Basic Traffic Models and Traffic Waves

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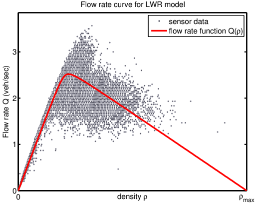
Fundamental Diagram



Ring Road

September 16–17, 2020

Mathematical Challenges and Opportunities for Autonomous Vehicles
Tutorials

Institute for Pure and Applied Mathematics, UCLA

TEMPLE UNIVERSITY

Overview

1. Traffic Flow Theory and Traffic Models
2. Macroscopic Traffic Models
3. Cellular Traffic Models
4. Microscopic Traffic Models

Overview

The Point of Traffic Models

One could study traffic flow purely empirically, i.e., observe and classify what one sees and measures.

So why study (principled) models?

- Reduce system complexity, e.g.: replace different drivers by one effective average driver type, while preserving system behavior.
- Remove/add specific effects (lane switching, vehicle inhomogeneities, road conditions, etc.) $\longrightarrow$ understand which effects play which role.
- Can study effect of model parameters (driver aggressiveness, etc.).
- Can be analyzed theoretically (to a certain extent).
- Can use computational resources to simulate.
- Yield quantitative predictions ($\longrightarrow$ traffic forecasting).
- We actually do not know (exactly) how we drive. Models that reproduce correct emergent phenomena help us understand our driving behavior.

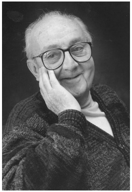
The Point of Traffic Models in the Context of AVs

Why do we need traffic flow modeling in light of AVs?

Because we (as a society) are fundamentally changing the transportation system, by introducing automation and connectivity (and electrification and shared mobility).

To predict the impacts of autonomous vehicles (and prevent the worst pitfalls), we must have a good principled understanding of traffic flow without vehicle automation.

Key message about flow modeling



1) "All models are wrong, but some are useful."
(George Box)

2) Whether a model is useful depends on what is needed in the specific situation.

See Traffic Flow Data Yourself

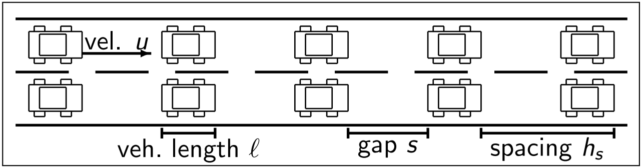
Visualize Real Traffic Data

The seminal NGSIM (Next Generation Simulation) data set:
`https://ops.fhwa.dot.gov/trafficanalysistools/ngsim.htm`
Here Interstate 80 Freeway Dataset near Emeryville, CA.

1. Download `https://www.math.temple.edu/~seibold/NGSIM.zip`
2. Unzip `NGSIM.zip`
3. Open Matlab
4. `>> A = load('trajectories-0500-0515.txt');`
5. `>> animate_ngsim`

Additional files:
`trajectories-0400-0415.txt`
`trajectories-0515-0530.txt`

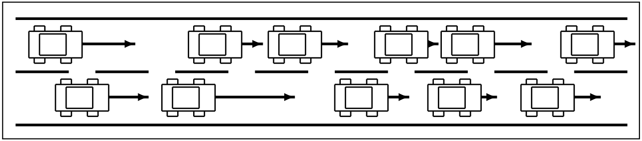
Uniform traffic flow



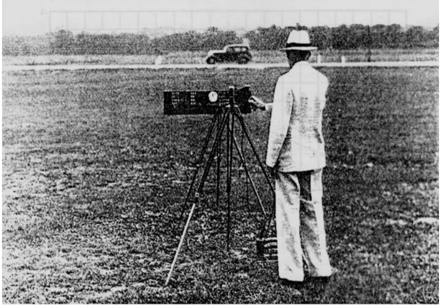
Fundamental quantities

- density ρ: # vehicles per unit length; $\rho_{\max}$: bumper to bumper + safety
- flow rate (throughput) q: # vehicles passing fixed position per time
- velocity u: distance traveled per unit time
- bulk-velocity $u = q/\rho$: correct notion in non-uniform flow
- spacing h_s: road length per vehicle; gap $s = h_s - \ell$

Non-uniform traffic flow

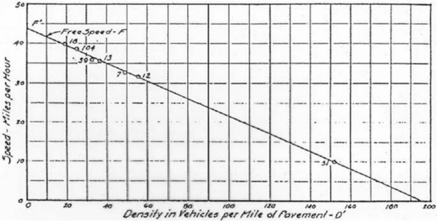


Bruce Greenshields collecting data (1933)



[This was only 25 years after the first Ford Model T (1908)]

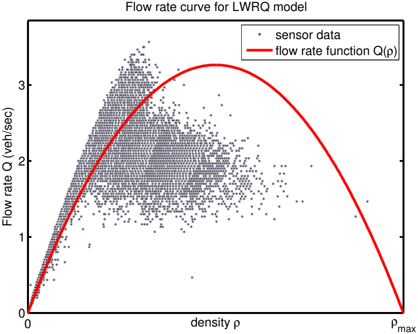
Postulated density–velocity relationship



Deduced relationship

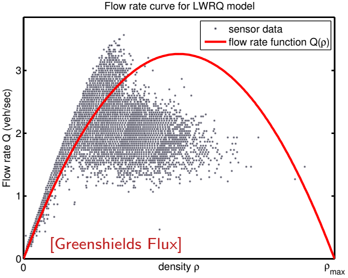
- $u = U(\rho) = u_{\max}(1 - \rho/\rho_{\max})$,
  $\rho_{\max} \approx 195$ veh/mi; $u_{\max} \approx 43$ mi/h
- Flow rate
  $q = Q(\rho) = u_{\max}(\rho - \rho^2/\rho_{\max})$

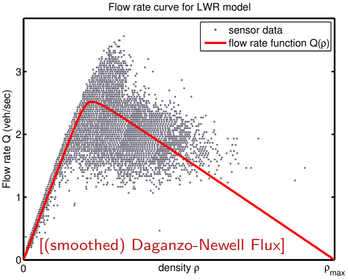
Contemporary measurements (q vs. ρ)



