Fundamentals of optimization
Part I

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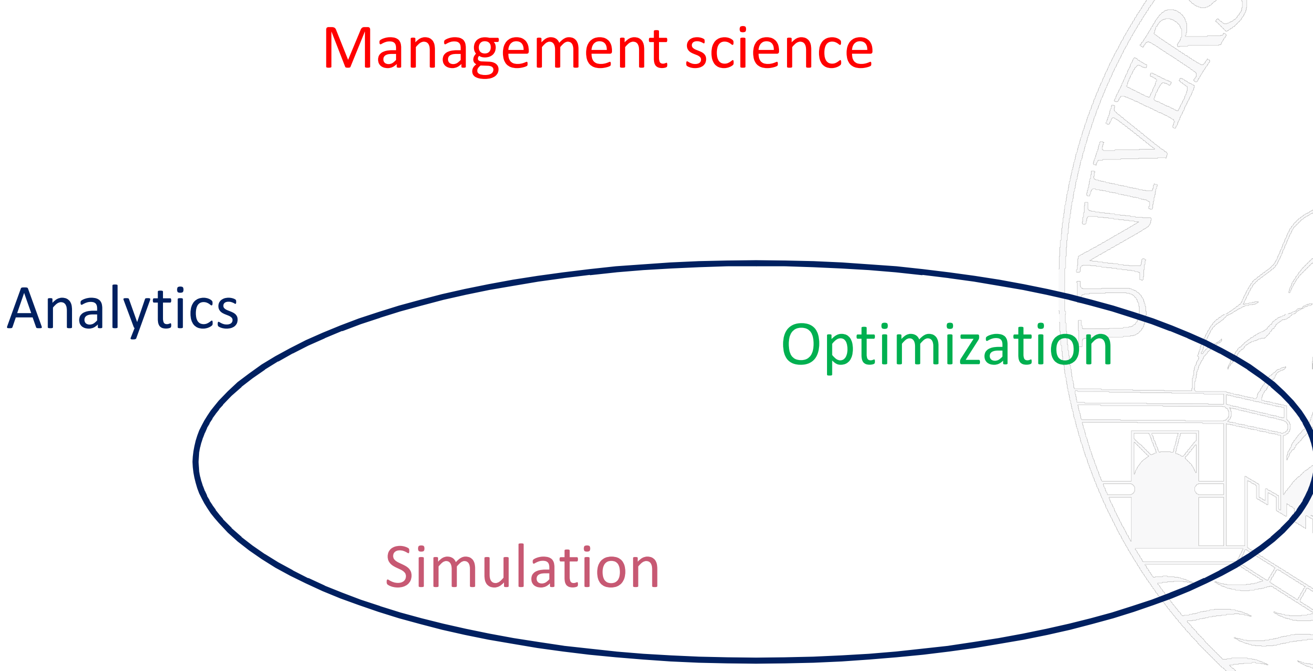
Operations research



Operations research

Management science

Analytics

Optimization

Simulation

What if analysis

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Operations research

Management science

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Optimization

Simulation

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Optimization

What is it?

Why do we care?

When do we use it?

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Optimization

$$\boxed{\begin{array}{c} \min c(x) \\ \\ x \in F \end{array}}$$

Decision variables

One objective (objective function)

Constraints (feasible region)

Optimization

One objective only?

