IPAM Tutorials 2020 on Autonomous Vehicles

Macroscopic models for Autonomous Vehicles

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Outline of the talk

Traffic flow modeling: approaches

Microscopic

- individual agents
- ODE system
- many parameters
- low and high densities
- comp. cost $\sim$ number of agents

@fotolia - MurielleB

Macroscopic

- continuous flow
- PDEs
- few parameters
- high densities
- analytical theory
- comp. cost $\sim$ domain size

@fotolia - VRD

Macroscopic variables

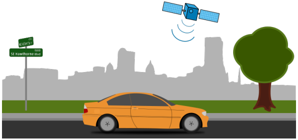
Along a road (network) we need to measure aggregate quantities:

- the (mean) traffic density ρ: number of vehicles per unit space
- the (mean) velocity v: distance covered by vehicles per unit time
- the traffic flow $q = \rho v$: number of vehicles per unit time

Data sources:

@Wikipedia

magnetic loop detectors, video recordings, floating car data, etc



Loop detector data

Example:

355#M3E;	2;	255;	Ma;	01/09/15;	00:00;	1240;	C;	109;	10;
355#M3E;	2;	255;	Ma;	01/09/15;	00:00;	1407;	D;	96;	54;
305#M3e;	2;	255;	Ma;	01/09/15;	00:00;	1449;	D;	100;	10;
709#M7i;	264;	687;	Ma;	01/09/15;	00:00;	1874;	D;	104;	37;
709#M7i;	264;	687;	Ma;	01/09/15;	00:00;	2248;	D;	83;	36;
709#M7i;	264;	687;	Ma;	01/09/15;	00:00;	2368;	D;	78;	39;
577#M6C;	16;	474;	Ma;	01/09/15;	00:00;	4237;	D;	120;	40;
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

where:

loop code
road milestone
distance from the milestone (m)
day of the week
date
time (hours:minutes)
seconds and hundredth
lane
vehicle speed (km/h)
vehicle length (dm)

Floating Car Data

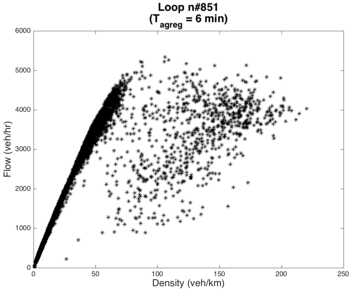
Example:

AVE34e...	2013-03-19T06:09:32	110	43.5452	6.94092	38	A8-Mandelieu-StIsidore	1
COY8c6...	2013-03-19T07:06:58	128	43.5454	6.94102	36	A8-Mandelieu-StIsidore	1
COYf79...	2013-03-19T07:09:20	121	43.5449	6.94064	36	A8-Mandelieu-StIsidore	1
COY3cd...	2013-03-19T07:10:50	124	43.5446	6.94014	36	A8-Mandelieu-StIsidore	1
AVE61d...	2013-03-19T07:16:30	123	43.5448	6.94056	39	A8-Mandelieu-StIsidore	1
AVE61d	2013-03-19T07:16:33	125	43.5455	6.94135	36	A8-Mandelieu-StIsidore	1
COYbdb	2013-03-19T07:23:30	124	43.5448	6.94045	36	A8-Mandelieu-StIsidore	1
COY2ba	2013-03-19T07:23:44	105	43.5458	6.94154	36	A8-Mandelieu-StIsidore	1
⋮	⋮	⋮	⋮			⋮	

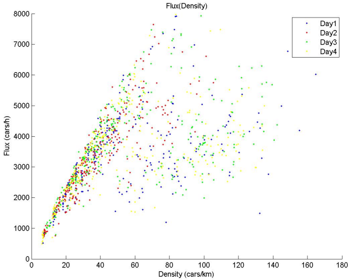
where:

device code
date and time
speed
GPS position and orientation
other location info

Typical fundamental diagram



loops



CFD

Macroscopic traffic flow models

$$\Big[\text{number of vehicles in } [a,b] \text{ at time } t\Big] = \int_a^b \rho(t,x)\, dx$$

must be conserved!

$$\int_a^b \rho(t_2,x)dx = \int_a^b \rho(t_1,x)dx + \int_{t_1}^{t_2} q(t,a)dt - \int_{t_1}^{t_2} q(t,b)dt$$