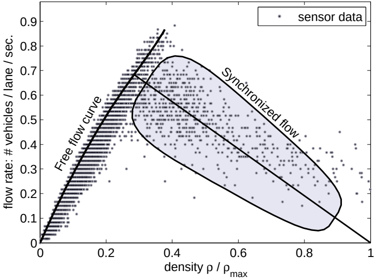
[Fundamental Diagram of Traffic Flow]

Fundamental Diagram (FD) of traffic flow (detector data)





Traffic phase theory (here: 2 phases) [Kerner]



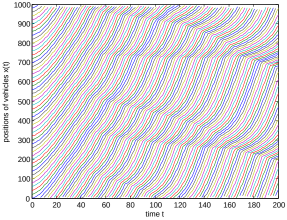
FDs around the world exhibit same features

- for ρ small (free-flow): small spread
- above a critical density (congestion): $Q(\rho)$ decreasing & FD set-valued

Key open question in traffic flow theory: precise phenomenological understanding of spread (role of sensor noise, inhomogeneities, non-equilibrium effects, etc.).

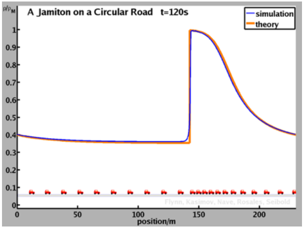
Microscopic Models

$$\ddot{x}_j = G(x_{j+1} - x_j, u_j, u_{j+1})$$

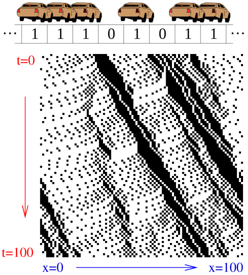


Macroscopic Models

$$\begin{cases} \rho_t + (\rho u)_x = 0 \\ (u+h)_t + u(u+h)_x = \frac{1}{\tau}(U-u) \end{cases}$$



Cellular Models



Idea

Describe behavior of individual vehicles (ODE system).

Micro ⟷ Macro

- macro = limit of micro when #vehicles $\to \infty$
- micro = discretization of macro in Lagrangian variables

Methodology and role

- Describe aggregate/bulk quantities via PDE.
- Natural framework for multiscale phenomena, traveling waves, and shocks.
- Suitable framework to incorporate sparse data [Mobile Millennium Project].

Idea

Cell-to-cell propagation (space-time-discrete).

Cellular ⟷ Macro

- macro = limit of cellular
- cellular = discretization of macro in Eulerian variables

Key Distinction for All Traffic Models

- **First-order dynamics:** System state is vehicle positions (or density). Obtain (instantaneous) vehicle velocities from positions.
- **Second-order dynamics:** System state is vehicle positions and velocities. Model vehicle accelerations (Newton's laws of motion).

- First-order dynamics can produce shock waves (moving upstream end of traffic jam; red/green light dynamics); but ...
- Second-order dynamics needed to produce instabilities and traveling waves (phantom traffic jams). [Or: first-order with delay; not treated here]

Microscopic Models

First-order: $\dot{x}_j = F(x_{j+1} - x_j)$

Second-order:

$$\ddot{x}_j = G(x_{j+1} - x_j, u_j, u_{j+1})$$

Macroscopic Models

First-order: $$\rho_t + (\rho U(\rho))_x = 0$$

Second-order:

$$\begin{cases} \rho_t + (\rho u)_x = 0 \\ (u+h(\rho))_t + u(u+h(\rho))_x = \frac{1}{\tau}(U(\rho)-u) \end{cases}$$

Overview

Macroscopic Traffic Models — Philosophy

Philosophy of macroscopic models

- Equations for macroscopic traffic variables (density, flow rate, etc.)
- Usually lane-aggregated ($\rho(x,t)$), but multi-lane models can also be formulated.
- Natural framework for multiscale phenomena, traveling waves, shocks.
- Established theory of control and coupling conditions for networks.
- Suitable framework to fill gaps in incorporated measurement data.
- Mathematically related with other models, e.g., microscopic models, mesoscopic (kinetic) models, cell transmission models, stochastic models.
- Good for estimation and prediction, and for mathematical analysis of emergent features. Not the best framework if vehicle trajectories are of interest. Also, analysis and numerical methods for PDE are more complicated than for ODE.

Macroscopic Traffic Models — Continuum Description

Continuity equation

Vehicle density $\rho(x,t)$. Number of vehicles in $[a,b]$: $m(t) = \int_a^b \rho(x,t)\mathrm{d}x$

Traffic flow rate (flux): $f = \rho u$

Change of number of vehicles equals inflow $f(a)$ minus outflow $f(b)$:

$$\frac{d}{dt}m(t) = \int_a^b \rho_t \mathrm{d}x = f(a) - f(b) = -\int_a^b f_x \mathrm{d}x$$

Equation holds for any choice of a and b: $\rho_t + (\rho u)_x = 0$

First-order models (Lighthill-Whitham-Richards)

Model: velocity uniquely given by density, $u = U(\rho)$. Yields flux function $f = Q(\rho) = \rho U(\rho)$. Scalar hyperbolic conservation law.

Second-order models (e.g., Payne-Whitham, Aw-Rascle-Zhang)

ρ and u are independent quantities; augment continuity equation by a second equation for velocity field (vehicle acceleration). System of hyperbolic conservation laws.

