Advances in Noise Modeling in Stochastic Systems and Control

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Outline

1. Some Gaussian processes for noise models
2. A non-Gaussian process (Roseblatt process) for a noise model
3. Ergodic linear-quadratic control for scalar equations driven by Rosenblatt processes
4. Some stochastic calculus for Rosenblatt processes
5. Optimal ergodic controls

Some of this work is joint with P. Coupek, T. E. Duncan and B. Maslowski.

Fractional Brownian Motions

A fractional Brownian motion $(B_H(t), t \geq 0)$ is a continuous, centered Gaussian process indexed by the Hurst parameter, $H \in (0,1)$. The covariance function, R_H, is

$$R_H(t) = \mathbb{E}[B_H(t)B_H(s)] = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right)$$

For $H = \frac{1}{2}$ the process is a standard Brownian motion.

Some properties

1. Self-similarity

$$(B^H(\alpha t), t \geq 0) \overset{L}{\sim} (\alpha^H B^H(t), t \geq 0)$$

   for $\alpha > 0$

2. Long range dependence for $H \in (\frac{1}{2}, 1)$

$$r(n) = \mathbb{E}[B^H(1)(B^H(n+1) - B^H(n))]$$

$$\Sigma_{n=0}^{\infty} r(n) = \infty$$

Gauss-Volterra Process

The process $(b(t), t \geq 0)$ is a centered Gauss-Volterra process, which is described by the covariance

(K1) $K(t, s) = 0$ for $s > t$, $K(0, 0) = 0$, and $K(t, \cdot) \in L^2(0, t)$ for each $t \in \mathbb{R}_+$.

$$R(t, s) = \mathbb{E}b(t)b(s) := \int_0^{\min(t,s)} K(t, r)K(s, r)dr$$

where the kernel $K : \mathbb{R}_+^2 \to \mathbb{R}$ satisfies

(K2) For each $T > 0$ there are positive constants C, β such that

$$\int_0^T (K(t, r) - K(s, r))^2 dr \leq C|t - s|^\beta$$

where $s, t \in [0, T]$.

(K3) (i) $K = K(t,s)$ is differentiable in the first variable in $\{0 < s < t < \infty\}$, both K and $\frac{\partial}{\partial t}K$ are continuous and $K(s+,s) = 0$ for each $s \in [0,\infty)$

(ii) $|\frac{\partial K}{\partial t}(t,s)| \leq c_T(t-s)^{\alpha-1}(\frac{t}{s})^{\alpha}$

(iii) $\int_0^t K(t,u)^2 du \leq c_T(t-s)^{1-2\alpha}$

on the set $\{0 < s < t < T\}$, $T < \infty$, for some constants $c_T > 0$ and $\alpha \in (0, \frac{1}{2})$.

For simplicity it is assumed that there is a real-valued Wiener process $(W(t), t \geq 0)$ such that $(b(t), t \geq 0)$ satisfies

$$b(t) = \int_0^t K(t,r)dW(r)$$

Three Examples

(i) A fractional Brownian motion (FBM) with the Hurst parameter $H > \frac{1}{2}$. In this case

$$\begin{aligned} K(t,s) &= C_H s^{1/2-H} \int_s^t (u-s)^{H-3/2} u^{H-1/2} du, \quad & s < t, \\ &= 0 & t \geq s. \end{aligned}$$

where C_H is a constant depending on H. The kernel satisfies conditions (K1)–(K3) (with $\alpha = H - \frac{1}{2}$).

(ii) A Liouville fractional Brownian motion (LFBM) for $H > \frac{1}{2}$, in which case

$$K(t,s) = C_H (t-s)^{H-\frac{1}{2}} 1_{(0,t]}(s), \;\; t,s \in \mathbb{R}_+$$

satisfies (K1)–(K3) with $\alpha = H - \frac{1}{2}$.

(iii) A multifractional Brownian motion (MBM). A simplified version analogous to LFBM in (ii) is considered. The kernel $K : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is defined as

$$K(t,s) = (t-s)^{H(t)-\frac{1}{2}} \, 1_{(0,t]}(s), \;\; t,s \in \mathbb{R}_+,$$

where $H : \mathbb{R}_+ \to [\frac{1}{2}, 1)$ is the “time-dependent Hurst parameter”. It is assumed that $H \in C^1(\mathbb{R}_+)$ and (a) there exists a constant $\varepsilon \in (0, \frac{1}{2})$ such that $H(t) \in [\frac{1}{2} + \varepsilon, 1)$, $t \in \mathbb{R}_+$
(b) for each $t > 0$ there is a constant $C_{\varepsilon,t}$ such that

$$|H'(t)| \leq C_{\varepsilon,t} \min_{u \in (0,t)} \left[\left(\frac{t}{u} \right)^{\varepsilon} \frac{1}{|\log(t-u)|(t-u)} \right].$$

It has been shown that the conditions (K1)-(K3) are satisfied in this case (the latter with $\alpha = \varepsilon$).