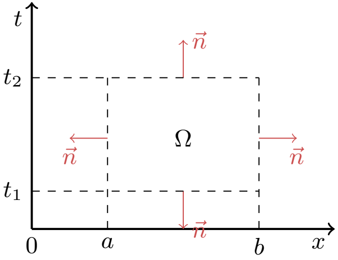
$$\Downarrow$$

divergence theorem

$$\int_\Omega \operatorname{div}_{(t,x)}(\rho,q) = \int_{\partial\Omega} (\rho,q)\cdot \vec{n}$$

$$\Downarrow$$

$$\int_{t_1}^{t_2}\int_a^b (\partial_t\rho + \partial_x q)\;dx\;dt = 0$$



conservation law

Basic requirements

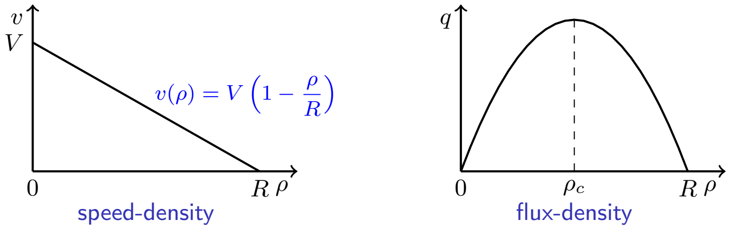
$$\partial_t \rho(t,x) + \partial_x q(t,x) = 0$$

- No information propagates faster than vehicles (anisotropy)
- Fundamental relation: $q(t,x) = \rho(t,x)v(t,x)$.
- Density and mean velocity must be non-negative and bounded: $0 \leq \rho(t,x),\ v(t,x) < +\infty,\ \forall x, t > 0$.
- Different from fluid dynamics:
  - preferred direction
  - no conservation of momentum / energy
  - no viscosity
  - continuum assumption?

The Lighthill-Whitham-Richards (LWR) model (mid 50s)

- Mass conservation equation: $\partial_t \rho + \partial_x(\rho v) = 0$
- Phenomenological speed-density relation: $v(t,x) = v(\rho(t,x))$

fundamental diagram



V maximal speed, R maximal or *jam* density, ρ_c critical density:

- flux is increasing for $\rho \leq \rho_c$: free-flow phase
- flux is decreasing for $\rho \geq \rho_c$: congestion phase

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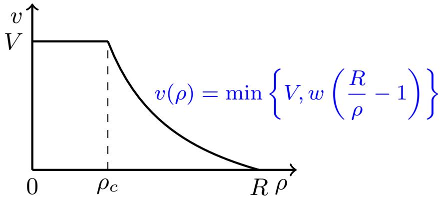
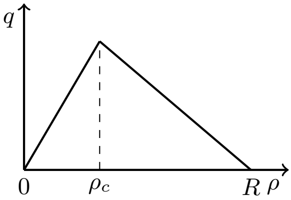
fundamental diagram



speed-density



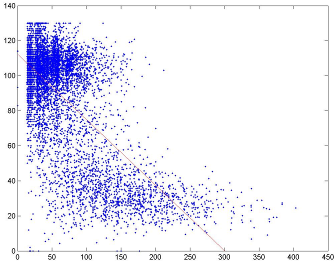
flow-density

V maximal speed, R maximal or *jam* density, ρ_c critical density:

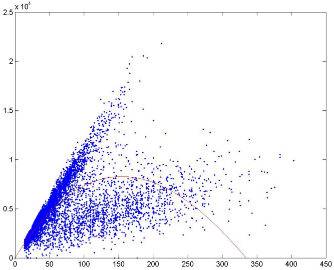
- flux is increasing for $\rho \leq \rho_c$: free-flow phase
- flux is decreasing for $\rho \geq \rho_c$: congestion phase

$v = v(\rho)$?

Experimental data:



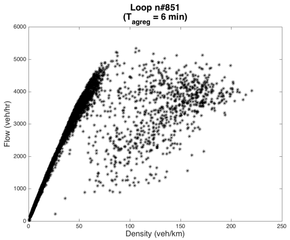
speed-density



flow-density

A8 (Antibes-Nice), March 2013

Improved models



- **Additional equation for v: second order models**
  [Payne (1971); Helbing (1996); Aw-Rascle (200 Zhang (2002); Colombo (2002); Lebacque-et-al (2005); ...]

- **Phase transition**
  [Colombo (2002); Goatin (2006); Blandin-et-al (2011); ...]

- **Stochastic and averaged models**
  [Boel-Mihaylova (2006); Wang-Ni-Chen-Li (2010); Chen-Wang (2011); Sumalee&al (2011); Jabari-Liu (2011); ...]

- **Multi-population**
  [Wong-Wong (2002); Benzoni-Colombo (2003); VanLint-Hoogendorn-Schreurer (2008); Nair-Mahmassani-Hooks (2011); Fan-Work (2015); Gashaw-Goatin-Harri (2016); ...]