Lighthill-Whitham-Richard (LWR) Model [Lighthill&Whitham: Proc. Roy. Soc. A 1955]

$$\left.\begin{aligned} \rho_t + (\rho U(\rho))_x &= 0 \\ \Longleftrightarrow \quad \rho_t + Q(\rho)_x &= 0 \end{aligned}\right\} \quad \text{where} \quad Q(\rho) = \rho U(\rho)$$

Model parameter: flow rate function $Q(\rho)$

First order model

Payne-Whitham (PW) Model [Whitham 1974], [Payne: Transp. Res. Rec. 1979]

$$\begin{cases} \rho_t + (\rho u)_x & = 0 \\ u_t + u u_x + \frac{1}{\rho} p(\rho)_x & = \frac{1}{\tau}(U(\rho) - u) \end{cases}$$

Parameters: pressure $p(\rho)$; desired velocity function $U(\rho)$; relaxation time τ

Second order model; vehicle acceleration: $u_t + u u_x = -\frac{p'(\rho)}{\rho}\rho_x + \frac{1}{\tau}(U(\rho) - u)$

Inhomogeneous Aw-Rascle-Zhang (ARZ) Model

[Aw&Rascle: SIAM J. Appl. Math. 2000], [Zhang: Transp. Res. B 2002]

$$\begin{cases} \rho_t + (\rho u)_x & = 0 \\ (u + h(\rho))_t + u(u + h(\rho))_x & = \frac{1}{\tau}(U(\rho) - u) \end{cases}$$

Parameters: hesitation function $h(\rho)$; velocity function $U(\rho)$; time scale τ

Second order model; vehicle acceleration: $u_t + u u_x = \rho h'(\rho) u_x + \frac{1}{\tau}(U(\rho) - u)$

Continuity equation

$$\rho_t + (\rho u)_x = 0$$

One equation, two unknown quantities ρ and u.

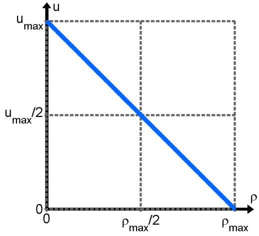
Simplest idea: model velocity u as a function of ρ.

(i) alone on the road $\Rightarrow$ drive with speed limit: $u(0) = u_{\max}$

(ii) bumper to bumper $\Rightarrow$ complete clogging: $u(\rho_{\max}) = 0$

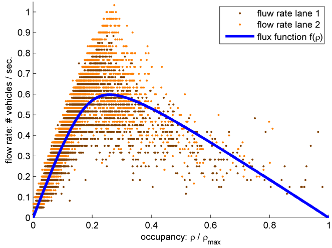
(iii) in between, use linear function: $u(\rho) = u_{\max}\left(1 - \frac{\rho}{\rho_{\max}}\right)$

Lighthill-Whitham-Richards model (1950)



$$f(\rho) = \frac{\rho}{\rho_{\max}}\left(1 - \frac{\rho}{\rho_{\max}}\right)u_{\max}$$

A more realistic $f(\rho)$



Method of characteristics

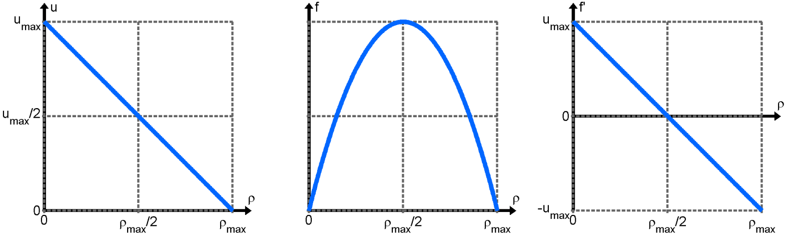
$$\rho_t + (f(\rho))_x = 0$$

Look at solution along a special curve $x(t)$. At this moving observer:

$$\frac{d}{dt}\rho(x(t),t) = \rho_x \dot{x} + \rho_t = \rho_x \dot{x} - (f(\rho))_x = \rho_x \dot{x} - f'(\rho)\rho_x = \left(\dot{x} - f'(\rho)\right)\rho_x$$

If we choose $\dot{x} = f'(\rho)$, then solution (ρ) is constant along the curve.

LWR flux function and information propagation



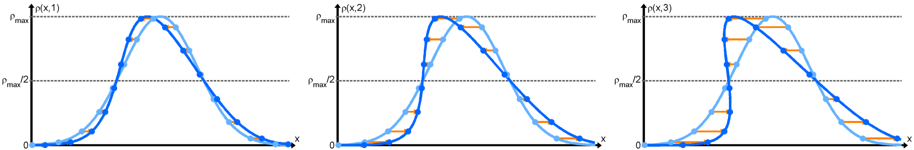
speed of vehicles

speed of information

Solution method

Let the initial traffic density $\rho(x,0) = \rho_0(x)$ be represented by points $(x, \rho_0(x))$. Each point evolves according to the characteristic equations

$$\begin{cases} \dot{x} = f'(\rho) \\ \dot{\rho} = 0 \end{cases}$$



Shocks

The method of characteristics eventually creates breaking waves.
In practice, a shock (= traveling discontinuity) occurs.
Interpretation: Upstream end of a traffic jam.

Note: A shock is a model idealization of a real thin zone of rapid braking.

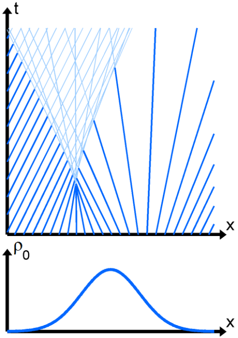
Characteristic form of LWR

LWR model $\rho_t + f(\rho)_x = 0$ (1)

in characteristic form: $\dot{x} = f'(\rho)$, $\dot{\rho} = 0$.

If initial conditions $\rho(x,0) = \rho_0(x)$ smooth (C^1), solution becomes non-smooth at time $t^* = -\frac{1}{\inf_x f''(\rho_0(x))\rho_0'(x)}$.

Reality exists for $t > t^*$, but PDE does not make sense anymore (cannot differentiate discont. function).



Weak solution concept

$\rho(x,t)$ is a weak solution if it satisfies

$$\int_0^\infty \int_{-\infty}^\infty \rho\phi_t + f(\rho)\phi_x \,\mathrm{d}x\mathrm{d}t = -\int_{-\infty}^\infty [\rho\phi]_{t=0}\,\mathrm{d}x \quad \forall \underbrace{\phi \in C_0^1}_{\text{test fct., } C^1 \text{ with compact support}} \qquad (2)$$

Theorem: If $\rho \in C^1$ ("classical solution"), then (1) $\iff$ (2).

