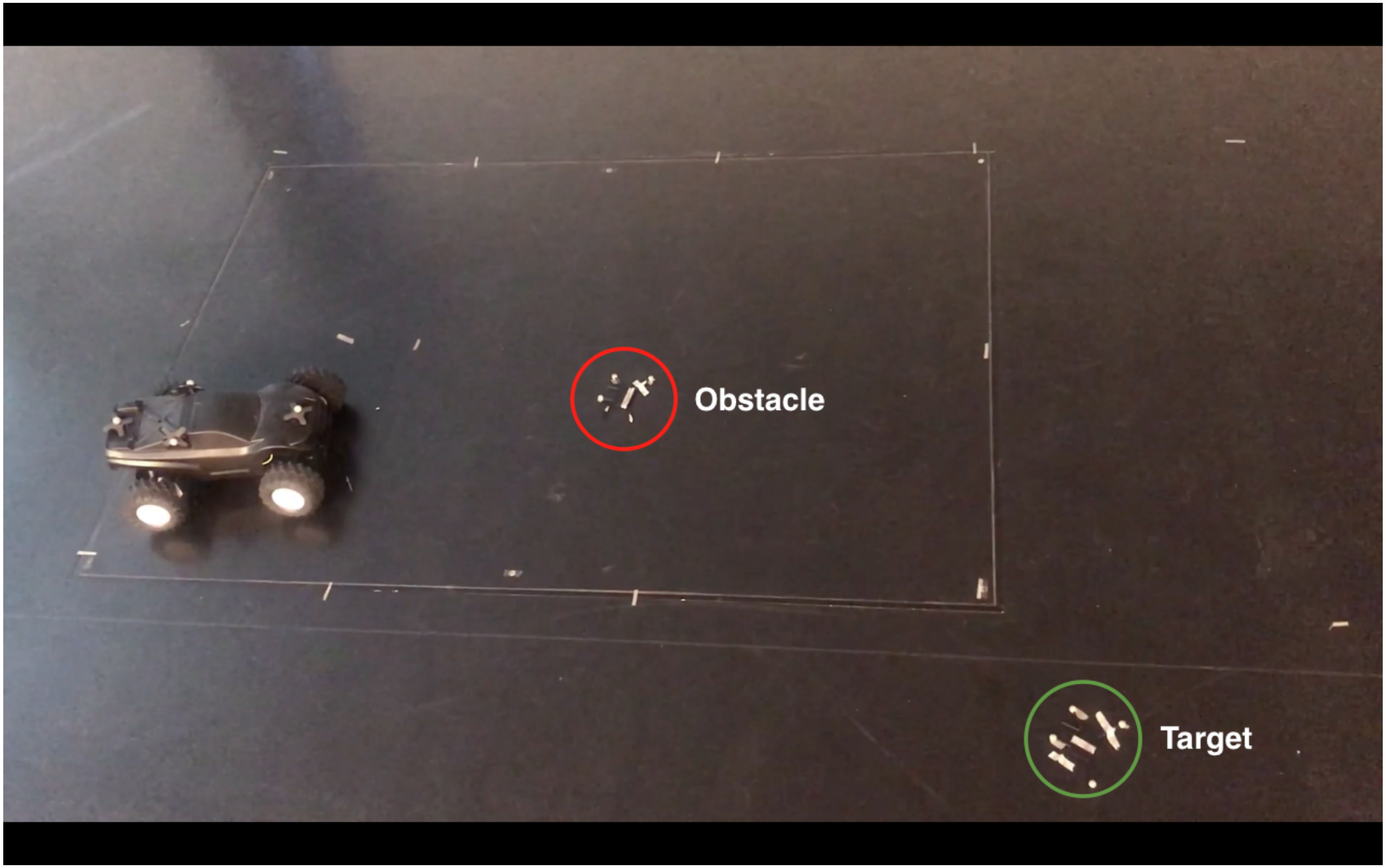
Experimental Setup



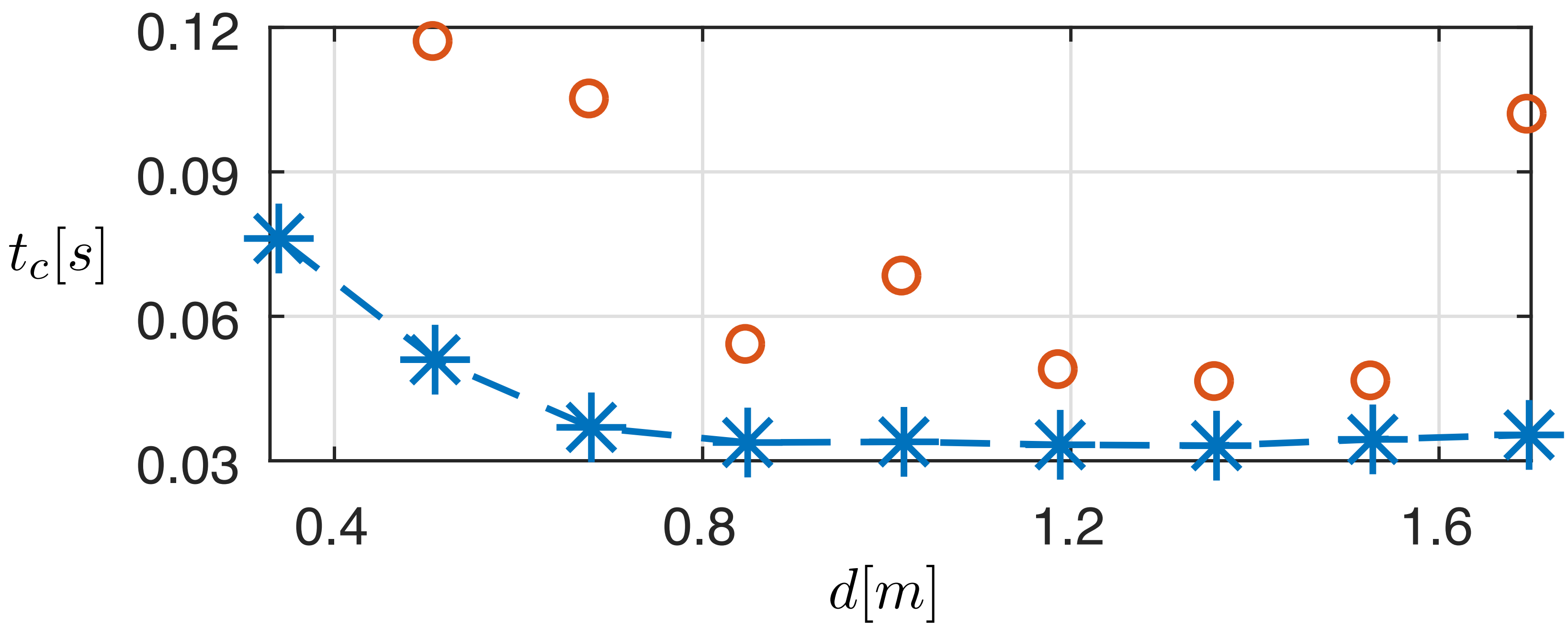
Static Obstacle Experiments



Static Obstacle Experiments



Computational Cost



- Circle - mean computation time
- Star - median computation time

Predictive Control for Motion Planning

Predictive Control for Robotic Manipulation

Predictive Control for Motion Planning

Predictive Control for Walking

Predictive Control for Gaming

Conclusion

A *Set-Based MPC framework* for discrete-time systems.

- The optimization is performed directly over set-based trajectories.
- The scheme is applicable to reachability computation, safety analysis, uncertainty propagation, and collision avoidance.

Conclusion

A *Set-Based MPC framework* for discrete-time systems.

- The optimization is performed directly over set-based trajectories.
- The scheme is applicable to reachability computation, safety analysis, uncertainty propagation, and collision avoidance.

Future work will include:

- *controllability* properties of set dynamical systems
- *numerical tools* for set-based optimization
- formal *feasibility* and *stability* guarantees

Thank you for your attention!

More details at https://hybrid.soe.ucsc.edu

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