- **Non-local models**
  [Blandin-Goatin (2016); Friedrich-Kolb-Göttlich (2018)]

Outline of the talk

1. Introduction to macroscopic models

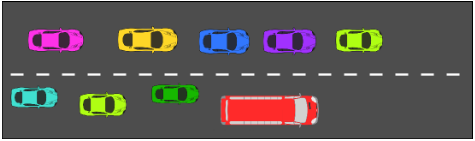
2. Macroscopic models for Autonomous Vehicles
   - PDE-ODE models
   - Non-local models

Moving bottleneck[1]

PDE-ODE model for interaction between a vehicle and surrounding traffic

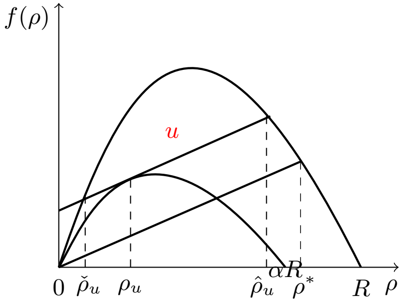
$$\begin{cases} \partial_t \rho + \partial_x(\rho\, v(\rho)) = 0, & x \in \mathbb{R}, t > 0, \\ \rho(0,x) = \rho_0(x), & x \in \mathbb{R}, \\ f(\rho(t,y(t))) - \dot{y}(t)\rho(t,y(t)) \leq F_\alpha(\dot{y}(t)), & t > 0, \\ \dot{y}(t) = \min\{u(t), v(\rho(t,y(t)+))\}, & t > 0, \\ y(0) = y_0. \end{cases}$$

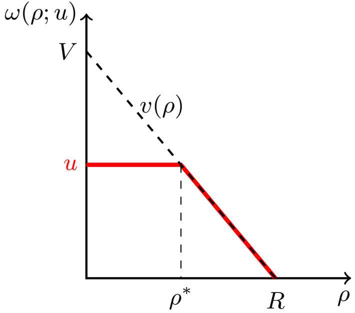
where $y = y(t) \in \mathbb{R}$ vehicle trajectory
$\alpha \in [0,1]$ road capacity reduction rate
$u(t) \in [0,V]$ AV desired velocity (control)



[1] [Lebacque-Lesort-Giorgi, TRR, 1998; DelleMonache-Goatin, JDE, 2014; Garavello-Goatin-Piccoli-Liard, JDE, 2020]

The model[2]





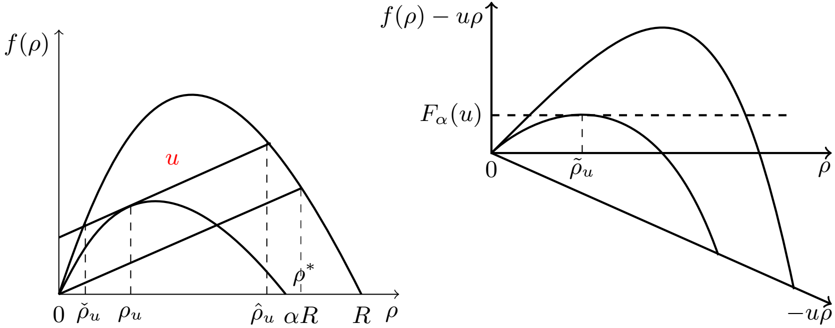
- $v(\rho) = V(1 - \frac{\rho}{R})$ mean traffic velocity
- $f(\rho) = \rho v(\rho)$ flux function, strictly concave
- $\omega(\rho; u) = \begin{cases} u & \text{if } \rho \leq \rho^* \doteq R(1 - u/V) \\ v(\rho) & \text{if } \rho > \rho^* \end{cases}$ AV speed

[2] [DelleMonache-Goatin, JDE, 2014]

The Riemann Problem

In the vehicle reference frame[3] $X = x - ut$ (u constant)

$$\begin{cases} \partial_t \rho + \partial_X \left(f(\rho) - u\rho \right) = 0, \\ \rho(0,x) = \begin{cases} \rho_L & \text{if } X < 0, \\ \rho_R & \text{if } X > 0, \end{cases} \\ f(\rho(t,0)) - u\rho(t,0) \leq f_\alpha(\rho_u) - u\rho_u = \dfrac{\alpha R}{4V}(V-u)^2 \doteq F_\alpha(u) \end{cases}$$



[3] [Colombo-Goatin, JDE, 2007]

The Riemann Problem

Definition (DelleMonache-Goatin, JDE 2014)

- If $f(\mathcal{R}(\rho_L,\rho_R)(u)) \leq F_\alpha + u\mathcal{R}(\rho_L,\rho_R)(u)$, then

$$\mathcal{R}^\alpha(\rho_L,\rho_R) = \mathcal{R}(\rho_L,\rho_R) \quad \text{and} \quad y(t) = \omega(\rho;u)t.$$

- If $f(\mathcal{R}(\rho_L,\rho_R)(u)) > F_\alpha + u\mathcal{R}(\rho_L,\rho_R)(u)$, then

$$\mathcal{R}^\alpha(\rho_L,\rho_R)(x) = \begin{cases} \mathcal{R}(\rho_L,\hat{\rho}_u) & \text{if } x < ut, \\ \mathcal{R}(\check{\rho}_u,\rho_R) & \text{if } x \geq ut, \end{cases} \quad \text{and} \quad y(t) = ut.$$

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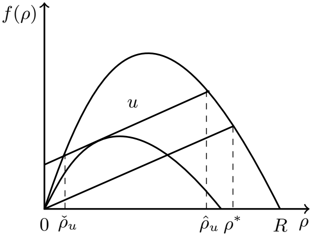
$\Longrightarrow$ $(\hat{\rho}_u, \check{\rho}_u)$ **non-classical shock** traveling with speed u

