Model problems in relaxation

(a) Relaxation limits

$$(1)_\varepsilon \quad \begin{cases} u_t + v_x = 0 \\ v_t + a u_x = -\frac{1}{\varepsilon}[v - f(u)] \end{cases}$$

"closure": $\varepsilon \approx 0$, $v = f(u)$

$\downarrow$

$$(1)_0 \qquad u_t + \partial_x f(u) = 0 \qquad \left[+ \varepsilon^1 \left((a - f'(u)^2) u_x \right)_x \right]$$

Q: Is the $\varepsilon \approx 0$ closure <u>valid</u>?

(b) Can we construct a stochastic particle model for

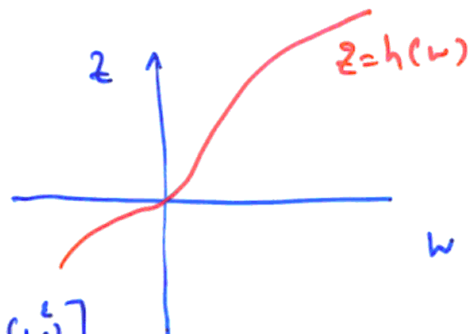
random fluctuations
- $(1)_\varepsilon$
- which approximates for $\varepsilon \approx 0$ $(1)_0$

Note: The same analysis applies to

$$(1)_0' \qquad u_t = \partial_x (K(u) \partial_x u)$$

[Tzavaras, K.]

- Comm. PDE 1997
- Compt. Rend. 1996



$$(1)_\varepsilon \quad \begin{cases} w^\varepsilon_t + c w^\varepsilon_x = \frac{1}{\varepsilon}\left[z^\varepsilon - h(w^\varepsilon)\right] \\ z^\varepsilon_t - c z^\varepsilon_x = \frac{1}{\varepsilon}\left[h(w^\varepsilon) - z^\varepsilon\right] \end{cases}$$

h is increasing

$$\overbrace{(w^\varepsilon + z^\varepsilon)}^{\text{conserved quant.}}{}_t + c\,(w^\varepsilon - z^\varepsilon)_x = 0$$

$\varepsilon \simeq 0$, $h(w) = z$ (local equilibrium)

$$(1)_0 \quad \left(w + h(w)\right)_t + c\left(w - h(w)\right)_x = 0$$

- $(1)_\varepsilon$ is similar to a toy model from kinetic theory : Carleman model
- $(1)_0$ can be written as $\boxed{u_t + f(u)_x = 0}$
- Multi-d / Nonlinear diffusion

Theorem If h is $\uparrow$

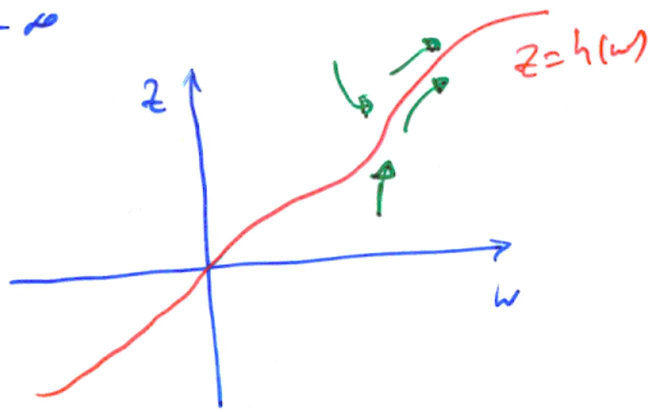
(analogous to the subcharacteristic condition)

(i) $w^\varepsilon + z^\varepsilon \xrightarrow{\varepsilon \downarrow 0} w + h(w)$ (error is $O(\sqrt{\varepsilon})$)

$u = w + h(w)$: entropy solution to $(1)_0$

(ii) Local equilibrium ("H – Theorem")

$$\int_0^T \int_{-\infty}^{\infty} |z^\varepsilon - h(w^\varepsilon)|^2 \, dx\, dt = O(\varepsilon)$$



Details (math...)

- Kruzhkov Theorem extends to $(1)_\varepsilon$
- L^1, BV contraction in $(1)_\varepsilon$
- Entropies for $(1)_\varepsilon$

[Tzavaras, K.] J. Stat. Phys '99

- Macroscopic description : conservation law
- Mesoscopic description : Relaxation system
- Microscopic description : Interacting Particles System

IPS set on a lattice, transport, collisions stochasticity.

w, z averaged # of particles with corresp. velocity

$$\begin{tikzcd}
\text{MICRO (IPS)} \arrow[r, dashed] \arrow[dr, dashed] & \text{MESO (CRS, kinetic)} \arrow[d, "\varepsilon \to 0"'] \\
& \text{MACRO (Cons. Law)}
\end{tikzcd}$$

