INTERNAL-WAVE PACKETS
+
TRANSITION TO TURBULENCE

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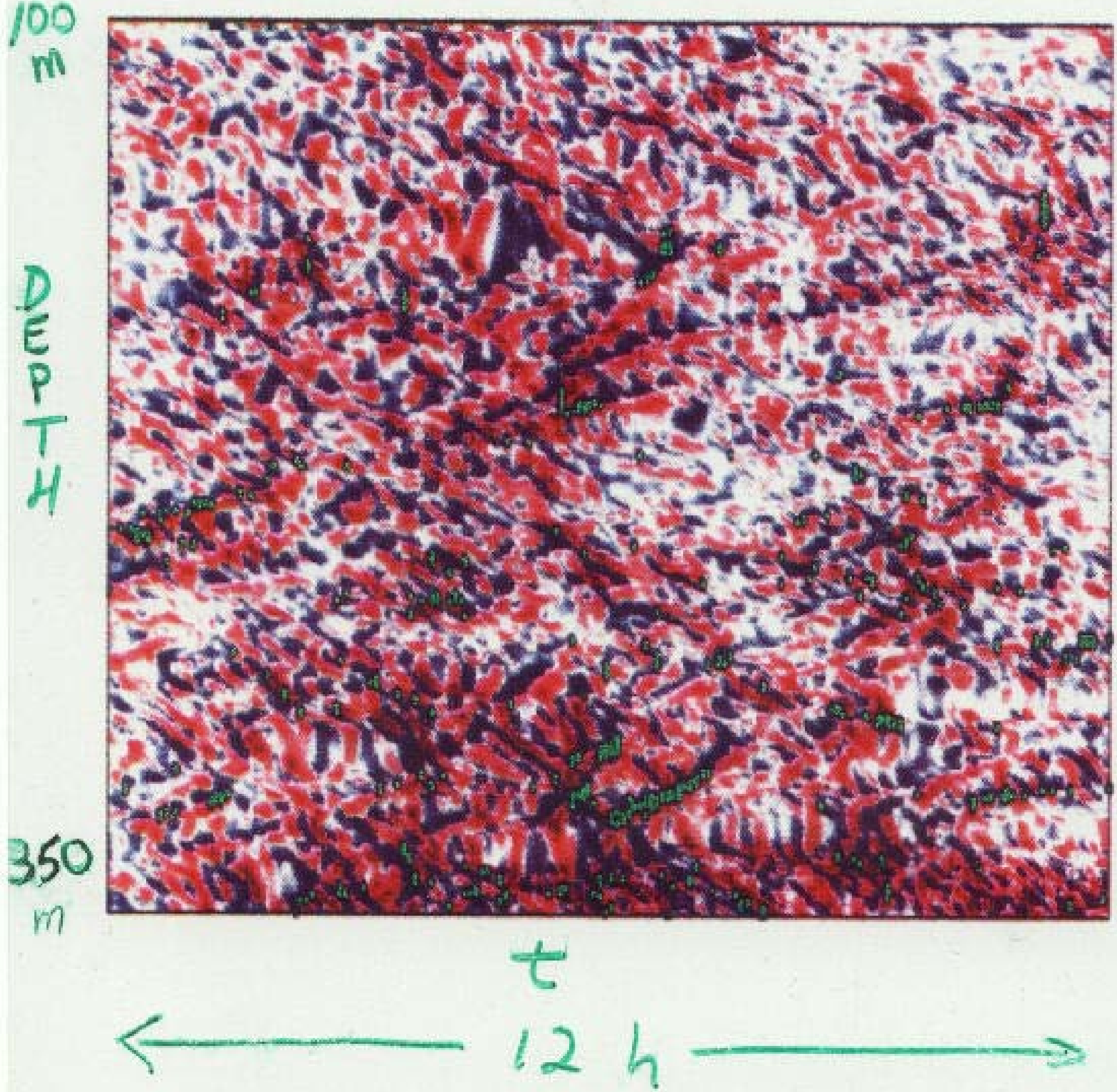
Alford & Pinkel 2000