The Riemann Problem: example



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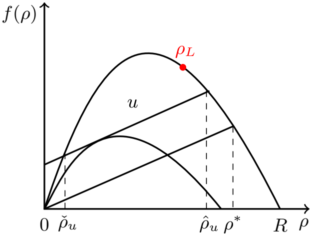
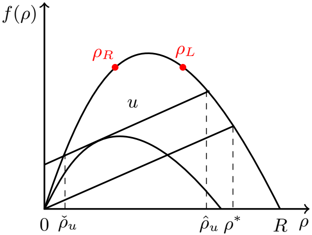


The Riemann Problem: example



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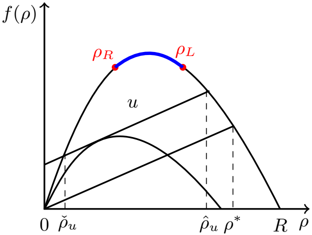

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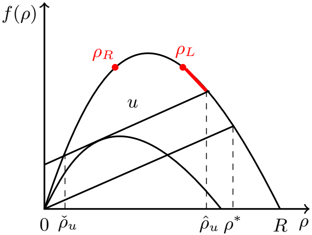

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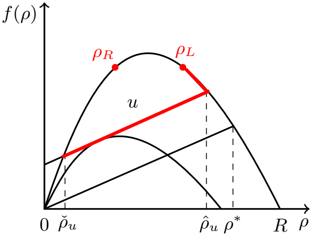

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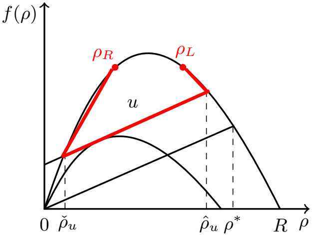

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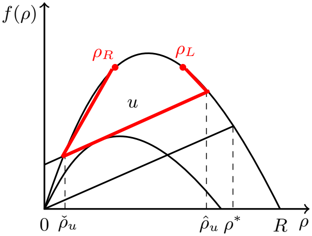

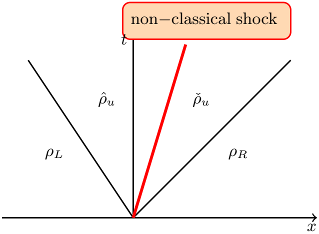

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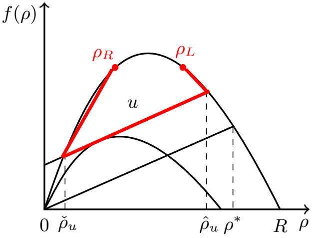
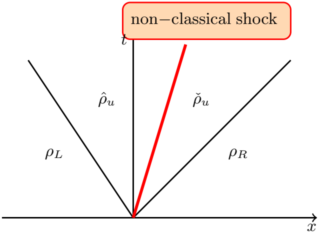


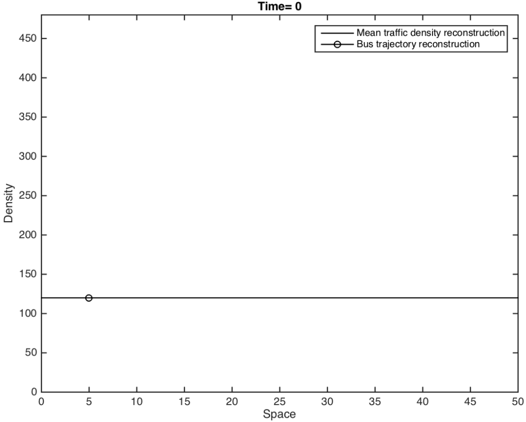


The non-classical shock $(\hat{\rho}_u, \check{\rho}_u)$ satisfies the Rankine-Hugoniot condition but violates the Lax entropy condition

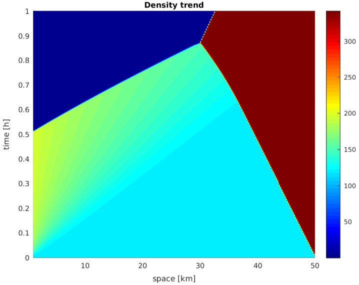
$$f'(\hat{\rho}_u) < u < f'(\check{\rho}_u)$$

Example

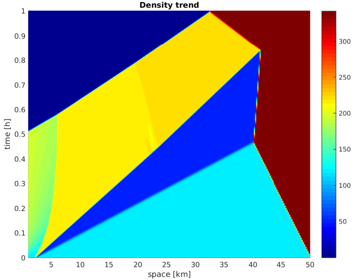
$V = 140$ km/h, $u = 80$ km/h, $R = 400$ vehicles/km, $\alpha = 0.6$:



Optimal control (MPC) [Piacentini-Goatin-Ferrara, IFAC CTS, 2018]



without control



with control

	ATT [h]	jam [km]	TFC [liters]	TFC reduction %
Uncontrolled	0.9107	10.18	$2.7413 \cdot 10^4$	0
Controlled	0.8579	7.66	$2.6852 \cdot 10^4$	2.05