Proof: integration by parts.

Weak formulation of LWR

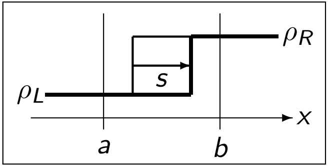
$$\int_0^\infty \int_{-\infty}^\infty \rho\phi_t + f(\rho)\phi_x \,\mathrm{d}x\mathrm{d}t = -\int_{-\infty}^\infty [\rho\phi]_{t=0}\,\mathrm{d}x \quad \forall\,\phi \in C_0^1$$

Every classical (C^1) solution is a weak solution.

In addition, there are discontinuous weak solutions (i.e., with shocks).

Riemann problem (RP)

$$\rho_0(x) = \begin{cases} \rho_L & x < 0 \\ \rho_R & x \geq 0 \end{cases}$$



Speed of shocks

The weak formulation implies that shocks move with a speed such that the number of vehicles is conserved:

RP: $(\rho_L - \rho_R)\cdot s = \frac{\mathrm{d}}{\mathrm{d}t}\int_a^b \rho(x,t)\,\mathrm{d}x = f(\rho_L) - f(\rho_R)$

Yields: $\boxed{s = \frac{f(\rho_R)-f(\rho_L)}{\rho_R-\rho_L} = \frac{[f(\rho)]}{[\rho]}}$ Rankine-Hugoniot condition

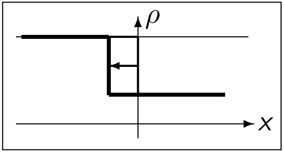
Weak formulation and Rankine-Hugoniot shock condition

$$\int_0^\infty \int_{-\infty}^\infty \rho\phi_t + f(\rho)\phi_x \,\mathrm{d}x\mathrm{d}t = -\int_{-\infty}^\infty [\rho\phi]_{t=0}\,\mathrm{d}x \;\forall\phi\in C_0^1 \quad ; \quad s = \frac{[f(\rho)]}{[\rho]}$$

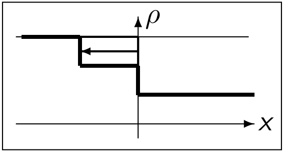
Problem

For RP with $\rho_L > \rho_R$, many weak solutions for same initial conditions.

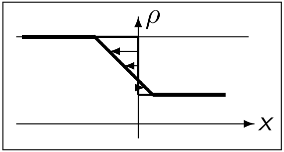
One shock



Two shocks



Rarefaction fan



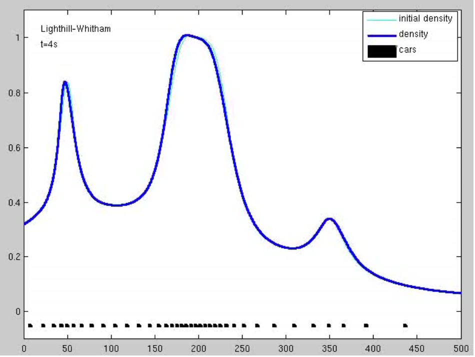
Entropy condition

Single out a unique solution (the dynamically stable one $\longrightarrow$ vanishing viscosity limit) via an extra “entropy” condition:

Characteristics must go **into** shocks, i.e., $f'(\rho_L) > s > f'(\rho_R)$.

For LWR ($f''(\rho) < 0$): shocks must satisfy $\rho_L < \rho_R$.

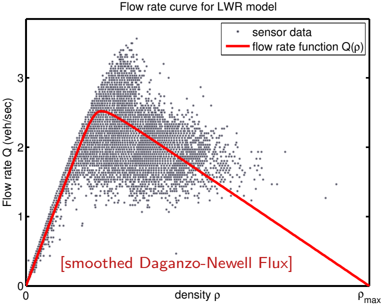
Evolution of Traffic Density for LWR Model



Result

The LWR model quite nicely explains the shape of traffic jams (vehicles run into a shock).

Data-Fitted Flow Rate Curve



Shortcomings of LWR

Cannot explain FD spread.

Cannot explain phantom traffic jams (perturbations never grow due to maximum principle).

Payne-Whitham (PW) Model [Analysis for ARZ Model is Very Similar]

$$\begin{cases} \rho_t + (\rho u)_x &= 0 \\ u_t + u u_x + \frac{1}{\rho} p(\rho)_x &= \frac{1}{\tau}(U(\rho) - u) \end{cases}$$

Mathematical Structure: System of Balance Laws

$$\underbrace{\begin{pmatrix} \rho \\ u \end{pmatrix}_t + \begin{pmatrix} u & \rho \\ \frac{1}{\rho}\frac{dp}{d\rho} & u \end{pmatrix} \cdot \begin{pmatrix} \rho \\ u \end{pmatrix}_x}_{\text{hyperbolic part}} = \underbrace{\begin{pmatrix} 0 \\ \frac{1}{\tau}(U(\rho) - u) \end{pmatrix}}_{\text{relaxation term}}$$

Relaxation to Equilibrium

Formally, we can consider the limit $\tau \to 0$.
In this case: $u = U(\rho)$, i.e., the system reduces to the LWR model.

Important Fact

Solutions of the 2×2 system converge to solutions of LWR, only if a condition is satisfied $\longrightarrow$ next slide. . .

System of Balance Laws (e.g., PW Model)

$$\begin{pmatrix}\rho\\ u\end{pmatrix}_t + \begin{pmatrix}u & \rho\\ \frac{1}{\rho}\frac{dp}{d\rho} & u\end{pmatrix}\begin{pmatrix}\rho\\ u\end{pmatrix}_x = \begin{pmatrix}0\\ \frac{1}{\tau}(U(\rho)-u)\end{pmatrix}$$

Eigenvalues

$$\left\{\begin{array}{l}\lambda_1 = u - c\\ \lambda_2 = u + c\end{array}\right\} \quad c^2 = \frac{dp}{d\rho}$$

Linear Stability Analysis

(LS) When are constant base state solutions $\rho(x,t) = \tilde{\rho}$, $u(x,t) = U(\tilde{\rho})$ stable (i.e. infinitesimal perturbations do not amplify)?