$$-0.4\,N < \frac{\partial w}{\partial z} < +0.4\,N$$



$$N^2 = -\frac{g}{\rho_0}\overline{\frac{\partial \rho}{\partial z}}$$

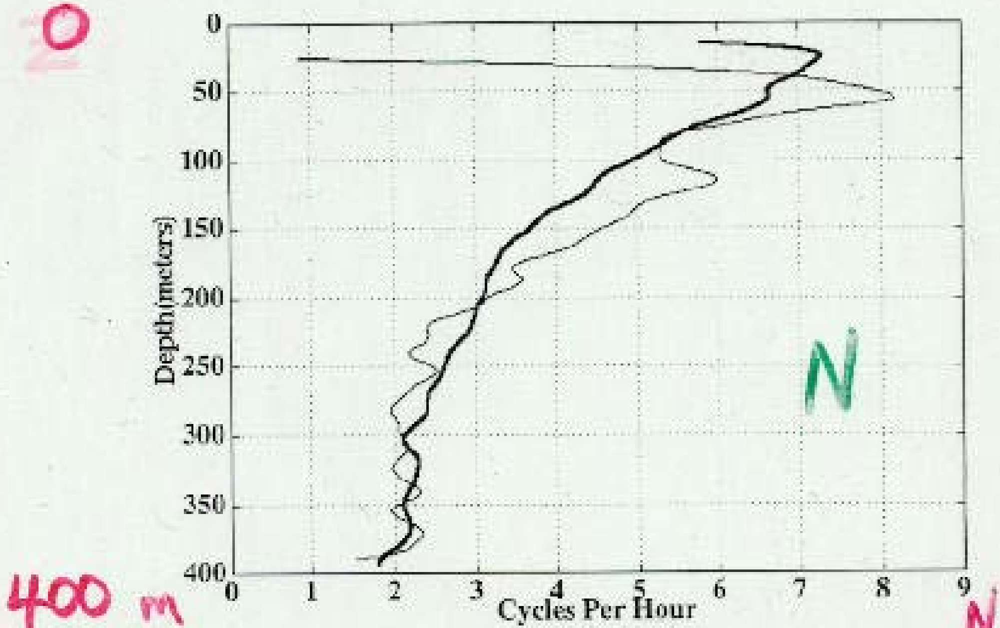
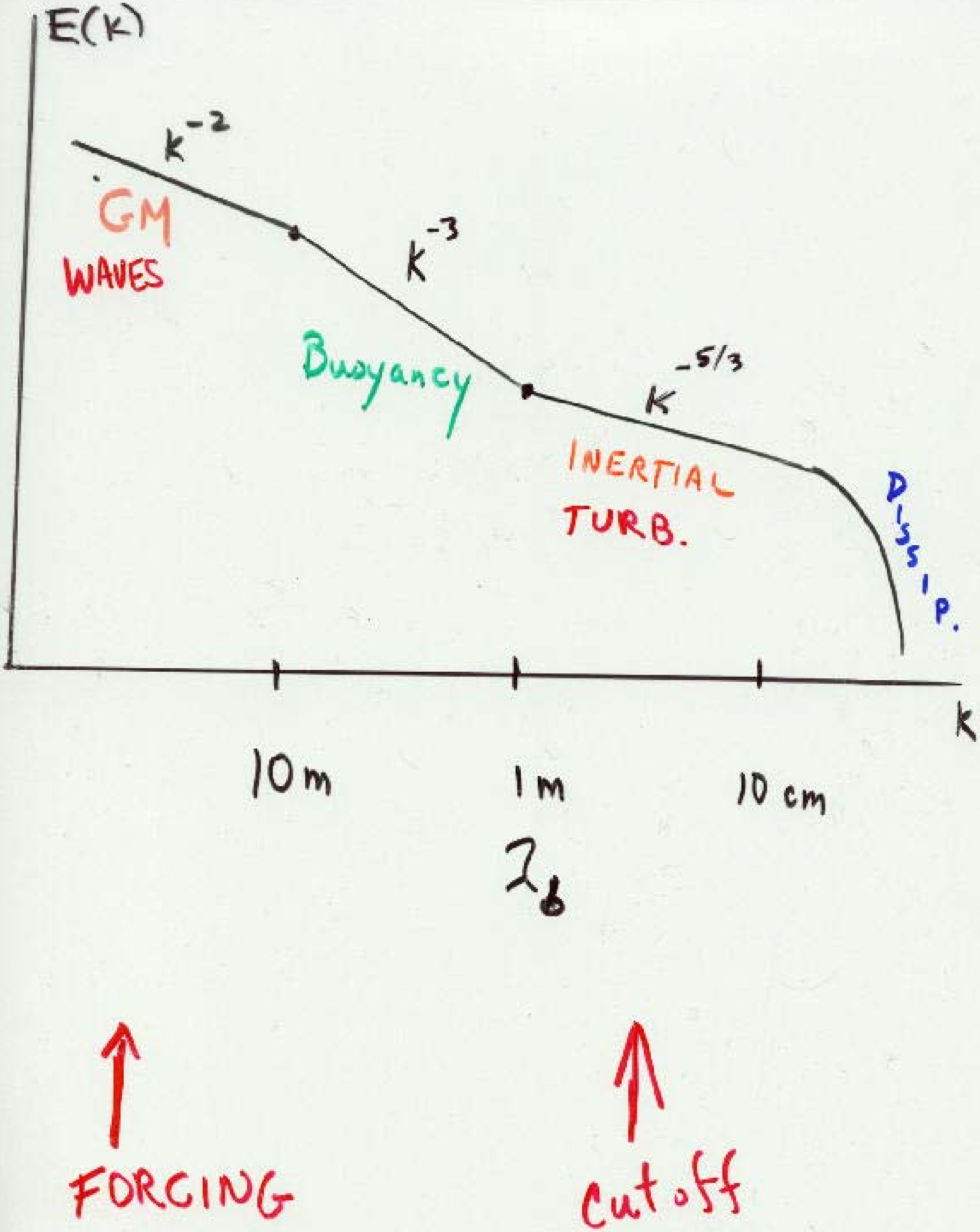


FIG. 3. MBL 9-day (solid line) and 12-h (yearday 67.5–68) mean (light line) buoyancy frequency profile.