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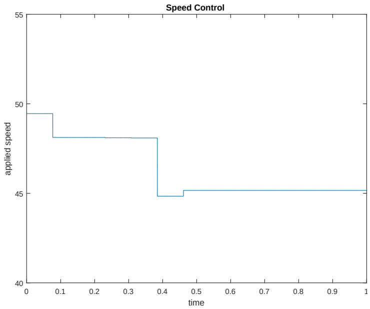
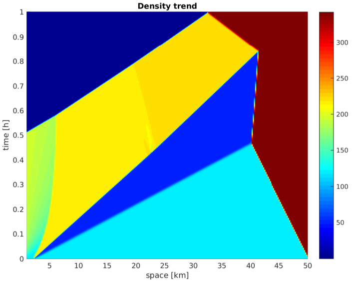


speed control law



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Model Predictive Control (MPC)

- Time $t = 0$: a prediction of the future evolution of the system is done over the prediction horizon $\Delta\tau = 15\text{min}$
- The **cost** is evaluated on the prediction as a function of the control variable and minimized wrt the controlled vehicle speed u
- The optimal value is applied for a time interval $\Delta t = 5\text{min} < \Delta\tau$
- At $t + \Delta t$ another prediction is done by considering the new situation and so on.

Some references

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- C. Chalons, M.L. Delle Monache, P. Goatin. A conservative scheme for non-classical solutions to a strongly coupled PDE-ODE problem, Interfaces and Free Boundaries (2017).
- M. Garavello, P. Goatin, T. Liard, B. Piccoli. A multiscale model for traffic regulation via autonomous vehicles, JDE, to appear.
- C. Lattanzio, A. Maurizi, B. Piccoli. Moving bottlenecks in car traffic flow: A PDE-ODE coupled model, SIAM J. Math. Analysis (2011).
- G. Piacentini, P. Goatin, A. Ferrara. A macroscopic model for platooning in highway traffic, SIAM J. Appl. Math. (2020).
- S. Villa, P. Goatin, C. Chalons. Moving bottlenecks for the Aw-Rascle-Zhang traffic flow model, DCDS B (2017).

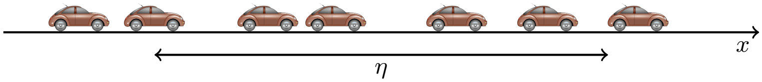
A model with non-local velocity[4]

LWR model with downstream non-local velocity

$$\partial_t \rho(t,x) + \partial_x \left(\rho(t,x) V(t,x)\right) = 0$$

where

$$V(t,x) = v\left(\int_x^{x+\eta} \rho(t,y) w_\eta(y-x)\, dy\right), \quad \eta > 0$$



with $w_\eta \in \mathbf{C^1}([0,\eta];\mathbb{R}^+)$ $w_\eta' \leq 0$ and $\int_0^\eta w_\eta(x)dx = 1$

$v : [0,\rho_{\max}] \to \mathbb{R}^+$ s.t. $-A \leq v' \leq 0$, $v(0) = v_{\max}$, $v(\rho_{\max}) = v_{\min}$

[4] [Blandin-Goatin, NumMath 2016; Goatin-Scialanga, NHM 2016]

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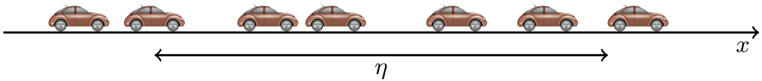

with $w_\eta \in \mathbf{C^1}([0,\eta];\mathbb{R}^+)$ $w'_\eta \leq 0$ and $\int_0^\eta w_\eta(x)dx = 1$
$v : [0,\rho_{\max}] \to \mathbb{R}^+$ s.t. $-A \leq v' \leq 0$, $v(0) = v_{\max}$, $v(\rho_{\max}) = v_{\min}$

[Friedrich-Kolb-Göttlich, NHM 2018]: weighted average velocity

$$V(t,x) = \int_x^{x+\eta} v(\rho(t,y)) w_\eta(y-x)\, dy, \quad \eta > 0$$

(better suited on networks)

[4] [Blandin-Goatin, NumMath 2016; Goatin-Scialanga, NHM 2016]

Well-posedness

Theorem
[Blandin-Goatin, NumMath 2016; Goatin-Scialanga, NHM 2016; Chiarello-Goatin, M2AN 2018]

Let $\rho_0 \in \mathrm{BV}(\mathbb{R}; [0, \rho_{\max}])$. Then the Cauchy problem

$$\begin{cases} \partial_t \rho + \partial_x \left(\rho V(t,x)\right) = 0 & x \in \mathbb{R}, t > 0 \\ \rho(0,x) = \rho_0(x) & x \in \mathbb{R} \end{cases}$$

admits a unique weak (entropy) solution ($\rho \in \mathbf{L^1} \cap \mathbf{L}^\infty \cap \mathrm{BV}$), such that

$$\min_{\mathbb{R}}\{\rho_0\} \leq \rho(t,x) \leq \max_{\mathbb{R}}\{\rho_0\} \quad \text{for a.e. } x \in \mathbb{R},\ t > 0$$