Rosenblatt Processes

Let $(u)_+ = \max(u, 0)$ be the positive part of u and define

$$h_k^H(u, y) = \prod_{j=1}^{k} (u - y_j)_+^{\frac{H}{k} - \left(\frac{1}{k} + \frac{1}{2}\right)}$$

for $H \in (\frac{1}{2}, 1), u \in \mathbb{R}$ and $y = (y_1, y_2, \ldots, y_k) \in \mathbb{R}^k$.

Definition. Let $H \in (1/2, 1)$. The *fractional Brownian motion* $B^H = (B_t^H)_{t \in \mathbb{R}}$ is defined by

$$B_t^H = C_H^B \int_{\mathbb{R}} \left(\int_0^t h_1(u, y) \underset{.}{u} \right) dW_y$$

for $t \geq 0$ (and similarly for $t < 0$) where C_H^B is a constant such that $\mathbb{E}(B_1^H)^2 = 1$ and W is a standard Wiener process.

Definition. Let $H \in (1/2, 1)$. The *Rosenblatt process* $R^H = (R_t^H)_{t \in \mathbb{R}}$ is defined by

$$R_t^H = C_H^R \int_{\mathbb{R}^2} \left(\int_0^t h_2(u, y_1, y_2) du \right) dW_{y_1} dW_{y_2}$$

The normalizing constants C_H^B and C_H^R are given explicitly by

$$C_H^B = \sqrt{\frac{H(2H-1)}{\mathrm{B}\left(2-2H, H-\frac{1}{2}\right)}}, \qquad C_H^R = \frac{\sqrt{2H(2H-1)}}{2\mathrm{B}\left(1-H, \frac{H}{2}\right)}$$

where B is the Beta function. For the Itô-type formula given below, it is also convenient to denote

$$c_H^B = C_H^B \, \Gamma\left(H - \frac{1}{2}\right), \qquad c_H^R = C_H^R \, \Gamma\left(\frac{H}{2}\right)^2,$$

and

$$c_H^{B,R} = \frac{c_R^H}{c^B_{\frac{H}{2}+\frac{1}{2}}} = \sqrt{\frac{(2H-1)}{(H+1)} \frac{\Gamma\left(1-\frac{H}{2}\right)\Gamma\left(\frac{H}{2}\right)}{\Gamma(1-H)}}$$

where Γ is the Gamma function.

Stochastic Control System with Rosenblatt Noise

$$\begin{aligned} dX(t) &= aX(t)dt + bU(t)dt + dR^H(t) \\ X(0) &= x_0 \end{aligned}$$

The solution of this scalar stochastic system is

$$X(t) = x_0 + \int_0^t (aX(s) + bU(s))ds + R^H(t)$$

where R^H is a Rosenblatt process with $H \in (\frac{1}{2}, 1)$ and $a, b, x_0 \in \mathbb{R}$, $b \neq 0$, are known constants.

The family of **admissible controls,** $\mathcal{U}$, is the collection of constant scalar linear feedback operators K, that is,

$$\mathcal{U} = \{(U_t)_{t \geq 0} : \; U_t = KX_t \text{ with } K \in \mathbb{R}\},$$

Ergodic Cost

The ergodic quadratic cost, J_∞, is the following

$$J_\infty(U) = \limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\int_0^T (qX_s^2 + rU_s^2)ds.$$

where q, r are strictly positive, known constants.

Let $H \in (\frac{1}{2}, 1)$, $y_0 \in \mathbb{R}$ be a constant, $\vartheta \in L^1_{loc}(0,\infty; D^{2,2})$, $D^{2,2}$ is an appropriate space for Malliavin calculus and define the process $(y_t)_{t\geq 0}$ by

$$y_t = y_0 + \int_0^t \vartheta_s ds + R_t^H.$$

An Itô Formula

If some conditions are satisfied for every $T > 0$, then the process $(Y_t)_{t \geq 0}$ defined by $Y_t = f(t, y_t)$ satisfies the equation

$$Y_t = Y_0 + \int_0^t \tilde{\vartheta}_s ds + 2c_H^{B,R} \int_0^t \tilde{\varphi}_s \delta B_s^{\frac{H}{2}+\frac{1}{2}} + \int_0^t \tilde{\psi}_s \delta R_s^H$$