ALFORD + PINKEL 2000


E(k)
k^-2
GM
WAVES
k^-3
Buoyancy
k^-5/3
INERTIAL
TURB.
DISSIP.
k
10 m
1 m
10 cm
λ_b
FORCING
Cutoff

Buoyancy Range

advection $\approx$ stratification

$$T_{advect} \approx \frac{1}{\sqrt{k^3 E(k)}}$$

$$T_{IW} \approx \frac{1}{N}$$

$$T_{advect} = T_{IW}$$

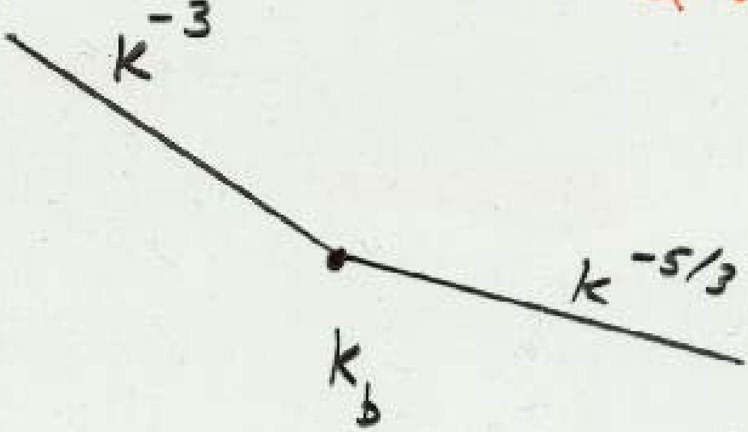
$$E(k) = \alpha N^2 k^{-3}$$

$C_k \approx 1.5$

INERTIAL Range $\quad E(k) = C_k \varepsilon^{2/3} k^{-5/3}$

Buoyancy Range $\quad E(k) = \alpha N^2 k^{-3}$

$\alpha \approx 0.47$



$$k_b \approx \sqrt{\frac{N^3}{\varepsilon}} \cdot \left(\frac{\alpha}{C_K}\right)^{3/4}$$

OZMIDOV

Boussinesq Equations with SGS model

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} + \frac{1}{\rho_0}\nabla p' - \frac{\rho'}{\rho_0}\mathbf{g} = \mathbf{F}_u + \nu(\nabla^2)\nabla^2\mathbf{u}$$

$$\nabla\cdot\mathbf{u} = 0.$$

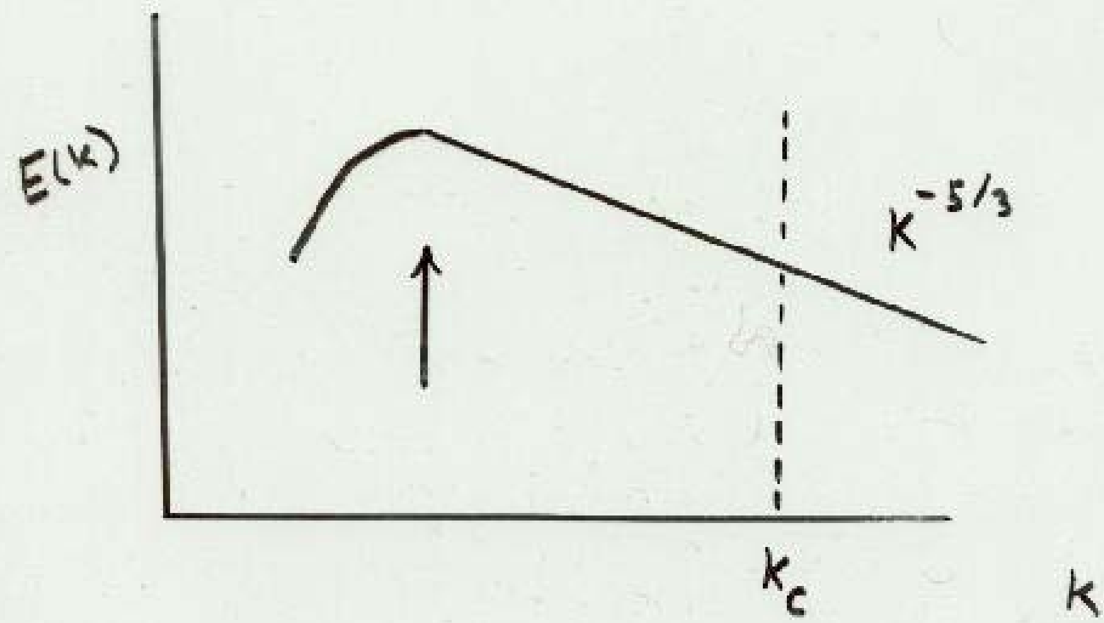
$$\frac{\partial \rho'}{\partial t} + \mathbf{u}\cdot\nabla\rho' + w\frac{\partial\bar{\rho}}{\partial z} = \mathbf{F}_\rho + \kappa(\nabla^2)\nabla^2\rho'$$