Reduced Equation

(RE) When do solutions of the 2×2 system converge (as $\tau \to 0$) to solutions of the reduced equation

$$\rho_t + (\rho U(\rho))_x = 0 \quad ?$$

Sub-Characteristic Condition

(SCC) $\lambda_1 < \mu < \lambda_2$, where $\mu = (\rho U(\rho))'$

Theorem [Whitham: Comm. Pure Appl. Math 1959]

(LS) $\iff$ (RE) $\iff$ (SCC)

Example: Stability for PW Model

(SCC) $\iff U(\rho) - c(\rho) \leq U(\rho) + \rho U'(\rho) \leq U(\rho) + c(\rho) \iff \frac{c(\rho)}{\rho} \geq -U'(\rho)$.

For $p(\rho) = \frac{\beta}{2}\rho^2$ and $U(\rho) = u_\text{m}\left(1 - \frac{\rho}{\rho_\text{m}}\right)$: stability iff $\rho < \rho_\text{c}$, where $\rho_\text{c} = \frac{\beta\rho_\text{m}^2}{u_\text{m}^2}$.

Phase transition: If enough vehicles on the road, uniform flow is unstable.

PW Model

$$\begin{cases} \rho_t + (\rho u)_x = 0 \\ u_t + u u_x + \frac{1}{\rho} p(\rho)_x = \frac{1}{\tau}(U(\rho) - u) \end{cases}$$

Traveling Wave Ansatz

$\rho = \rho(\eta)$, $u = u(\eta)$, with self-similar variable $\eta = \frac{x - st}{\tau}$.

Then $\rho_t = -\frac{s}{\tau}\rho'$, $\rho_x = \frac{1}{\tau}\rho'$, $u_t = -\frac{s}{\tau}u'$, $u_x = \frac{1}{\tau}u'$

and $p_x = \frac{1}{\tau}c^2\rho'$, $c^2 = \frac{dp}{d\rho}$

Continuity Equation

$$\rho_t + (u\rho)_x = 0$$

$$-\frac{s}{\tau}\rho' + \frac{1}{\tau}(u\rho)' = 0$$

$$(\rho(u - s))' = 0$$

$$\rho = \frac{m}{u - s}$$

$$\rho' = -\frac{\rho}{u - s}u'$$

Momentum Equation

$$u_t + u u_x + \frac{p_x}{\rho} = \frac{1}{\tau}(U - u)$$

$$-\frac{s}{\tau}u' + \frac{1}{\tau}u u' + \frac{dp}{d\rho}\frac{\rho'}{\rho} = \frac{1}{\tau}(U - u)$$

$$(u - s)u' - c^2\frac{1}{u - s}u' = U - u$$

$$u' = \frac{(u - s)(U - u)}{(u - s)^2 - c^2}$$

Jamiton Ordinary Differential Equation for $u(\eta)$

$$u' = \frac{(u-s)(U(\rho)-u)}{(u-s)^2 - c(\rho)^2} \quad \text{where} \quad \rho = \frac{m}{u-s}$$

where

$$s = \text{travel speed of jamiton}$$

$$m = \text{mass flux of vehicles through jamiton}$$

Key Point

In fact, m and s can not be chosen independently:

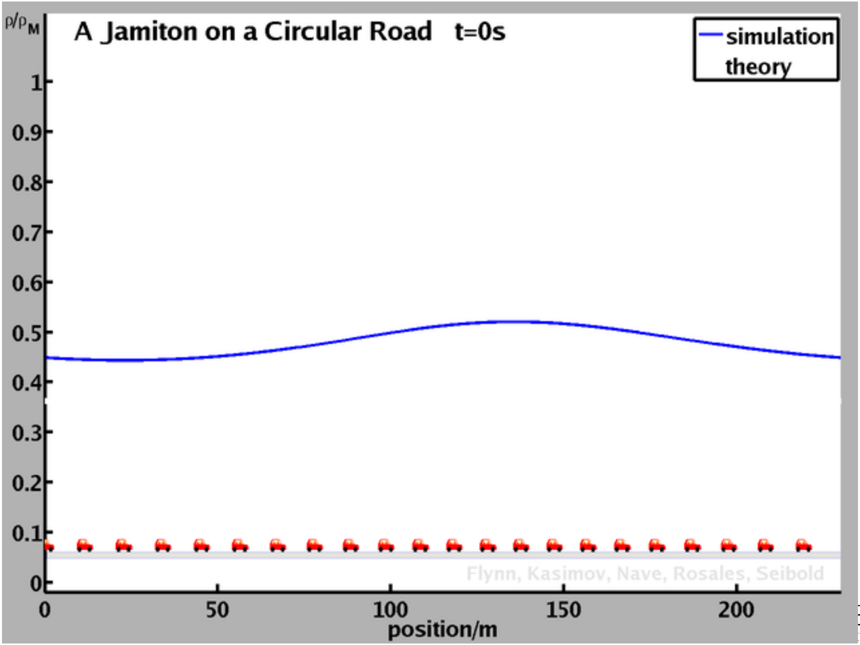
Denominator has root at $u = s + c$. Solution can only pass smoothly through this singularity (the sonic point), if $u = s + c$ implies $U = u$.