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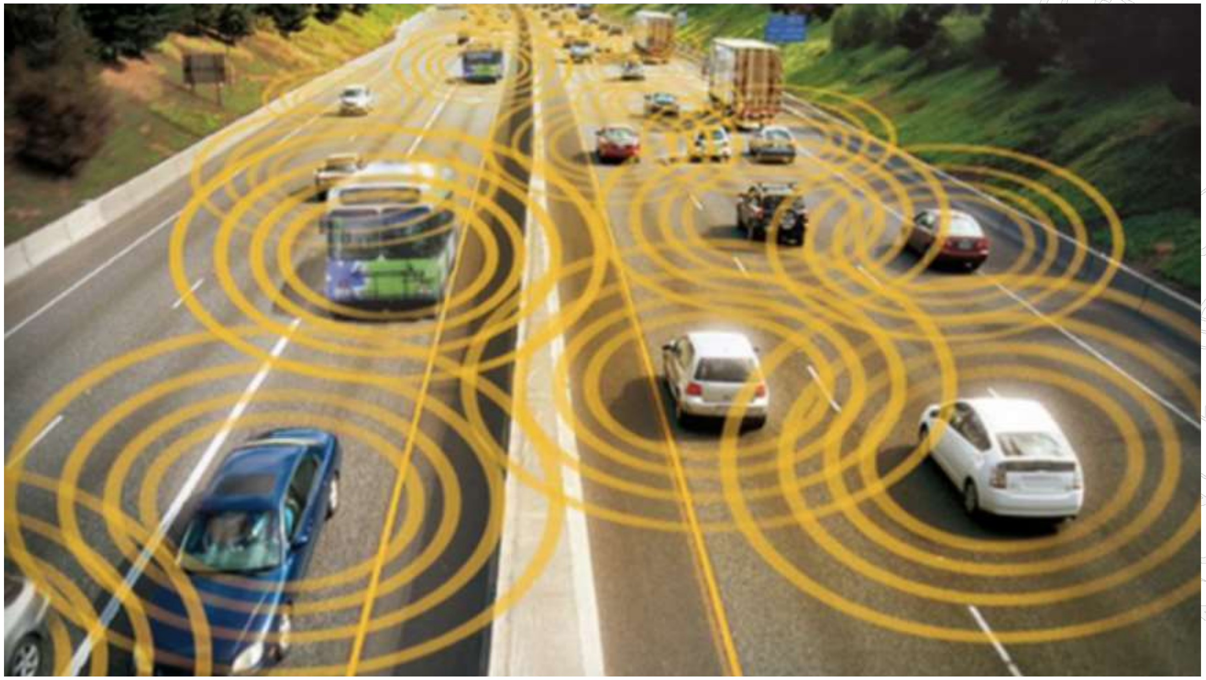
Optimization

One objective only?

Safety

‘Low’ cost

Small traveling time



Fair traffic assignment

Environmentally friendly

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Optimization

Multiple objectives

a) Combination of objectives into a single one

$$\min f(x) \qquad \max g(x) \quad \rightarrow \quad \max g(x)-p\,f(x)$$

Method not always appropriate

Optimization

Multiple objectives

b) One objective chosen as objective with the others into constraints with thresholds

min $f(x)$	$\rightarrow$	min $f(x)$
max $g(x)$		$g(x) \geq k$

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Optimization

a)

max g(x)-p f(x)

Choice of p

b)

min f(x)

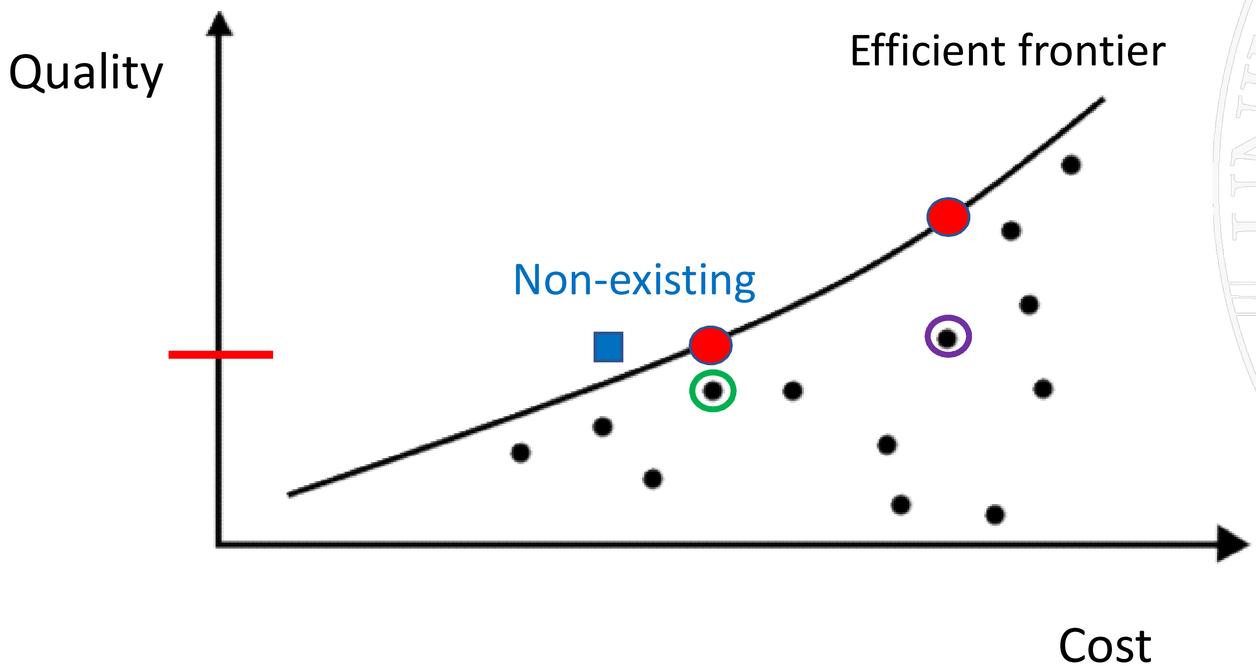
g(x)≥k

Choice of k

Multiple optimization problems, each with one objective

Efficient frontier

Optimization



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Optimization problem

$$\min c(x)$$

$$x \in F$$

Types of objectives

$$\boxed{min \sum_i \; c_i(x)}$$

Average

$$\boxed{minmax \; c_i(x)}$$

Worst

$$min \sum_i \; c_i(x)$$

sum over a given percentage of the worst i

CVaR-like
(can be used on scenarios)