for every $t \geq 0$ where

$$\begin{aligned}
\tilde{\vartheta}_s &= \frac{\partial f}{\partial s}(s, y_s) + \frac{\partial f}{\partial x}(s, y_s)\vartheta_s \\
&\quad + c_H^R \frac{\partial^2 f}{\partial x^2}(s, y_s)(\nabla^{\frac{H}{2},\frac{H}{2}} y_s)(s, s) \\
&\quad + c_H^R \frac{\partial^3 f}{\partial x^3}(s, y_s)[(\nabla^{\frac{H}{2}} y_s)(s)]^2, \\
\tilde{\varphi}_s &= \frac{\partial^2 f}{\partial x^2}(s, y_s)(\nabla^{\frac{H}{2}} y_s)(s), \\
\tilde{\psi}_s &= \frac{\partial f}{\partial x}(s, y_s).
\end{aligned}$$

A second-order fractional integral given by

$$(I^{\frac{H}{2},\frac{H}{2}}f)(x_1,x_2)$$
$$=\frac{1}{\Gamma(\frac{H}{2})^2}\int_{-\infty}^{x_1}\int_{-\infty}^{x_2}f(u,v)(x_1-u)^{\frac{H}{2}-1}(x_2-v)^{\frac{H}{2}-1}dudv,$$

for $f:\mathbb{R}^2\to\mathbb{R}$ and almost every $x_1,x_2\in\mathbb{R}$

A first-order fractional integral given by

$$(I_{tr}^{\frac{H}{2},\frac{H}{2}}f)(x_1,x_2)=\frac{1}{\Gamma\left(\frac{H}{2}\right)^2}\int_{x_1\vee x_2}^{\infty}f(u)(u-x_1)^{\frac{H}{2}-1}(u-x_2)^{\frac{H}{2}-1}du$$

for $f:\mathbb{R}\to\mathbb{R}$ and almost every $x_1,x_2\in\mathbb{R}$.

Define the fractional stochastic gradients $\nabla^{\frac{H}{2}}$ and $\nabla^{\frac{H}{2},\frac{H}{2}}$ by

$$\nabla^{\frac{H}{2}} = I_{+}^{\frac{H}{2}}D$$
$$\nabla^{\frac{H}{2},\frac{H}{2}} = I_{+,+}^{\frac{H}{2},\frac{H}{2}}D^2.$$

D is a Mallivin derivative and I is a fractional integral.

Theorem. The optimal gain $\hat{K}$ in the family of admissible feedbacks is given by

$$\hat{K} = -\frac{a + \sqrt{a^2 + 4H(1-H)(\frac{b^2 q}{r})}}{2b(1-H)},$$

and the optimal cost $J_\infty(\hat{K})$ is given by

$$J_\infty(\hat{K}) = \frac{\Gamma(2H)}{[-(a+b\hat{K})]^{2H-1}} \left(\frac{-r\hat{K}}{b}\right).$$

Proof. Apply the Itô formula to $(X^2(t), t \in [0, T])$.

$$X_T^2 = x_0^2 + \int_0^T [2(a + bK)X_s^2 + 2c_H^R \nabla^{\frac{H}{2},\frac{H}{2}} X_s(s,s)]ds$$

$$+2c_H^{B,R} \int_0^T 2\nabla^{\frac{H}{2}} X_s(s)\delta B_s^{\frac{H}{2}+\frac{1}{2}} + \int_0^T 2X_s \delta R_s^H$$

where $T > 0$. By taking the expectation

$$\mathbb{E}X_T^2 = x_0^2 + 2(a + bK)\mathbb{E}\int_0^T X_s^2 ds +$$

$$+2c_H^R \mathbb{E}\int_0^T \nabla^{\frac{H}{2},\frac{H}{2}} X_s(s,s)ds$$

$$\frac{1}{T}\mathbb{E}X_T^2 = \frac{1}{T}x_0^2 + 2(a + bK)\frac{1}{T}\mathbb{E}\int_0^T X_s^2 ds$$

$$+2H(2H-1)\frac{1}{T}\int_0^T \int_0^s exp^{(a+bK)r} r^{2H-2} drds$$

By the definition of the Gamma function there is the convergence

$$\int_0^s \mathrm{e}^{(a+bK)r} r^{2H-2} dr \quad \underset{s\to\infty}{\longrightarrow} \quad \frac{\Gamma(2H-1)}{[-(a+bK)]^{2H-1}}$$

which ensures that also the average

$$\frac{1}{T}\int_0^T \int_0^s \mathrm{e}^{(a+bK)r} r^{2H-2} drds$$

converges to the same limit as $T \to \infty$. This convergence can also be used to show that $\lim_{T\to\infty} \frac{1}{T}\mathbb{E}X_T^2 = 0$