Using $u = s + \frac{m}{\rho}$, we obtain for this sonic density ρ_S that:

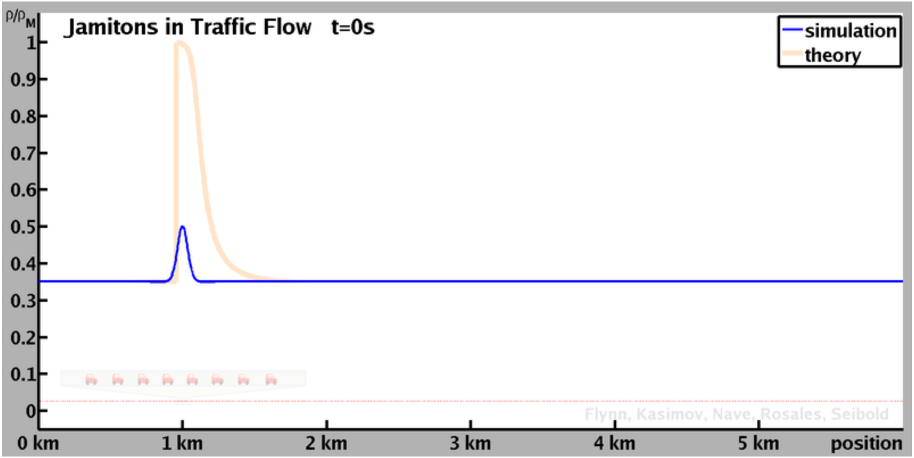
$$\begin{cases} \text{Denominator} & s + \frac{m}{\rho_S} = s + c(\rho_S) \implies m = \rho_S c(\rho_S) \\ \text{Numerator} & s + \frac{m}{\rho_S} = U(\rho_S) \implies s = U(\rho_S) - c(\rho_S) \end{cases}$$

Algebraic condition (Chapman-Jouguet condition [Chapman, Jouguet (1890)]) that relates m and s (and ρ_S). Jamitons described by ZND detonation theory.

Experiment: Jamitons on circular road [Sugiyama et al.: New J. of Physics 2008]


A Jamiton on a Circular Road t=0s
ρ/ρ_M
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
50
100
150
200
position/m
simulation
theory
Flynn, Kasimov, Nave, Rosales, Seibold

Infinite road; lead jamiton gives birth to a chain of "jamitinos".



Important practical lesson: traffic waves can arise as properties of the flow; no bad drivers needed to cause them.

Jamiton Fundamental Diagram

For each sonic density ρ_S that violates the SCC: construct maximal jamiton.
$\rightsquigarrow$ Line segment in FD.
Jamitons can explain spread in real FD.

Emulating Detector Data

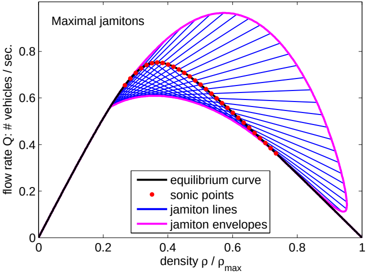
At fixed position, calculate all possible temporal averages of jamiton profiles.

Resulting aggregated jamiton FD is a subset of the maximal jamiton FD.

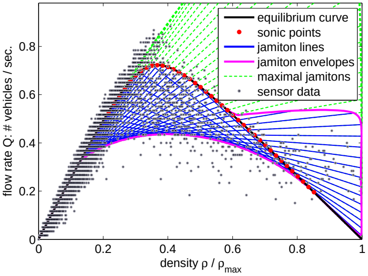
Good Agreement With Dectector Data

We can reverse-engineer model parameters, such that the aggregated jamiton FD shows a good qualitative agreement with sensor data.

Jamiton Fundamental Diagram



Sensor Data with Jamiton FD

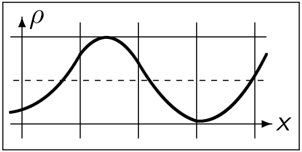


Overview

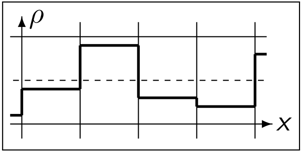
Back to LWR

$$\rho_t + (f(\rho))_x = 0$$

Initial condition



Cell-averaged initial cond.

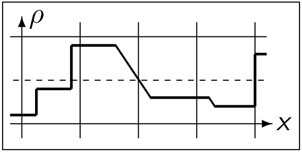


Godunov's method

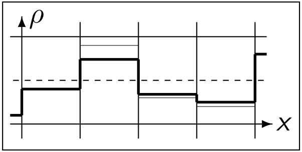
REA = reconstruct–evolve–average

1. Divide road into cells of width h.
2. On each cell, store the average density ρ_j.
3. Assume solution is constant in each cell.
4. Evolve this piecewise constant solution exactly from t to $t + \Delta t$.
5. Average over each cell to obtain a pw-const. sol. again.
6. Go to step 4.

Solution evolved exactly



Cell-averaged evolved solution



Godunov's method

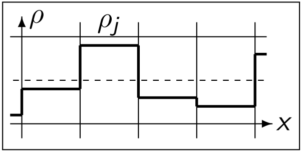
4. Evolve pw-const. sol. exactly from t to $t+\Delta t$.
5. Average over each cell.
6. Go to step 4.

Key Points

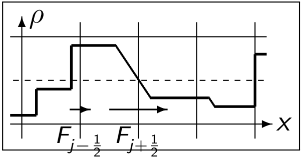
- If we choose $\Delta t < \frac{h}{2\max|f'|}$ ("CFL condition"), waves starting at neighboring cell interfaces never interact. Thus, can be solved as local Riemann problems.
- Because the exactly evolved solution is averaged again, all that matters for the change $\rho_j(t) \longrightarrow \rho_j(t+\Delta t)$ are the fluxes through the cell boundaries:

$$\rho_j(t+\Delta t) = \rho_j(t) + \frac{\Delta t}{h}(F_{j-\frac{1}{2}} - F_{j+\frac{1}{2}})$$

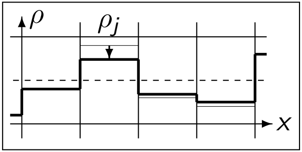
Solution at time t



Solution evolved exactly



Solution at time $t+\Delta t$

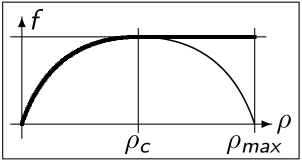


Godunov's method

$$\rho_j(t+\Delta t) = \rho_j(t) + \frac{\Delta t}{h}(F_{j-\frac{1}{2}} - F_{j+\frac{1}{2}})$$