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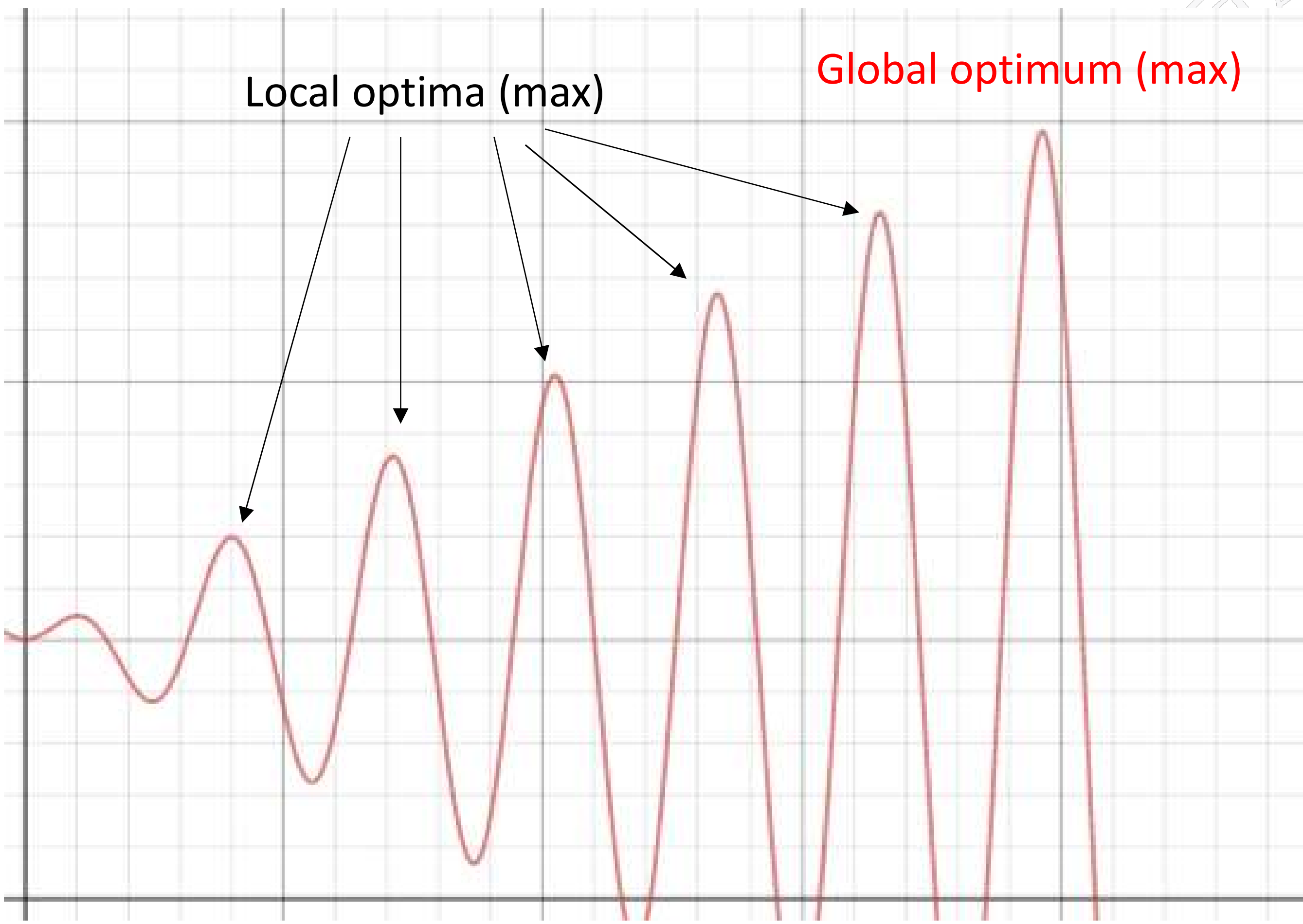
Local and global optima



What do we need?