$$\mathbf{g} = -g\hat{\mathbf{z}}$$

$$\rho = \bar{\rho}(z) + \rho'(x,y,z,t)$$

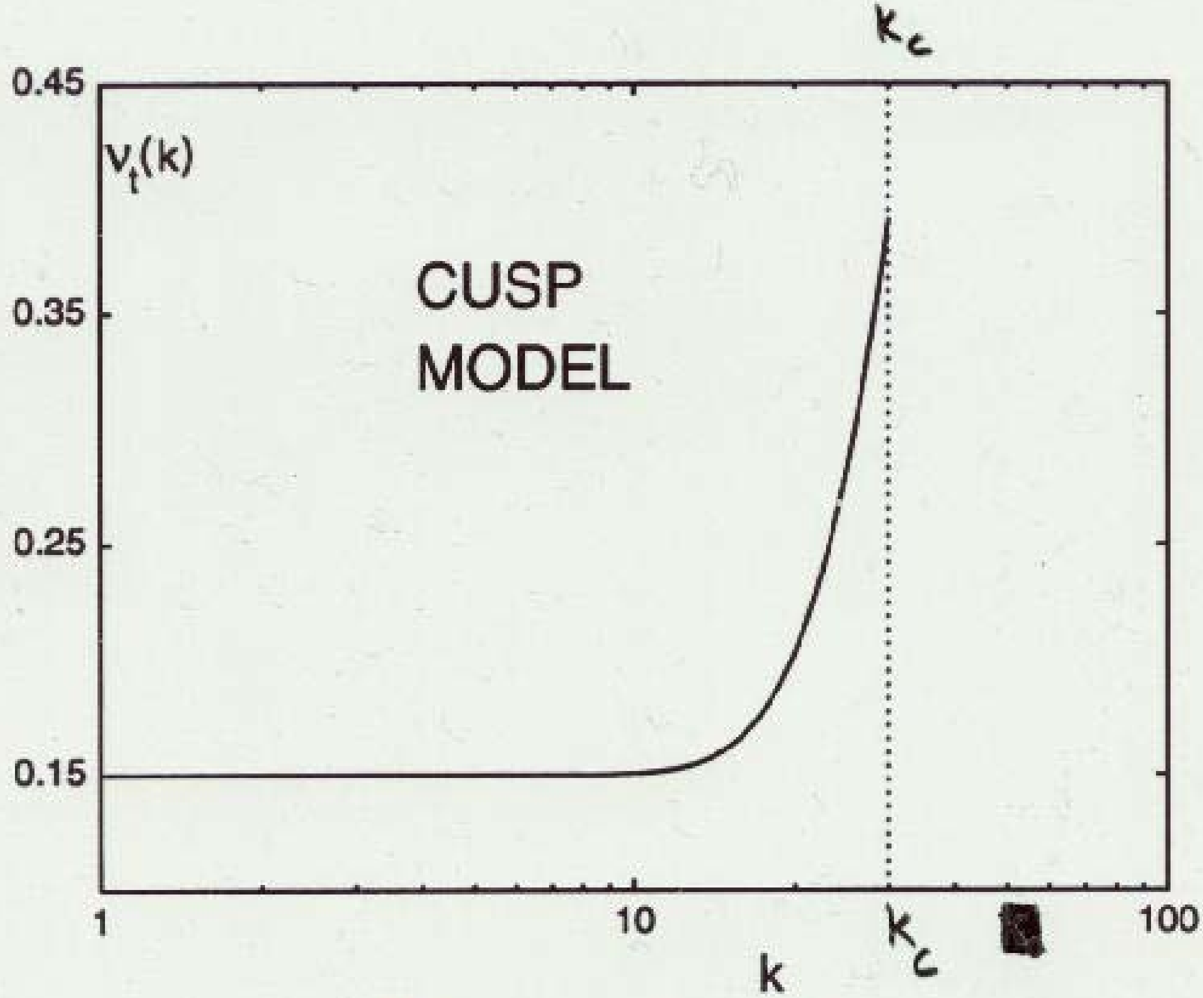
$$\bar{\rho}(z) = \rho_0 + z\frac{\partial\bar{\rho}}{\partial z}$$

$$\frac{\partial \vec{u}_k}{\partial t} + \cdots = -\nu_t(k) k^2 \vec{u}_k$$



Kraichnan '76
Lesieur + Rogallo '89

$$\nu_t(k) = \left(a + b\, e^{-c k_c / k}\right) \sqrt{\frac{E(k_c)}{k_c}}$$



$$Pr_t = \nu_t(k) / \kappa_t(k)$$

$$.5 < Pr_t < .6$$

Lesieur + Rogallo '89

Linear internal waves

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{1}{\rho_0}\nabla p' - \frac{\rho'}{\rho_0}\mathbf{g} = 0$$

$$\frac{\partial \rho'}{\partial t} + w\frac{\partial \bar{\rho}}{\partial z} = 0$$

single linear plane wave

$$(\omega_x, \omega_y, \omega_z, \rho') = A\mathbf{e_k}\exp i(\mathbf{k}\cdot\mathbf{r} - \sigma t)$$

$$\mathbf{e_k} = (gkk_y/Nk_h, -gkk_x/Nk_h, 0, \rho_0)$$

$$\sigma = N\frac{k_h}{k}$$

$$k^2 = k_x^2 + k_y^2 + k_z^2$$

$$k_h^2 = k_x^2 + k_y^2$$

Maintain Standing Wave

$$\sigma = \pm N \frac{k_h}{k}$$

$$\vec{u} = A \frac{g^*}{\sqrt{2}} (0, \sin y \sin z, \cos y \cos z) \sin \frac{t}{\sqrt{2}}$$