- right-going shock or raref.: $F_{j+\frac{1}{2}} = f(\rho_j)$
- left-going shock or raref.: $F_{j+\frac{1}{2}} = f(\rho_{j+1})$
- transsonic rarefaction: $F_{j+\frac{1}{2}} = f(\rho_c)$

Demand function



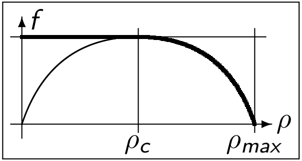
Equivalent formulation of fluxes ⟶ CTM

$F_{j+\frac{1}{2}} = \min\{D(\rho_j), S(\rho_{j+1})\}$

is the maximal flux that exceeds neither the

- demand $D(\rho) = f(\min(\rho, \rho_c))$, nor the
- supply $S(\rho) = f(\max(\rho, \rho_c))$.

Supply function

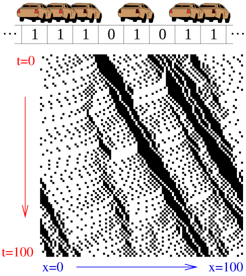


Generalizations

- Same concept for network coupling conditions.
- Principles generalize to (certain) second-order traffic models.

Cellular Automata

Concept



Nagel-Schreckenberg model

en.wikipedia.org/wiki/Nagel-Schreckenberg_model

1. Acceleration: Increase velocity by 1, up to a given maximum speed.
2. Slowing down: Reduce velocity to number of empty cells ahead (if necessary), to avoid collision.
3. Randomization: With probability p, reduce vehicle velocity by 1, not below 0.
4. Car motion: Move cars forward as many cells as their velocity is.

- Ease of simulation and parallelization
- PDE models as macroscopic limits

Simulation Code

- https://www.math.temple.edu/~seibold/teaching/2018_2100
- temple_abm_traffic_cellular.m

Overview

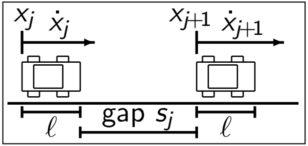
Microscopic Traffic Models — Philosophy

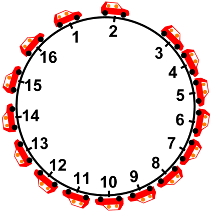
Philosophy of microscopic models

- Compute trajectories of each vehicle.
- Natural to extent to multiple lanes (lane switching model), different vehicle types, etc.
- At the core of most micro-simulators (e.g., Aimsun (Gipps' model); Vissim (Wiedemann model); SUMO (Krauss model)); usually with a discrete time-step and fail-safes.
- Many other car-following models, e.g., the intelligent driver model.
- May have many parameters, in particular free parameters that cannot be measured directly. Hence, calibration required.
- Natural to add randomness. Generally, ensembles of computations must be run.
- Good for off-line simulation ("How would a lane-closure affect this highway section?").

Microscopic Car-Following Models

- Vehicles at positions $x_1 < \ldots < x_N$.
- Car-following: car j affected only by $j+1$.
- Types of arrangements:
  a) Infinite road with one vehicle leading.
  b) Ring road (N follows 1): proxy for infinite road.





Possible Model Dynamics

- First order with delay: $\dot{x}_j(t+\tau) = V(s_j(t))$ with gap $s_j = x_{j+1} - x_j - \ell$.
- Second order: $\ddot{x}_j = f(s_j, \dot{s}_j, v_j)$. Here $\dot{s}_j = \dot{x}_{j+1} - \dot{x}_j$ velocity difference.

Perturbations to Uniform Flow

- Equilibrium: vehicles equi-spaced with identical velocities v^{eq}.
- Linearize: $x_j = x_j^{\text{eq}} + y_j$, where y_j infinitesimal perturbation.

Car-Following: String Stability

Linearized Dynamics

- First order: $\dot{y}_j(t+\tau) = V'(s^{\text{eq}})(y_{j+1}(t) - y_j(t))$.
- Second order: $\ddot{y}_j = \alpha_1\,(y_{j+1} - y_j) - \alpha_2\,\dot{y}_j + \alpha_3\,\dot{y}_{j+1}$,
  where $\alpha_1 = \frac{\partial f}{\partial s}$, $\alpha_2 = \frac{\partial f}{\partial \dot{s}} - \frac{\partial f}{\partial v}$, $\alpha_3 = \frac{\partial f}{\partial \dot{s}}$ (all eval. at equilibrium).

Frequency Response of Car-Following I/O Behavior

- Laplace transform ansatz $y_j(t) = c_j e^{\omega t}$, where $c_j, \omega \in \mathbb{C}$.
- Yields I/O system: $c_j = F(\omega) c_{j+1}$ with transfer function

$$F(\omega) = \left(1 + \tfrac{1}{V'(s^{\text{eq}})}\omega e^{\omega\tau}\right)^{-1} \quad \text{resp.} \quad F(\omega) = \frac{\alpha_1 + \alpha_3\,\omega}{\alpha_1 + \alpha_2\,\omega + \omega^2}\,.$$

- $\text{Re}(\omega)$: temporal growth/decay $\qquad |F|$: growth/decay across vehicles
  $\text{Im}(\omega)$: frequency of oscillation $\qquad \theta(F)$: phase shift across vehicles
- **Def.:** string stability means $|F(\omega)| \leq 1\ \forall \omega \in i\mathbb{R}$.
- The models above are string stable exactly if

$$2\tau V'(s^{\text{eq}}) \leq 1 \qquad \text{resp.} \qquad \alpha_2^2 - \alpha_3^2 - 2\alpha_1 \geq 0\,.$$

Two-Species Car-Following (Humans and AVs)