- We know how to analitically find local optima for ‘simple’ optimization problems
- We know how to analitically find the global optimum for some ‘very simple’ optimization problems
- We do not know for all the others – **algorithms needed**

Convex problem

$$\min_{x \in F} c(x)$$

F	Convex set
$c(x)$	Convex function on F

$\Downarrow$

Convex problem

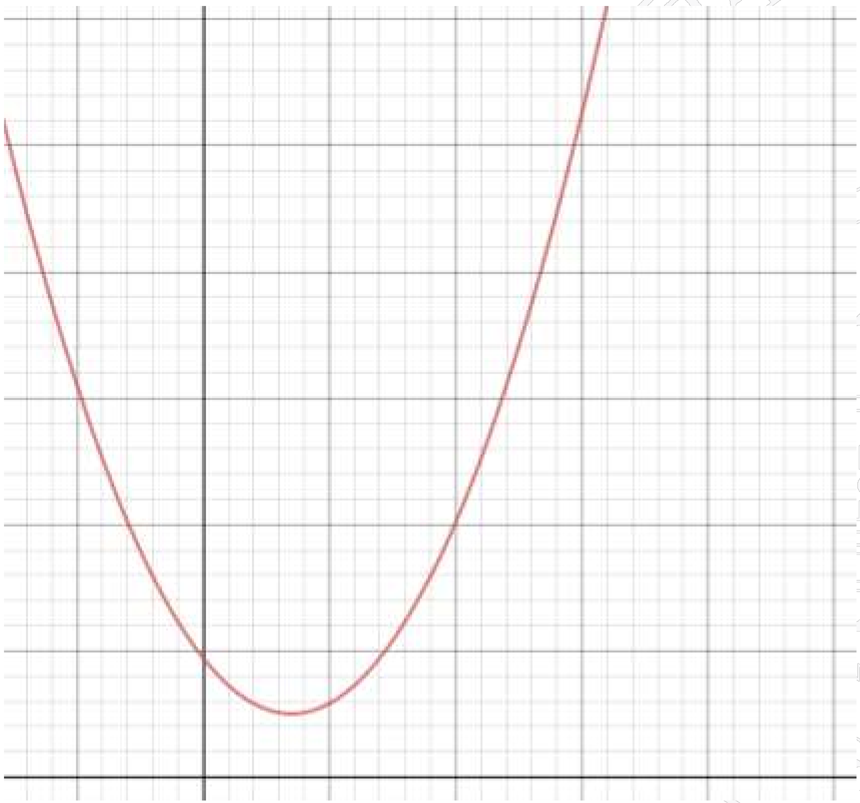
Convex problem

Property:

For a convex problem, any local optimum is a global optimum

↓

Effective and efficient algorithms



Convex problem

$$\begin{gathered}\min \varphi(x) \\ g_i(x) \geq 0 \qquad (i = 1, \dots, q) \\ h_j(x) = 0 \qquad (j = 1, \dots, p)\end{gathered}$$

Theorem:

If $\varphi(x)$ is convex, $g_i(x)$ are concave, $h_j(x)$ are linear, the optimization problem is a convex problem

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Linear programming

$$\min c'x$$
$$Ax = b$$
$$x \geq 0$$

Special case of convex problem

Model - algorithm

Mathematical programming formulation

⬇

Algorithm

(for the formulation and not for a specific problem)

Linear programming

$$\min c'x$$
$$Ax = b$$
$$x \geq 0$$

LP problems can be solved with efficient algorithms:

Simplex method

Interior point methods (Khachiyan, Karmarkar)

Computational complexity

Computational complexity or simply **complexity** of an algorithm is the amount of resources required to run it

Resources: time and memory

Computational complexity

Worst-case complexity of an algorithm

$n \rightarrow f(n)$

n: size of the input

f(n) = max f(I) over all instances I of size n

Problem: Sorting

Instance I: 6, 35, 7, 15, 27, 12, 18

Algorithm: Bubblesort

Size: 7

f(7) = max f(I) over all instances I of size 7

Computational complexity

f(n) very difficult to obtain

Asymptotic behavior (when *n* tends to the infinity)

The complexity is expressed by using big O notation

Complexity of Bubblesort: O(n^2)

Computational complexity

The complexity of a problem is the infimum of the complexities of the algorithms that solve the problem, including unknown algorithms.