Thus

$$0 = \frac{2(a+bK)}{q+rK^2} J_\infty(K) + \frac{\Gamma(2H+1)}{[-(a+bK)]^{2H-1}}$$

from which $J_\infty(K)$ can be determined as

$$J_\infty(K) = \frac{\Gamma(2H+1)(q+rK^2)}{2[-(a+bK)]^{2H}}.$$

It is necessary to minimize $J_\infty(K)$ with respect to K. Differentiate $J_\infty(K)$ with respect to K to obtain

$$DJ_\infty(K) = 0 = -2ra\hat{K} - 2rb\hat{K}^2 + 2Hbq + 2Hbr\hat{K}^2$$

which simplifies to

$$0 = b(1-H)\hat{K}^2 + a\hat{K} - \frac{Hbq}{r}$$

Optimal Feedback and Cost

$\hat{K}$ is the optimal feedback where

$$\hat{K} = -\frac{a + \sqrt{a^2}}{+} 4H(1-H)\frac{b^2 q}{r} 2b(1-H),$$

The optimal cost is

$$J_\infty(\hat{K}) = \frac{\Gamma(2H + 1(q + r\hat{K}^2)}{2[-(a + b\hat{K})]^{2H}}.$$

It follows from the equation for the feedback K that if P is the positive solution to the following algebraic Riccati equation

$$(1 - H)\frac{b^2}{r}P^2 - aP - Hq = 0,$$

then the optimal feedback gain can be expressed as:

$$\hat{K} = -\frac{b}{r}P.$$

Future Work: An Adaptive Control Problem

$$\begin{aligned} dX(t) &= aX(t)dt + bU(t)dt + dR^H(t) \\ X(0) &= x_0 \end{aligned}$$

where R^H is a Rosenblatt process with $H \in (\frac{1}{2}, 1)$ and $a, b, x_0 \in \mathbb{R}$, $b \neq 0$, are known constants.
The family of **admissible controls,** $\mathcal{U}$, is the collection of constant scalar linear feedback operators K, that is,

$$\mathcal{U} = \{(U_t)_{t \geq 0} : \; U_t = KX_t \text{ with } K \in \mathbb{R}\},$$

Assume that the parameter a in the stochastic equation is unknown. Find a family estimators for a and show that the family of adaptive controls minimize the ergodic cost. This adaptive control problem has been solved for Brownian motion and for fractional Brownian motions.

An Adaptive Control Problem with a Fractional Brownian Motion

A real-valued process $(B(t), t \geq 0)$ on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is called a (real-valued) standard fractional Brownian motion with the Hurst parameter $H \in (0, 1)$ if it is a Gaussian process with continuous sample paths that satisfies

$$\begin{aligned} \mathbb{E}\left[B(t)\right] &= 0 \\ \mathbb{E}\left[B(s)B(t)\right] &= \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right) \end{aligned} \qquad (1)$$

for all $s, t \in \mathbb{R}_+$.

Let $H \in (1/2, 1)$ be fixed and B be a fractional Brownian motion with Hurst parameter H. If f satisfies

$$|f|^2_{L^2_H} = \rho(H) \int_0^T \left(u_{1/2-H}(s) \left| I^{H-1/2}_{T-} \left(u_{H-1/2} f \right)(s) \right| \right)^2 ds < \infty$$

then $f \in L^2_H$ and $\int_0^T f \, dB$ is a zero mean Gaussian random variable with second moment

$$\mathbb{E}\left[\left(\int_0^T f \, dB \right)^2 \right] = |f|^2_{L^2_H} \tag{2}$$

where $u_a(s) = s^a$ for $a > 0$ and $s \geq 0$, $I^{H-1/2}_{T-}$ is a fractional integral defined almost everywhere and given by

$$\left(I^{H-1/2}_{T-} f \right)(x) = \frac{1}{\Gamma(\alpha)} \int_x^T \frac{f(t)}{(t-x)^{3/2-H}} \, dt \tag{3}$$

for $x \in [0, T)$, $f \in L^1([0, T])$ and $\Gamma(\cdot)$ is the gamma function and

Let $(X(t), t \geq 0)$ be the real-valued process that satisfies the stochastic differential equation

$$\begin{aligned} dX(t) &= \alpha_0 X(t)\, dt + bU(t)\, dt + dB(t) \\ X(t) &= X_0 \end{aligned} \tag{4}$$

where X_0 is a constant, $(B(t), t \geq 0)$ is a standard fractional Brownian motion with the Hurst parameter $H \in (1/2, 1)$, $\alpha_0 \in [a_1, a_2]$ where $a_2 < 0$, $b \in \mathbb{R} \setminus \{0\}$.