Well-posedness

Theorem

[Blandin-Goatin, NumMath 2016; Goatin-Scialanga, NHM 2016; Chiarello-Goatin, M2AN 2018]

Let $\rho_0 \in \mathrm{BV}(\mathbb{R}; [0, \rho_{\max}])$. Then the Cauchy problem

$$\begin{cases} \partial_t \rho + \partial_x (\rho V(t,x)) = 0 & x \in \mathbb{R}, t > 0 \\ \rho(0,x) = \rho_0(x) & x \in \mathbb{R} \end{cases}$$

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$$\min_{\mathbb{R}}\{\rho_0\} \leq \rho(t,x) \leq \max_{\mathbb{R}}\{\rho_0\} \quad \text{for a.e. } x \in \mathbb{R},\ t > 0$$

Remark

Space-time dependent flux function

$$f = f(t,x,\rho) = \rho V(t,x)$$

Dependence on the location of the kernel support

We set $v(\rho) = 1 - \rho$ and

$$\text{downstream:} \quad V_d(t,x) = 1 - \int_x^{x+\eta} \rho(t,y) w_\eta(y-x)\, dy$$

$$\text{center:} \quad V_c(t,x) = 1 - \int_{x-\eta/2}^{x+\eta/2} \rho(t,y) w_\eta(y-x)\, dy$$

$$\text{upstream :} \quad V_u(t,x) = 1 - \int_{x-\eta}^{x} \rho(t,y) w_\eta(y-x)\, dy$$

Dependence on the location of the kernel support

Rarefaction

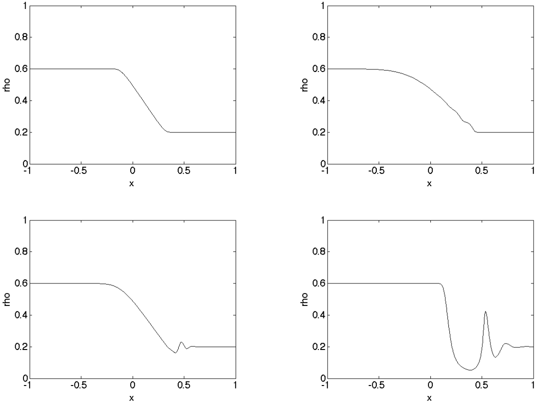


Figure: $w_\eta(x) = 1/\eta$ with downstream, central and upstream supports respectively and initial data $\rho_L = 0.6$, $\rho_R = 0.2$

Dependence on the kernel support

Shock

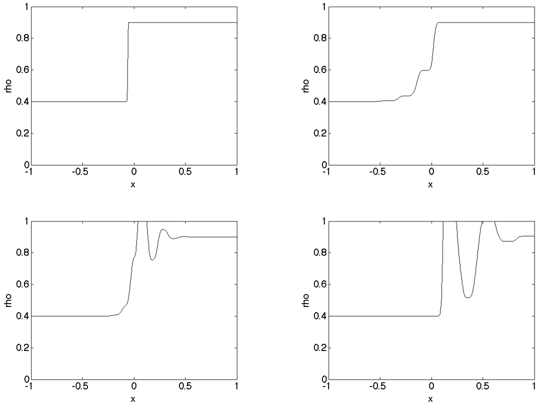


Figure: $w_\eta(x) = 1/\eta$ with downstream, central and upstream supports respectively and initial data $\rho_L = 0.4$, $\rho_R = 0.9$

Dependence on the kernel support

Oscillating initial datum

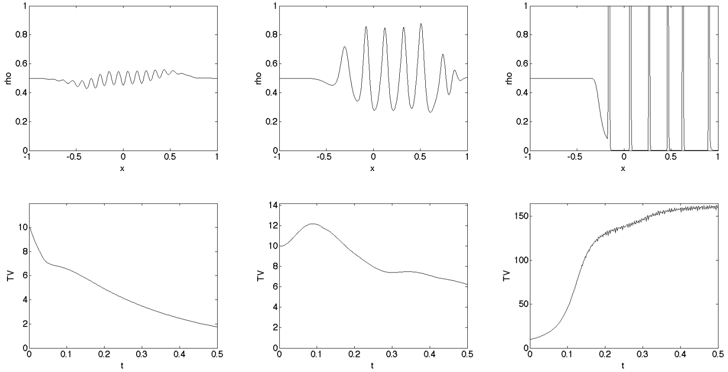
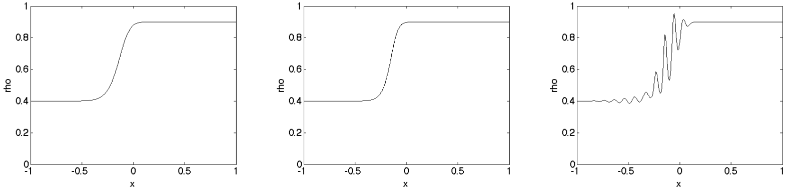