L^1 - contraction
BV theory

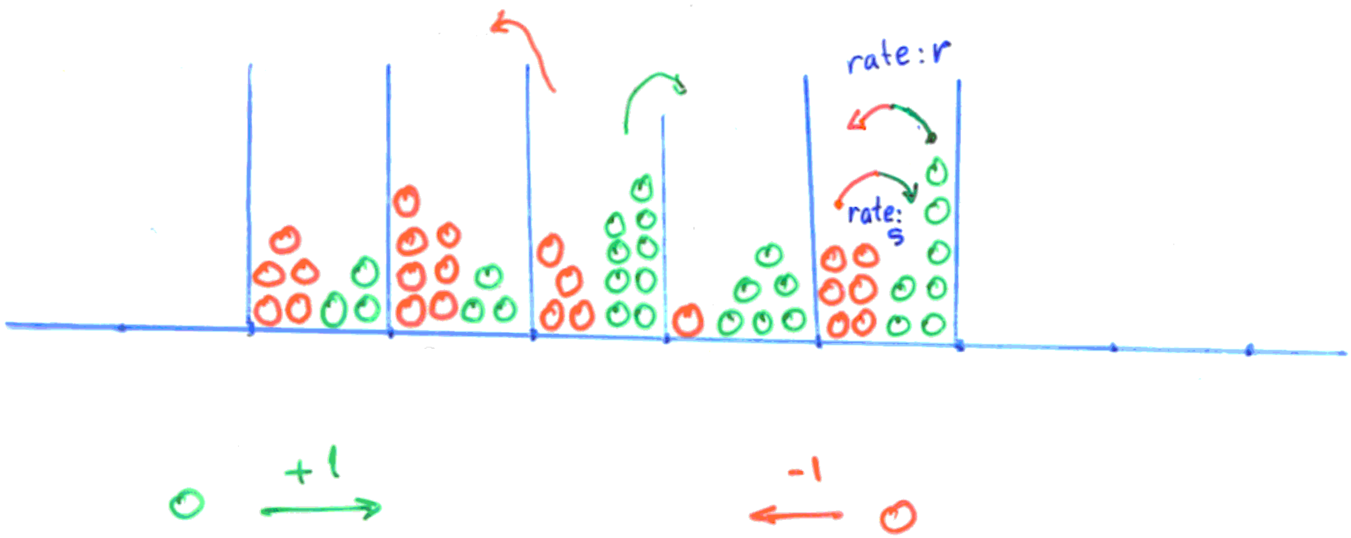
The Interacting Particle System (IPS)

1-d lattice : $\mathbb{Z}$

"velocities" : $\{-1, 1\}$

$i \in \mathbb{Z}$, $\sigma \in \{-1, 1\}$.

$n(i;\sigma)$ # particles at the site i with "velocity" σ



Mechanisms of particle motion

(a) Streaming [FAST] (b) Species change [SLOW]

Ref Carleman, Broadwell systems:
Caprino, De Masi, Presutti, Pulvirenti ~ '89, '91.

Construction of the process

(rates to be chosen later)

- +1 velocity converts to −1

$n_t(i,1)$ = # particles at $i \in \mathbb{Z}$ with veloc. 1

$$t \longmapsto t+\Delta t$$

$$n_{t+\Delta t}(i,1) = \begin{cases} n_t(i,1) & \text{with prob. } 1-r(n,i)\Delta t \\ n_t(i,1)-1 & \text{—''—} \quad r(n,i)\Delta t \end{cases}$$

(also $n_{t+\Delta t}(i,-1) = n_t(i,-1)+1$)

- −1 vel. converts to +1
- transport left/right with prob. $\frac{c}{\gamma} n_t(i,\pm 1)\Delta t$

$\boxed{\text{fast ! } \gamma \ll 1}$

Species Change:

- $(i,1)$ converts to $(i,-1)$ with rate $r(\eta,i)$
- $(i,-1)$ converts to $(i,1)$ with rate $s(\eta,i)$

$\{\eta_t\}_{t\geq 0}$ is a $\mathbb{N}^{\mathbb{Z}\times\{-1,1\}}$ - valued Markov process, with generator

$$\gamma^{-1}L_s + L_{coe}\,,$$

$$L_s f(\eta) = \sum_{i,\sigma} \eta(i,\sigma)\left[f(\eta+\delta_{(i+\sigma,\sigma)}-\delta_{(i,\sigma)}) - f(\eta)\right]$$

$$L_{coe} f(\eta) = \sum_i r(\eta,i)\left[f(\eta+\delta_{(i,-1)}-\delta_{(i,1)}) - f(\eta)\right] +$$
$$+\, s(\eta,i)\left[f(\eta+\delta_{(i,1)}-\delta_{(i,-1)}) - f(\eta)\right]$$

all $f \in C(\mathbb{X};\mathbb{R})$, cylindrical. $\left(\mathbb{X} = \mathbb{N}^{\mathbb{Z}\times\{-1,1\}}\ \text{configuration space}\right)$

Correlation Functions
BBGKY Hierarchy

$$\frac{d}{dt} E\, n(i, \pm 1) = \dots$$

$$\frac{d}{dt} E\, n(i_1, \pm 1)\, n(i_2, \pm 1) = \dots$$

$$\frac{d}{dt} E\, n(i_1, \pm 1)\, n(i_2, \pm 1)\, n(i_3, \pm 1) = \dots$$

$$\vdots$$

Closure? (Estimates)

Local Stat. Equilibrium

- Invariant measure for the streaming mechanism

Product Poisson probab. distn.