Consider the optimal control problem where the state X satisfies the above equation and the ergodic (or average cost per unit time) cost function J is

$$J(U) = \limsup_{T\to\infty} \frac{1}{T} \int_0^T \left(qX^2(t) + rU^2(t)\right)\, dt \tag{5}$$

where $q > 0$ and $r > 0$ are constants. The family $\mathcal{U}$ of admissible controls is all $(\mathcal{F}_t)$ adapted processes such that (4) has one and only one solution.

If α_0 is unknown, then it is important to find a family of strongly consistent estimators of the unknown parameter α_0 in (4). A method is used that is called pseudo-least squares because it uses the least squares estimate for α_0 assuming $H = 1/2$, that is, B is a standard Brownian motion. It can be shown that the family of estimators $(\hat{\alpha}(t), t \geq 0)$ is strongly consistent for $H \in (1/2, 1)$ where

$$\hat{\alpha}(t) = \alpha_0 + \frac{\int_0^t X^0(s)\, dB(s)}{\int_0^t (X^0(s))^2\, ds} \tag{6}$$

where

$$\begin{aligned} dX^0(t) &= \alpha_0 X^0(t)\, dt + dB(t) \\ X^0(0) &= X_0 \end{aligned} \tag{7}$$

An adaptive control $(U^{\wedge}(t), t \geq 0)$, is obtained from the certainty equivalence principle, that is, at time t, the estimate $\alpha(t)$ is assumed to be the correct value for the parameter. Thus the stochastic equation for the system (4) with the control $U^{\wedge}$ is

$$
\begin{aligned}
dX^{\wedge}(t) &= (\alpha_0 - \alpha(t) - \delta(t))X^{\wedge}(t)\,dt \\
&\quad - \frac{b\rho(t)}{r} V^{\wedge}(t)\,dt + dB(t) \\
&= (-\alpha_0 - \alpha(t) - \delta(t))X^{\wedge}\,dt \\
&\quad - (\alpha(t) + \delta(t))V^{\wedge}(t)\,dt + dB(t) \\
X^{\wedge}(0) &= X_0
\end{aligned}
\tag{8}
$$

$$
\begin{aligned}
\delta(t) &= \sqrt{\alpha^2(t) + \frac{b^2}{r} q} \\
U^\wedge(t) &= -\frac{b\rho(t)}{r} \left[X^\wedge(t) + V^\wedge(t)\right] \\
\rho(t) &= \frac{r}{b^2} [\alpha(t) + \delta(t)] \\
V^\wedge(t) &= \int_0^t \tilde{\delta}(s) V^\wedge(s)\, ds \\
&\quad + \int_0^t [\tilde{k}(t,s) - 1] \\
&\qquad [dX^\wedge(s) - \alpha(s) X^\wedge(s)\, ds - bU^\wedge(s)\, ds] \\
&= \int_0^t \tilde{\delta}(s) V^\wedge(s)\, ds \\
&\quad + \int_0^t [\tilde{k}(t,s) - 1][dB(s) + (\alpha_0 - \alpha(t)) X^\wedge(s)\, ds] \\
\tilde{\delta}(t) &= \delta(t) + \alpha(t) - \alpha_0 \qquad (9)
\end{aligned}
$$

Theorem

Let the scalar-valued control system satisfy the equation given above. Let $(\alpha(t), t \geq 0)$ be the family of estimators of α_0 given above, let $(U^{\wedge}(t), t \geq 0)$ be the associated adaptive control in , and let $(X^{\wedge}(t), t \geq 0)$ be the solution with the control $U^{\wedge}$. Then

$$\lim_{t \to \infty} \frac{1}{t} \mathbb{E} \int_0^t \left| U^*(s) - U^{\wedge}(s) \right|^2 ds = 0$$

and

$$\lim_{t \to \infty} \frac{1}{t} \mathbb{E} \int_0^t \left| X^*(s) - X^{\wedge}(s) \right|^2 ds = 0$$

so

$$\lim_{t \to \infty} \frac{1}{t} \mathbb{E} \int_0^t \left(q(X^{\wedge}(s))^2 + r(U^{\wedge}(s))^2 \right) ds = \lambda$$

where λ is the optimal for the known system.

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Thank You