Complexity of problem Sorting: O(n log n)
(thanks to Heapsort)

Computational complexity

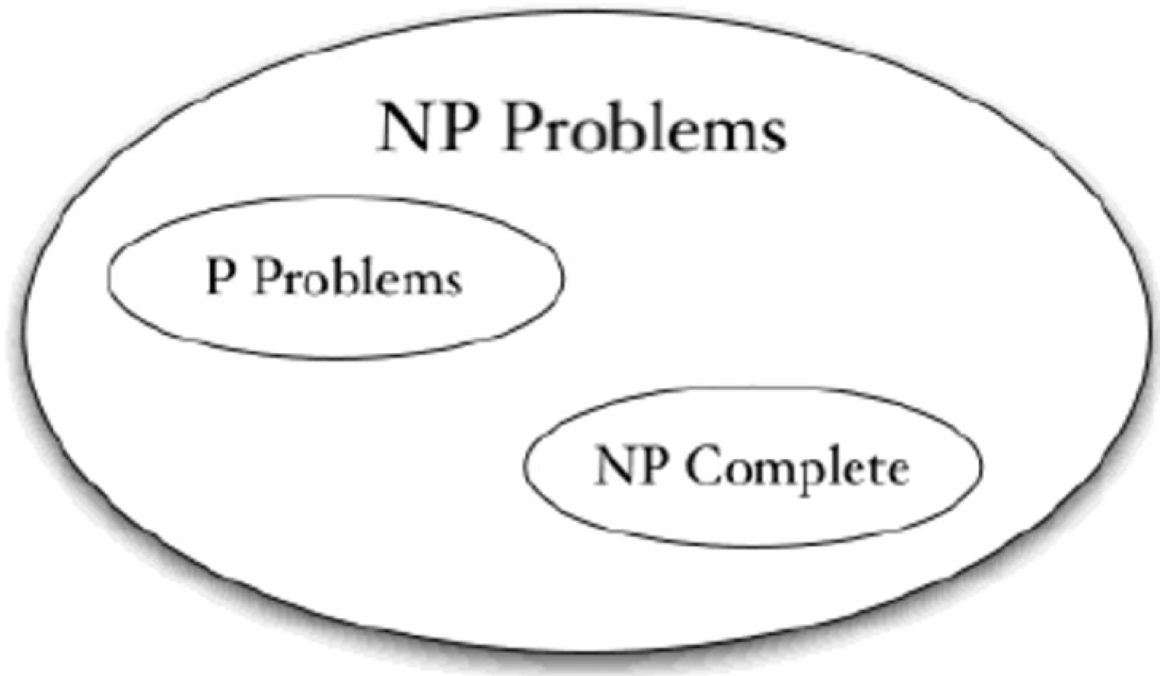
Why do we care about computational complexity?

O(n) O (n log n) O(n^2) O(2^n)

	$n=10$	$n=20$	$n=30$	$n=40$	$n=50$	$n=60$
$O(n)$	$10^{-5''}$	$2 \cdot 10^{-5''}$	$3 \cdot 10^{-5''}$	$4 \cdot 10^{-5''}$	$5 \cdot 10^{-5''}$	$6 \cdot 10^{-5''}$
$O(n^2)$	$10^{-4''}$	$4 \cdot 10^{-4''}$	$9 \cdot 10^{-4''}$	$16 \cdot 10^{-4''}$	$25 \cdot 10^{-4''}$	$36 \cdot 10^{-4''}$
$O(2^n)$	$10^{-3''}$	$1''$	$17.9'$	12.7 *days*	35.7 *years*	366 *centuries*

CPU speed *1000 O(2^n) $\longrightarrow$ size solved increases by 10

Computational complexity: classes



P=NP?

Conjecture: no

Computational complexity: LP

George Dantzig (1914-2005) in 1947 invented the **simplex method**

Worst-case complexity: O($2^{\frac{m}{2}-1}$)

In practice: Rarely time required is greater than O(m log n)

Complexity of LP unknown until 1979

Leonid Khachiyan (1952-2005) in 1979 invented the **ellipsoid method**

Worst-case complexity: O(n^6)

In practice: O(n^6)