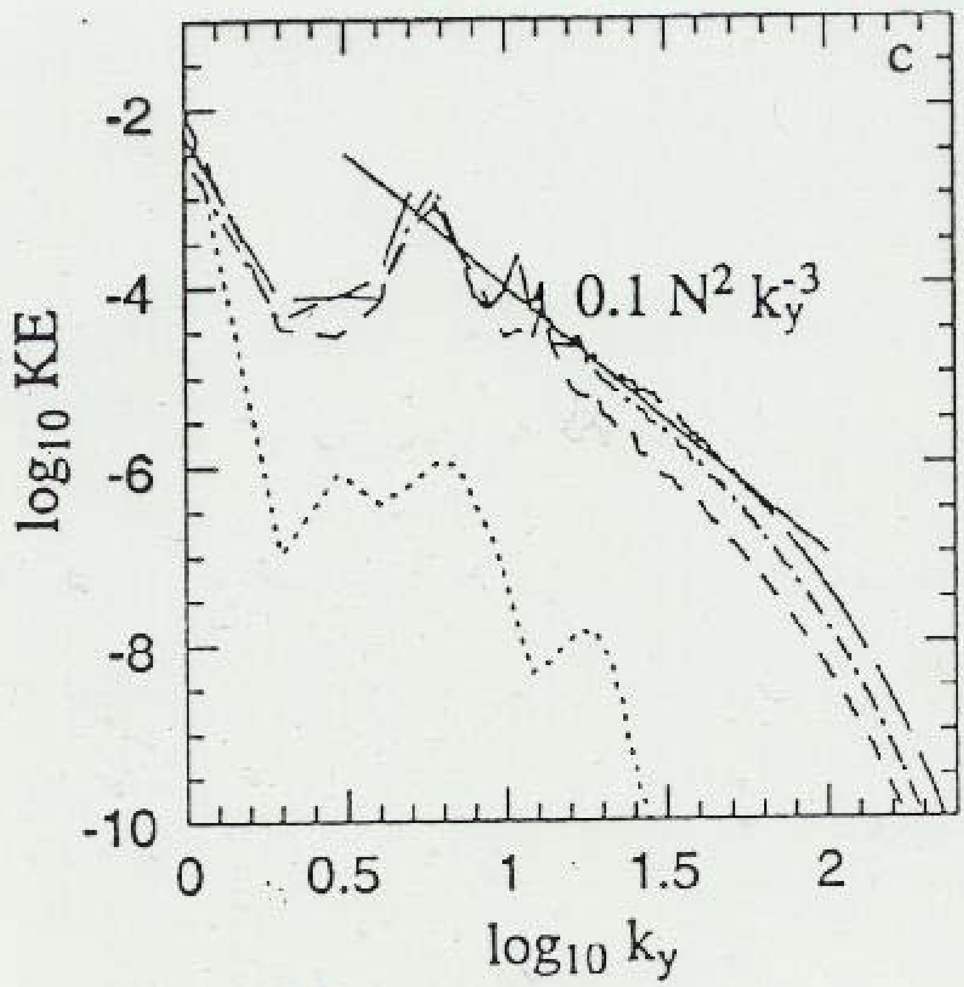
$$\frac{\rho'}{\rho_0} = A \cos y \cos z \cos \frac{t}{\sqrt{2}}$$