- Slightly unstable human driver model, i.e. $\alpha_2^2 - \alpha_3^2 - 2\alpha_1 < 0$.
- What changes when a few automated vehicles are added to the flow? (that drive slightly differently than humans)
  Can the few AVs stabilize traffic flow, and thus prevent traffic waves?
- Humans: $\ddot{x}_j = f(h_j, \dot{h}_j, v_j)$; AVs: $\ddot{x}_j = g(h_j, \dot{h}_j, v_j)$.
- Let AVs leave same equilibrium spacing as humans. Linearize.
- Humans: $\ddot{y}_j = \alpha_1\,(y_{j+1} - y_j) - \alpha_2\,u_j + \alpha_3\,u_{j+1}$
  AVs: $\ddot{y}_j = \beta_1\,(y_{j+1} - y_j) - \beta_2\,u_j + \beta_3\,u_{j+1}$
- Transfer functions: $F(\omega) = \frac{\alpha_1+\alpha_3\,\omega}{\alpha_1+\alpha_2\,\omega+\omega^2}$ and $G(\omega) = \frac{\beta_1+\beta_3\,\omega}{\beta_1+\beta_2\,\omega+\omega^2}$.
- Stability criterion with AV penetration rate γ:

$$\boxed{|F(\omega)|^{1-\gamma} \cdot |G(\omega)|^{\gamma} \leq 1\ \forall \omega \in i\mathbb{R}}$$

- Problem with this result: It states that any number of human-driven vehicles can be stabilized with any number of AVs, and any spatial arrangement. That cannot be true in reality.

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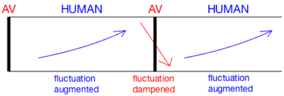
Resolution of Modeling Problem

- Linear stability only captures $t \to \infty$ behavior.
- For transient t, a small perturbation may produce a large deviation.
- Instability of human driving: perturbations grow from car to car.
- Stability of coupled system: AV(s) reduce(s) perturbation by more than amplification caused by all humans.
- Just before hitting the AV, perturbation could be amplified a lot.
- System with noise yields needed failure to remain close to equilibrium:

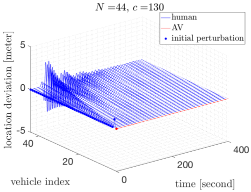
$$\mathrm{d}u_j = [\alpha_1(y_{j+1} - y_j) - \alpha_2 u_j + \alpha_3 u_{j+1}]\mathrm{d}t + s_j \mathrm{d}B_t$$

Amplification and decay of perturbation

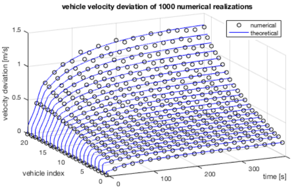


Max. possible: 1 AV stabilizes $\approx$ 25 humans.

System response to single perturbation



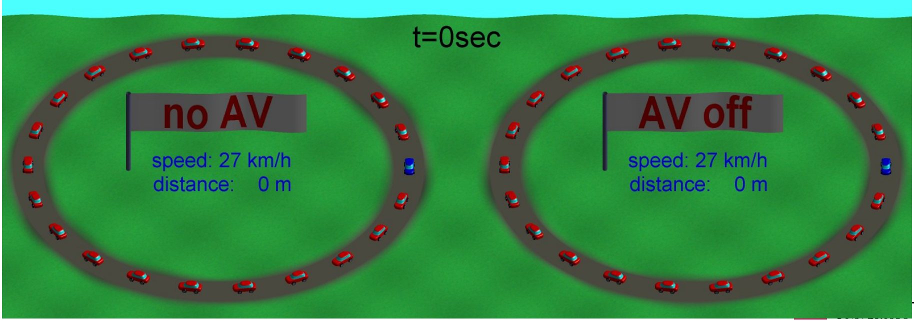
With noise: system's mean deviation



Flow Smoothing via a Sparse Autonomous Vehicles (AVs)

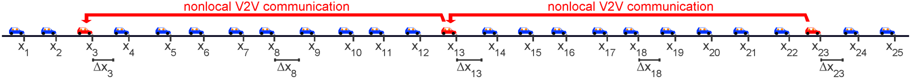
- Traditional highway traffic controls (ramp metering, variable speed limits) lack resolution to dissipate traffic waves. Use AVs.
- Ring road of N vehicles with a single AV; proxy for long road with AV penetration rate $1/N$.
- AV control law: local (safety) + global (smart avg. speed).

Simulation: uncontrolled vs. AV-controlled traffic flow



Some Key Modeling Extensions

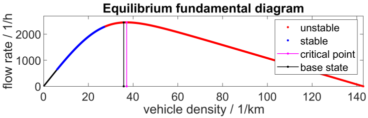
- Many different drivers/vehicles, up to: $\ddot{x}_j = f_j(h_j, \dot{h}_j, v_j))$.
- Space-dependent driving laws (e.g., road features, speed limits).
- Multiple lanes (lane-switching models); ramps, intersections, etc.
- **Connected Automated Vehicles (CAVs):** Non-local effects: $\ddot{x}_j = g(v_j, h_j, \dot{h}_j, h_{j+1}, \dot{h}_{j+1}, \dots)$.



- Vehicle-to-Infrastructure communication.

Boundary Conditions

- Need to spawn/remove cars at inflows/outflows.
- Must adhere to macroscopic laws:



- Can only prescribe inflow state (ρ_L, q_L) at $x = 0$ if $s = \frac{q(0) - q_L}{\rho(0) - \rho_L} > 0$.
- Must prescribe condition at outflow if analogously $s < 0$ there.

Simulation Codes

- https://www.math.temple.edu/~seibold/teaching/2018_2100
  - Follow-the-leader model: $\ddot{x}_j = \dot{h}_j / h_j$
    temple_abm_traffic_follow_the_leader.m
  - optimal velocity model: $\ddot{x}_j = \frac{1}{\tau}(V(h_j) - \dot{x}_j)$
    temple_abm_traffic_car_following.m
- Simple traffic simulator (highway):
  https://www.traffic-simulation.de
- Simple traffic simulator (urban):
  http://volkhin.com/RoadTrafficSimulator

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