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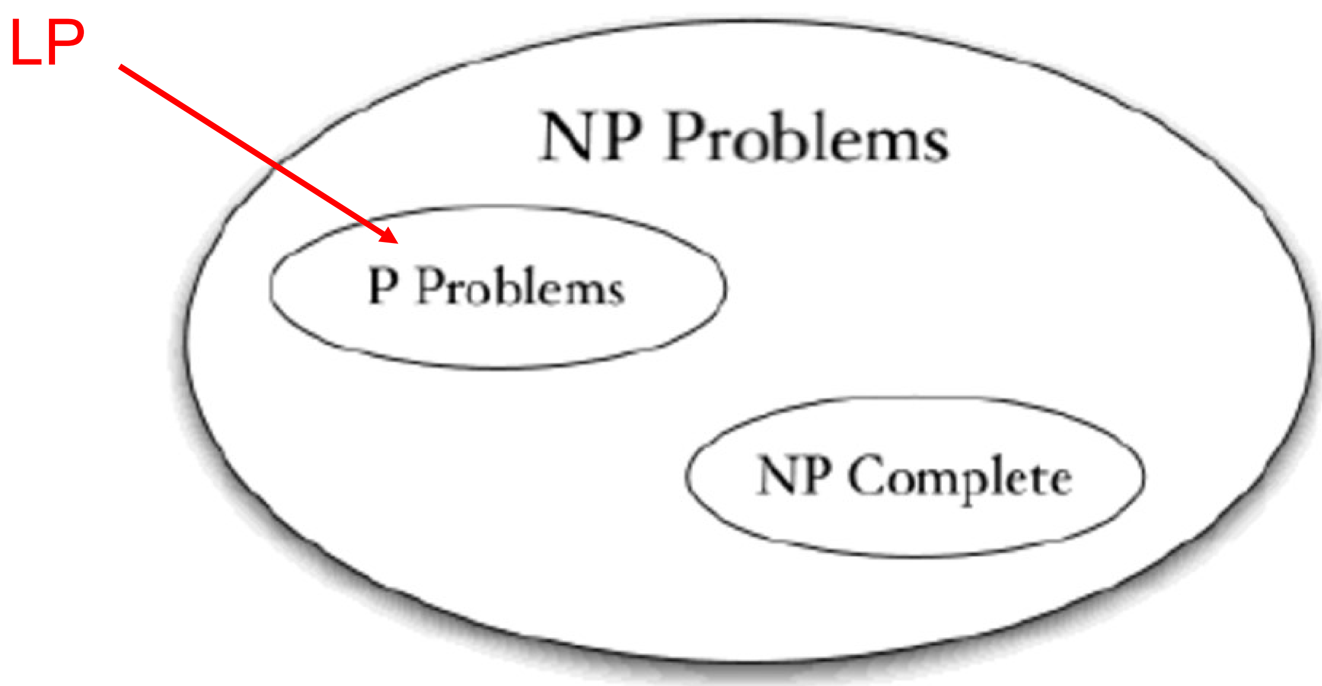
Computational complexity: LP