$$T = 1/N \qquad L = \frac{2\pi}{l}$$

2-D

Bouruet-Aubertot, Sommeria + Staquet '96

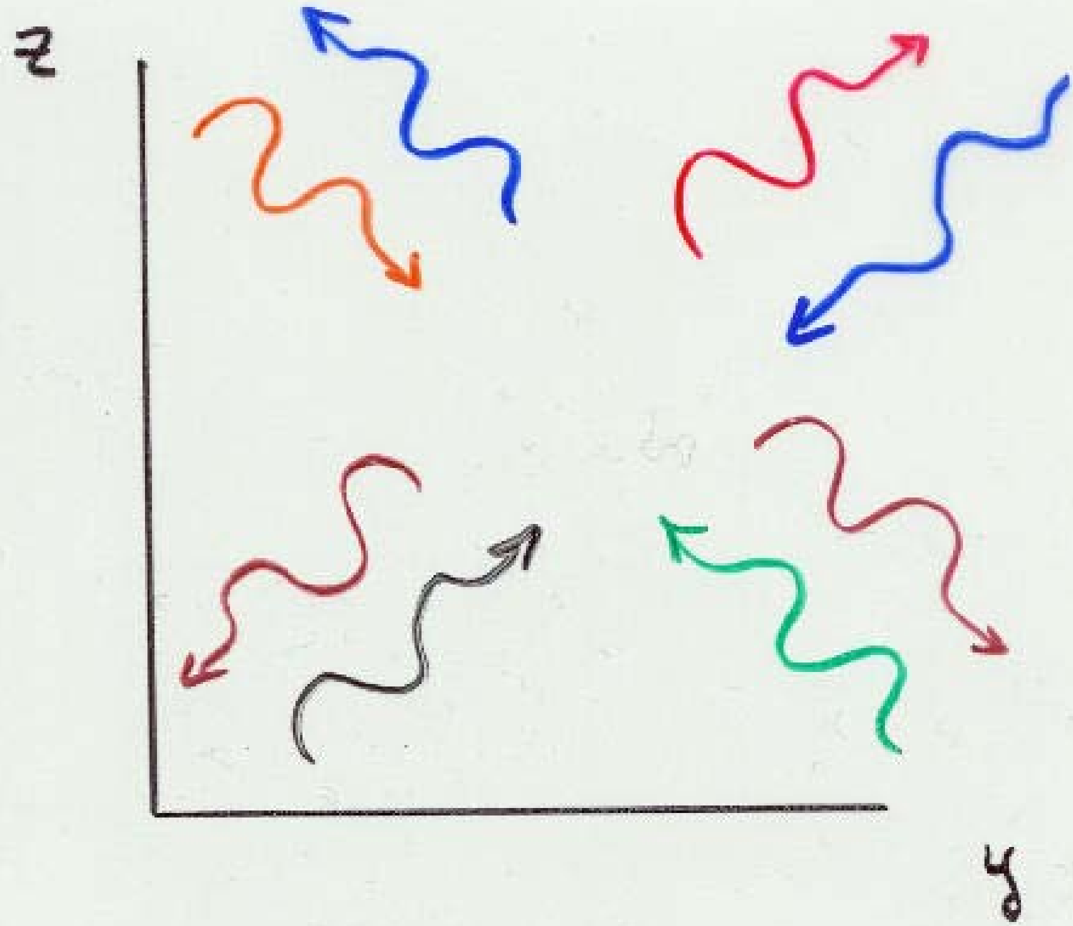
2D SIMULATION



512^2 Simulation

Bournet-Aubertot, Sommeria & Staquet '96

Decay