$\lambda = (a, b)$, $\quad k$: # of particles from one species

$$\mu_\lambda (\eta(i,1)=k) = e^{-a} \frac{a^k}{k!}$$

$$\mu_\lambda (\eta(i,-1)=k) = e^{-b} \frac{b^k}{k!}$$

- No detailed balance ...

But approximate detailed balance:

Fast streaming dominates
Slow species changes

- Local Statistical Equilibrium

λ varies slowly on the lattice and in time :

$$\lambda = (w(\gamma i, t),\ z(\gamma i, t)) := \lambda(\gamma i, t)$$

$$\mu_{\lambda(\cdot, t)}(n(i, 1) = k) = e^{-w(\gamma i, t)} \frac{w(\gamma i, t)^k}{k!}$$

$\vdots$ $\qquad\qquad$ $\vdots$

Similarly: $n(i, -1)$ $\qquad$ $z(\gamma i, t)$

Back to the PDE: $(\underline{\varepsilon = 1})$

$$(1)_1 \quad \begin{cases} w_t + c\, w_x = z - h(w) \\ z_t - c\, z_x = h(w) - z \end{cases}$$

NOTE: space/time rescaling $\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}\right) \Rightarrow (1)_\varepsilon$

SPECIES change rates + Local Stat. Equil

$$\downarrow$$

$$(1)_1$$

- If at local equil: $E_\mu\, n(i,1) = w(\gamma i, t)$,

  $E_\mu\, n(i,-1) = z(\gamma i, t)$

- $\frac{d}{dt} E_\mu\, n(i,1) + \frac{c}{\gamma} E_\mu\left(n(i,1) - n(i-1,1)\right) =$

  $= E_\mu\, s(n,i) - E_\mu\, r(n,i)$

NOT CLOSED (yet)

WHAT RATES ?

- By the loc. equil. assumption

• $s(n,i) = n(i,-1)$

$$\Downarrow$$

$$E_p\, s(n,i) = z(ri,t)$$

• $E_p\, r(n,i) = h(w(ri,t))$

↑ polynomial in $n(i,1)$

Equation closes at the 0-th moment

- Note: If $r(n,i) = n(i,1)^2 \Rightarrow h(w) = w^2 + w$:

$$E_p\, r(n,i) = E\, n(i,1)^2 = (E\, n(i,1))^2 + E\, n(i,1)$$

$$\neq (E\, n(i,1))^2$$

We select r, s so that

WRAP-UP

- $s(\eta, x) = \eta(x, -1)$
- $E_\mu r(\eta, x) = h(w)$, $h \uparrow$

where μ Poisson measure

- $E_\mu \eta(x, 1) = w$
- $E_\mu \eta(x, -1) = z$

Theorem (mesoscopic scale)

$$E\,\eta_t(x, 1) = w(\gamma x, t) + o(1)$$

$$E\,\eta_t(x, -1) = z(\gamma x, t) + o(1)$$

, $\gamma << 1$

where w, z solve

$$\begin{cases} w_t + c w_x = z - h(w) \\ z_t - c z_x = h(w) - z \end{cases}$$

"Macroscopic" Theorem

Let u the unique entropy solution of

$$\begin{cases} u_t + f(u)_x = 0 \\ u(x,0) = u_0(x) \end{cases}$$

If $\quad E\, n_0(x,1) = w_0(\gamma\varepsilon x)\,, \quad E\, n_0(x,-1) = -z_0^{\varepsilon}(\gamma\varepsilon x)$,

$$E\,(n_t(x,1) + n_t(x,-1)) = u(\gamma\varepsilon x, \varepsilon t) + o(1)\,, \quad \gamma \ll 1$$

where $\quad \varepsilon = \varepsilon(\gamma) = \gamma^{\rho}$.

- Propagation of chaos

Stochastic Cellular Automata

This IPS can take only two values, 0 and 1 at each lattice site $x \in \mathbb{Z}$ and for each velocity $e \in \{-1, 1\}$.

We define the Markov process $\eta_\cdot = \{\eta_t\}_{t \geq 0}$, $\eta_t(x,e) \in \{0,1\}$, $t \geq 0$, $(x,e) \in \mathbb{Z} \times \{-1,1\}$ with generator

$$\gamma^{-1} L_s + L_{col}$$

$$L_s f(\eta) = \sum_{(x,e)} \eta(x,e) \left[f(\eta^{(x,x+e)}) - f(\eta) \right]$$

$$L_{coll} f(\eta) = \sum_{(x,e)} c(x,\eta,e)\, \eta(x,e)\left[1-\eta(x,-e)\right] \left[f(\eta + \delta_{(x,-e)} - \delta_{(x,e)}) - f(\eta) \right]$$