Figure: $w_\eta(x) = 1/\eta$ with downstream, central and upstream supports respectively

Kernel monotonicity



(a) $w_\eta(x) = 1/\eta$ (b) $w_\eta(x) = 2(\eta - x)/\eta^2$ (c) $w_\eta(x) = 2x/\eta^2$

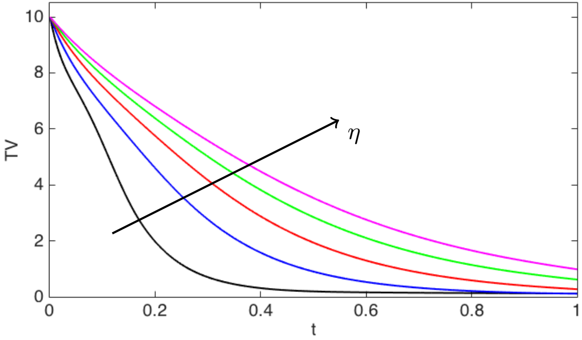
Figure: $\rho(t = 0.5, \cdot)$ corresponding to $\rho_L = 0.4$, $\rho_R = 0.9$

What this non-local model teaches us

- You should always look forward
- You should care more what is closer

What this non-local model teaches us

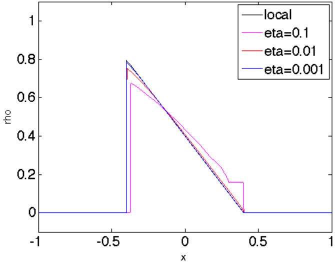
- You should always look forward
- You should care more what is closer
- The farther you see, the worse it is



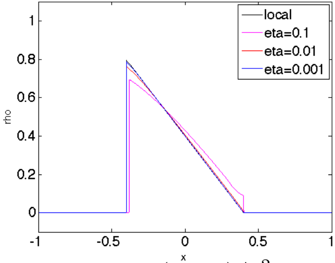
Limit $\eta \searrow 0$[5]

$$\partial_t \rho + \partial_x \left(\rho v(\rho * w_\eta)\right) = 0 \quad \rightarrow \quad \partial_t \rho + \partial_x \left(\rho v(\rho)\right) = 0 \quad ??$$

We consider $v(\rho) = 1 - \rho$ and $\rho_0(x) = \begin{cases} 0.8 & \text{if } -0.5 < x < -0.1 \\ 0 & \text{otherwise} \end{cases}$



$w_\eta = 1/\eta$



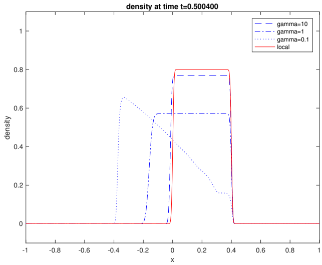
$w_\eta = 2(\eta - x)/\eta^2$

[5] [Colombo-Crippa-Spinolo, 2018; Colombo-Crippa-Marconi-Spinolo, 2019]

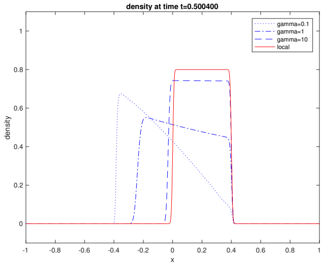
Limit $\eta \to +\infty$[6]

$$\partial_t \rho + \partial_x \left(\rho v(\rho * w_\eta)\right) = 0 \quad \rightarrow \quad \partial_t \rho + \partial_x \left(\rho v(0)\right) = 0$$

We consider $v(\rho) = 1 - \rho$ and $\rho_0(x) = \begin{cases} 0.8 & \text{if } -0.5 < x < -0.1 \\ 0 & \text{otherwise} \end{cases}$



$w_\eta = 1/\eta$



$w_\eta = 2(\eta - x)/\eta^2$

[6] [Chiarello-Goatin, M2AN 2018]

A multi-class model with non-local velocity[7]

Multi-class traffic model with downstream non-local velocity

$$\partial_t \rho_i(t,x) + \partial_x \left(\rho_i(t,x) v_i((r * \omega_i)(t,x))\right) = 0, \qquad i = 1, ..., M,$$

where

$$r(t,x) := \sum_{i=1}^{M} \rho_i(t,x), \quad v_i(\xi) := v_i^{\max} \psi(\xi),$$

$$(r * \omega_i)(t,x) := \int_x^{x+\eta_i} r(t,y)\omega_i(y-x)dy,$$

$$\textbf{(H)} \quad \begin{array}{ll} \omega_i \in \mathbf{C^1}([0,\eta_i];\mathbb{R}^+), & \omega_i' \leq 0, \quad \int_0^{\eta_i} \omega_i(y) \dot{y} = J_i. \\ W_0 := \max_{i=1,\dots,M} \omega_i(0). & 0 < v_1^{\max} \leq v_2^{\max} \leq \dots \leq v_M^{\max}. \\ \psi : \mathbb{R}^+ \to \mathbb{R}^+ \text{ smooth} & \psi' \leq 0 \text{ s.t. } \psi(0) = 1 \text{ and } \psi(r) = 0 \text{ for } r \geq 1. \end{array}$$