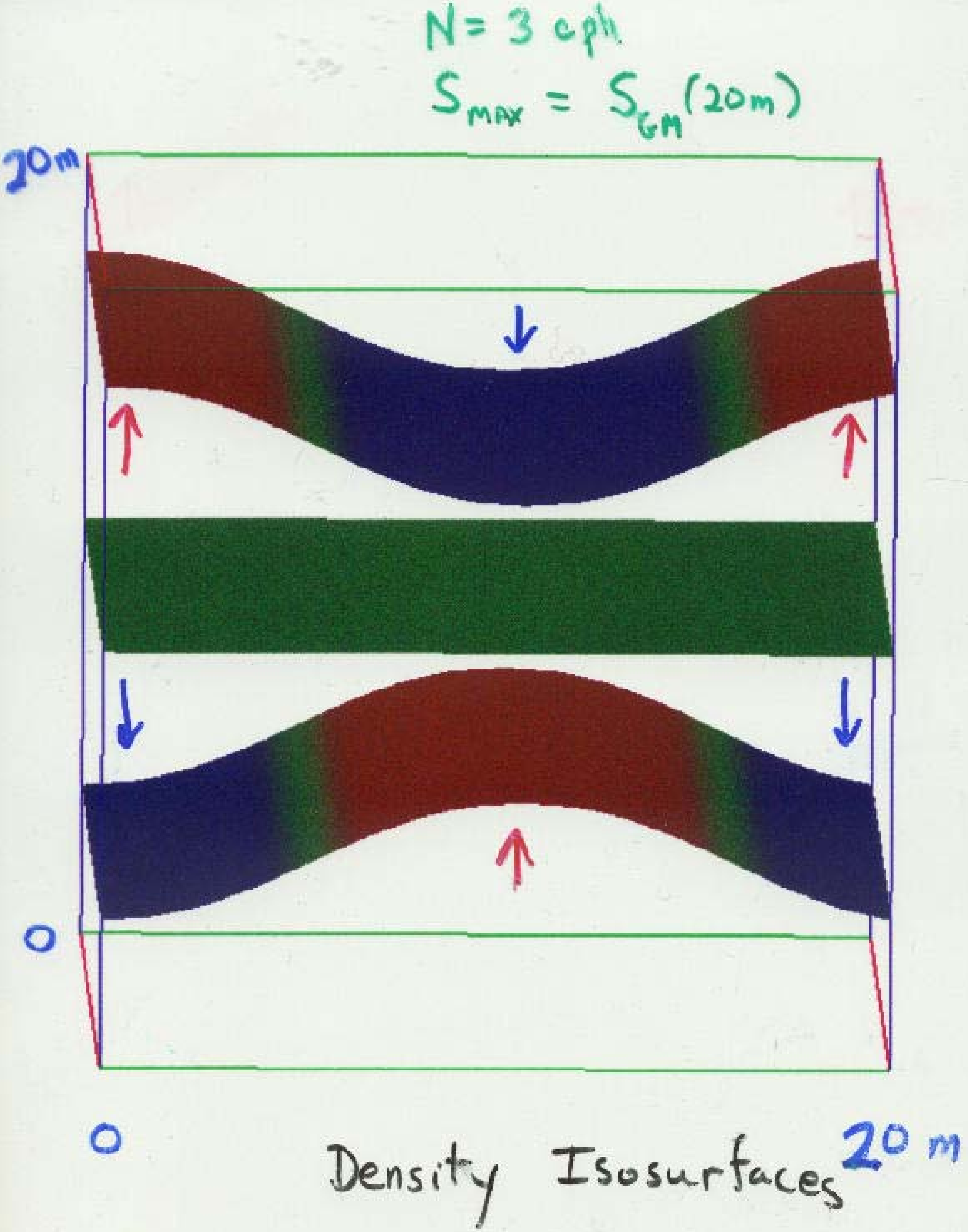


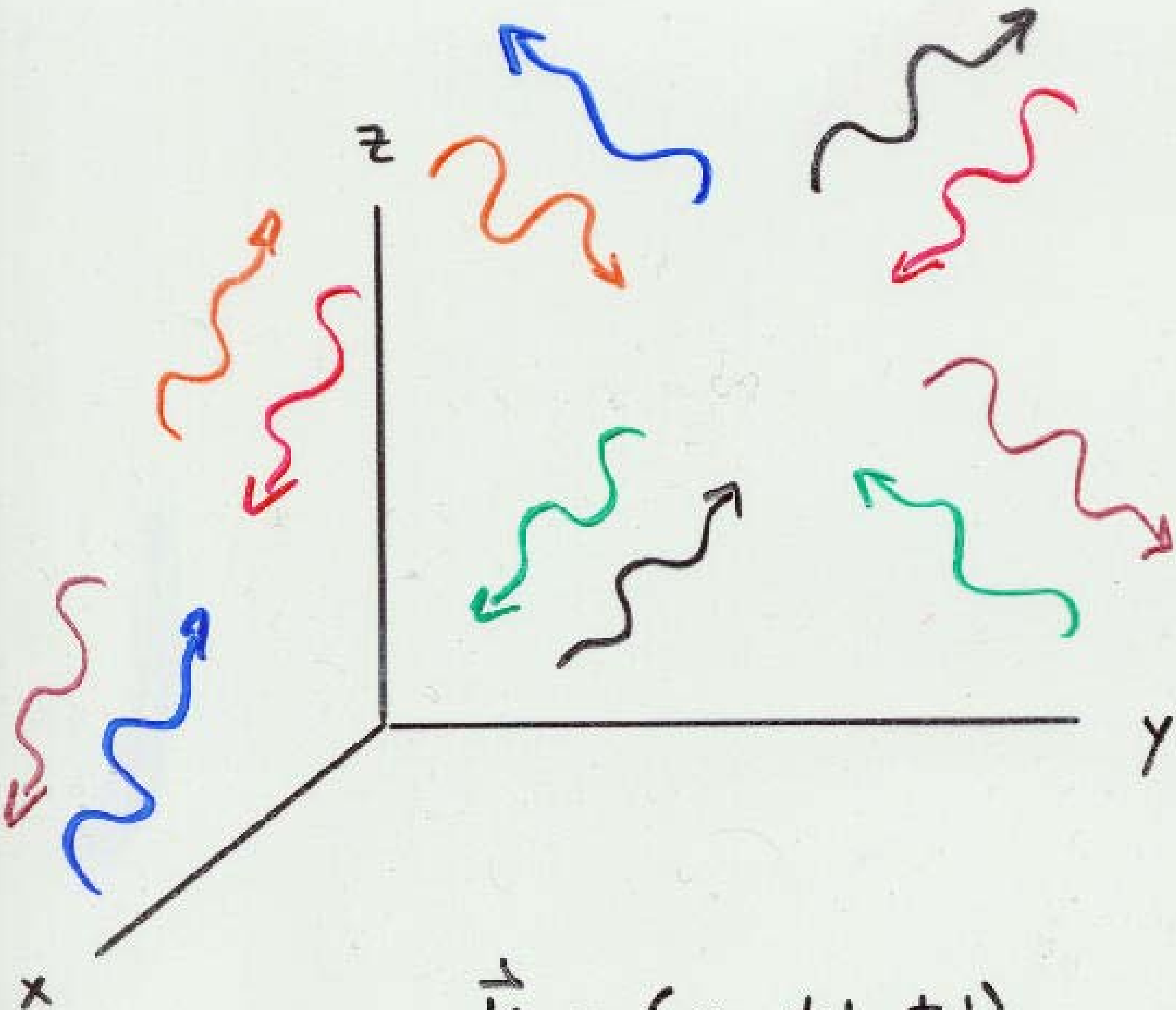
$$\vec{k} = (0, \pm 1, \pm 1)$$

$$\vec{k} = (0, \pm 1, \mp 1)$$

$$\omega = \pm N/\sqrt{2}$$

2D Bouruet-Aubertot, Sommeria + Staquet '96