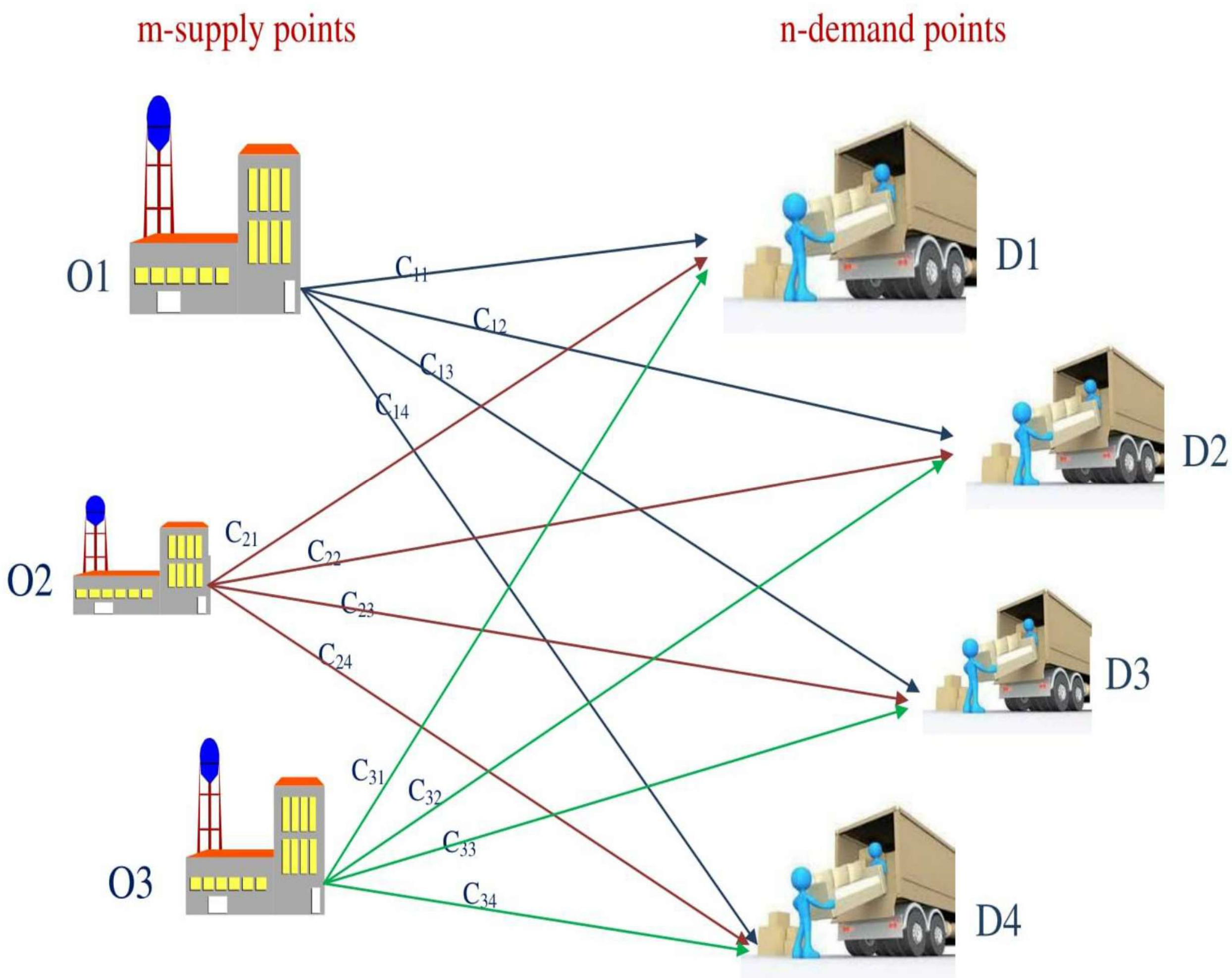


Solution of LP

More and more powerful software available

(CPLEX, Gurobi)
(GLPK, LP-SOLVE)

Planning problems



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Planning problems

$$\min \sum_{i=1}^{i=m} \sum_{j=1}^{j=n} c_{ij} x_{ij}$$

$$\text{s.t.} \quad \sum_{j=1}^{j=n} x_{ij} = s_i \quad (i = 1, 2, \ldots, m) \qquad \text{(Supply constraints)}$$

$$\sum_{i=1}^{i=m} x_{ij} = d_j \quad (j = 1, 2, \ldots, n) \qquad \text{(Demand constraints)}$$