- Local multi-class model: [Benzoni-Colombo, EJAM 2003]

[7] [Chiarello-Goatin, NHM 2019]

AV and human-driven mixed traffic

Circular road modeled by the space interval $[-1,1]$ with periodic boundary conditions at $x = \pm 1$.

$$\begin{cases} \partial_t \rho_1(t,x) + \partial_x \left(\rho_1(t,x) v_1^{\max} \psi((r * \omega_1)(t,x))\right) = 0, & \textit{autonomous}, \\ \partial_t \rho_2(t,x) + \partial_x \left(\rho_2(t,x) v_2^{\max} \psi((r * \omega_2)(t,x))\right) = 0, & \textit{human-driven} \\ \rho_1(0,x) = \beta \left(0.5 + 0.3 \sin(5\pi x)\right), \\ \rho_2(0,x) = (1-\beta)\left(0.5 + 0.3 \sin(5\pi x)\right), \end{cases}$$

with

$$\begin{aligned} \omega_1(x) &= \frac{1}{\eta_1}, & \eta_1 &= 1, \\ \omega_2(x) &= \frac{2}{\eta_2}\left(1 - \frac{x}{\eta_2}\right), & \eta_2 &= 0.01, \\ \psi(\xi) &= \max\{1-\xi, 0\}, & \xi &\geq 0, \\ v_1^{max} &= v_2^{max} = 1. \end{aligned}$$

$\beta \in [0,1]$ AV penetration rate

AV and human-drive mixed traffic

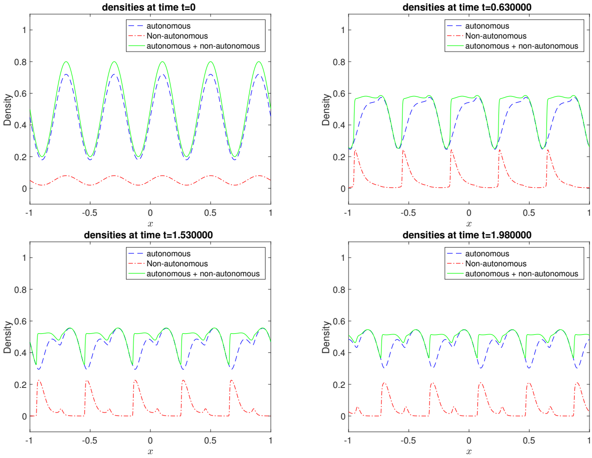


Figure: Density profiles corresponding to $\beta = 0.9$ at different times.

Impact of connected autonomous vehicles[8]

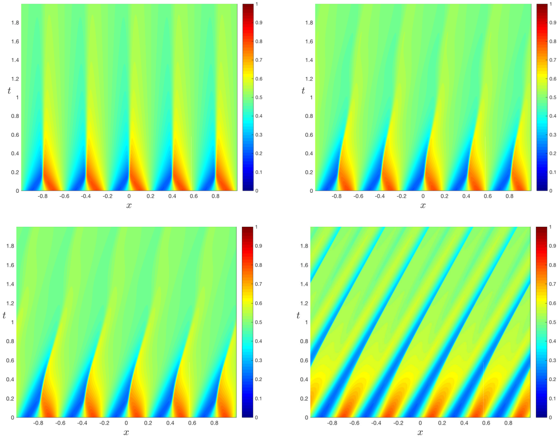


Figure: $(t, x)-$plots of the total traffic density $r(t, x)$ corresponding to $\beta = 0, 0.2, 0.3, 0.9$.

[8] [Chiarello-Goatin, NHM 2019]

Related results

- **Numerical schemes**
  - First order
    - Lax-Friedrichs
      [Colombo-Amorim-Texeira,ESAIM M2AS 2015; Blandin-Goatin, NumMath 2016; Aggarwal-Colombo-Goatin, SINUM 2015; ...]
    - Upwind
      [Friedrich-Kolb-Göttlich, NHM 2018; Chiarello-Goatin, NHM 2019]
    - Lagrangian-Remap
      [Chiarello-Goatin-Villada, Comput. Appl. Math. 2020]
  - High order
    - Discontinous Galerkin and Finite Volume WENO
      [Chalons-Goatin-Villada, SISC 2018; Chiarello-Goatin-Villada, HYP2018 Proceedings (2020)]
- **Extension to networks**
  [Chiarello-Friedrichs-Goatin-Göttlich-Kolb, European J. Appl. Math. (2019)]
- **Micro-macro limits**
  [Goatin-Rossi, CMS 2017; Chiarello-Friedrich-Goatin-Göttlich, SIAM J. Appl. Math.]