$$x_{ij} \geq 0 \quad (i = 1, 2, \ldots, m; j = 1, 2, \ldots, n)$$

Optimization problems

Mixed integer linear programming

Non-linear programming

Global optimization

Non-convex optimization

Stochastic programming

Robust optimization

Mixed integer linear programming

Fixed costs

Scheduling

Location

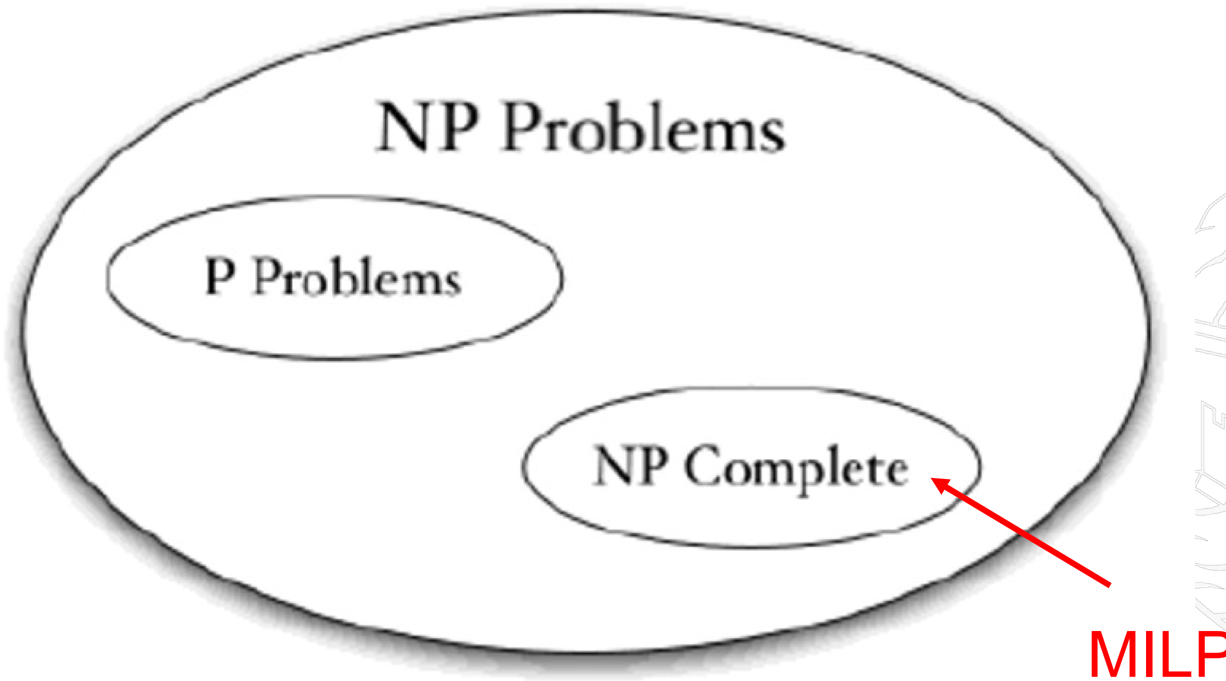
$$\min c'x$$
$$Ax = b$$
$$x \geq 0$$

Integer

Selection

Routing

Mixed integer linear programming



Mixed integer linear programming

Branch-and-bound

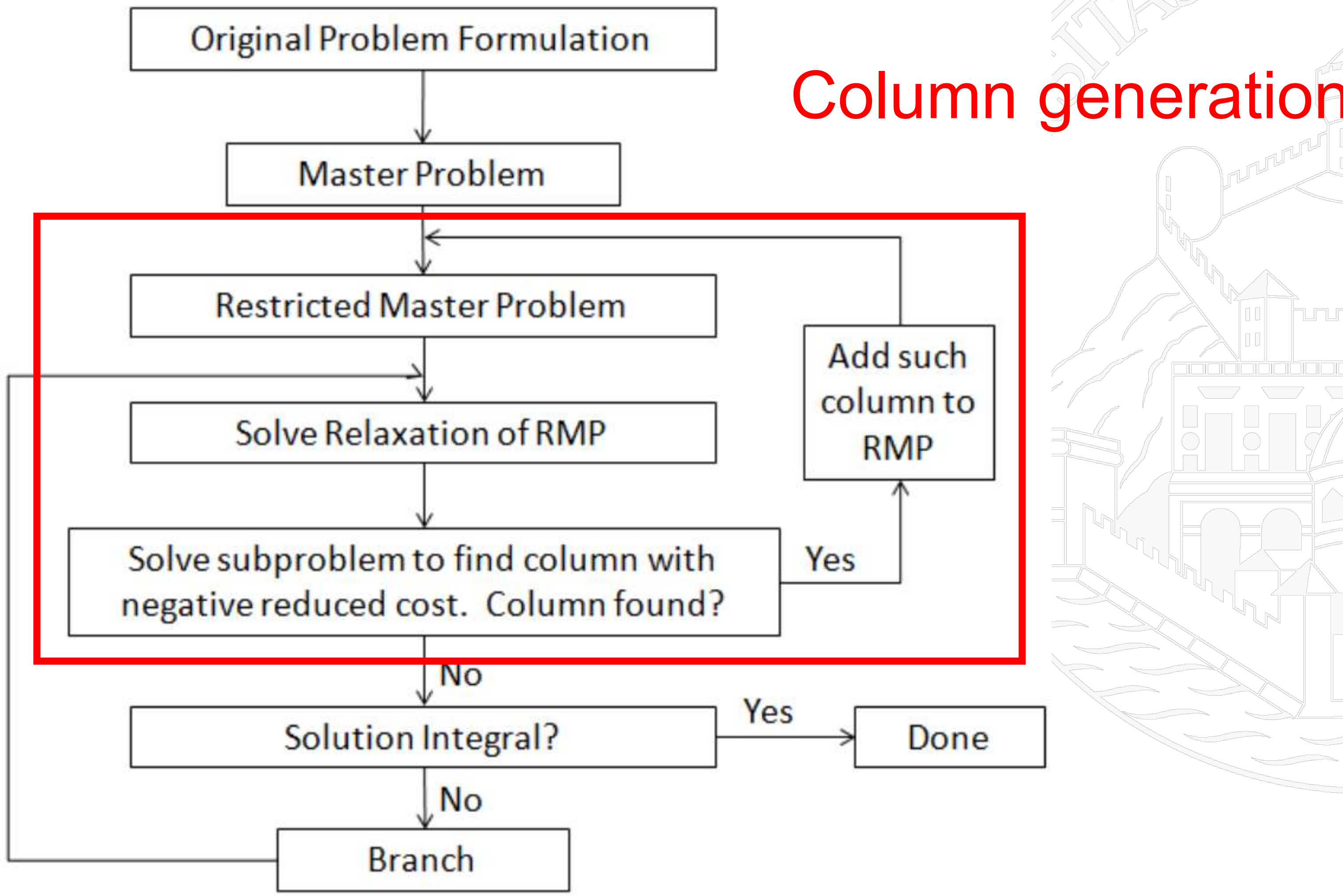
Branch-and-cut

Branch-and-price

$$\min c'x$$
$$Ax = b$$
$$x \geq 0 \;\; binary$$

Exponential number of variables

Branch-and-price



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MILP

More and more powerful software

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Mixed integer linear programming

Fixed costs

Scheduling

Location

Specialized algorithms
(exact and heuristic)

Selection

Routing

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