N = 3 cph
S_MAX = S_GM(20m)
20m
0
0
20 m
Density Isosurfaces



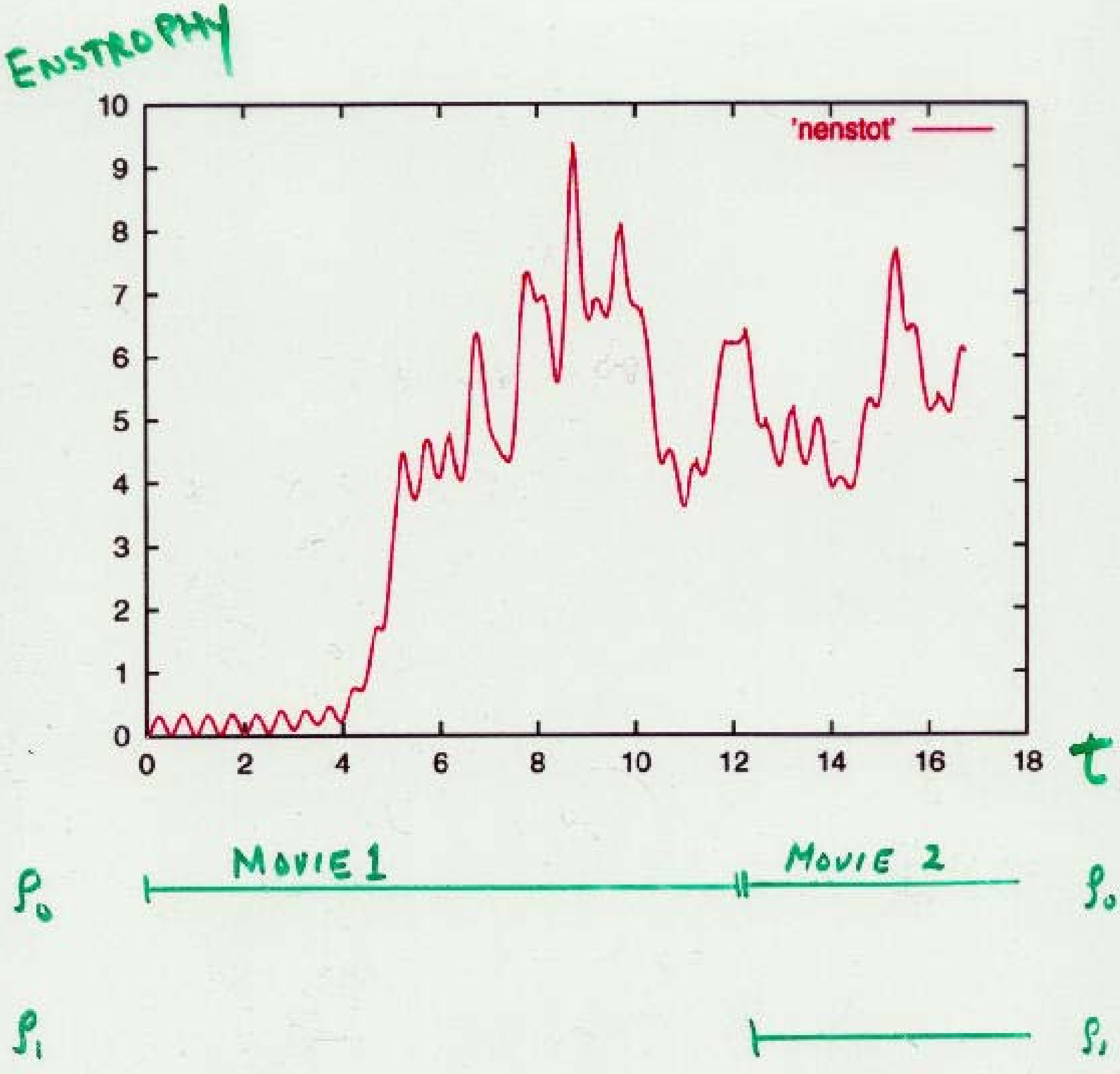
$$\vec{k} = (0, \pm 1, \pm 1)$$

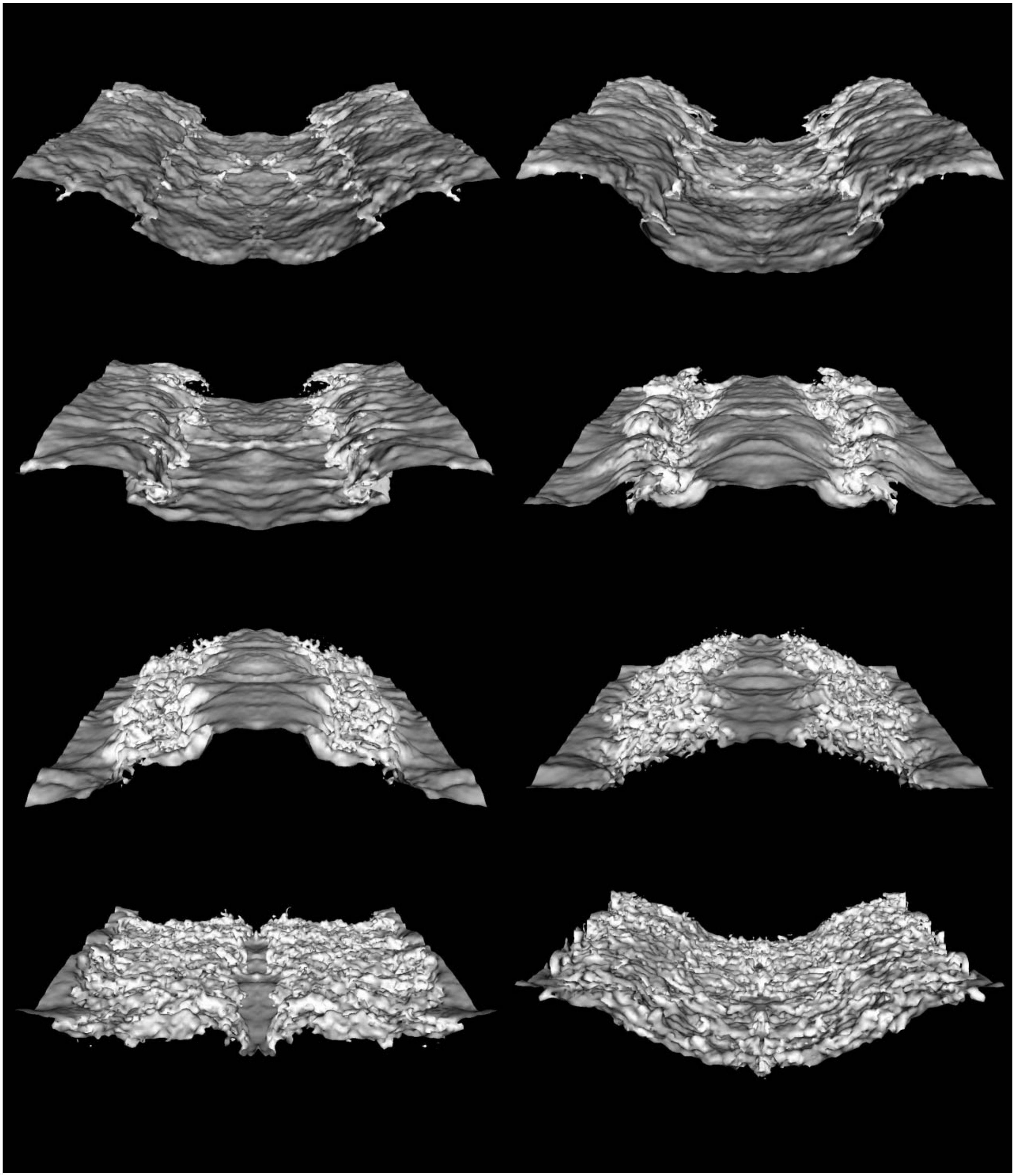
$$\vec{k} = (0, \pm 1, \mp 1)$$

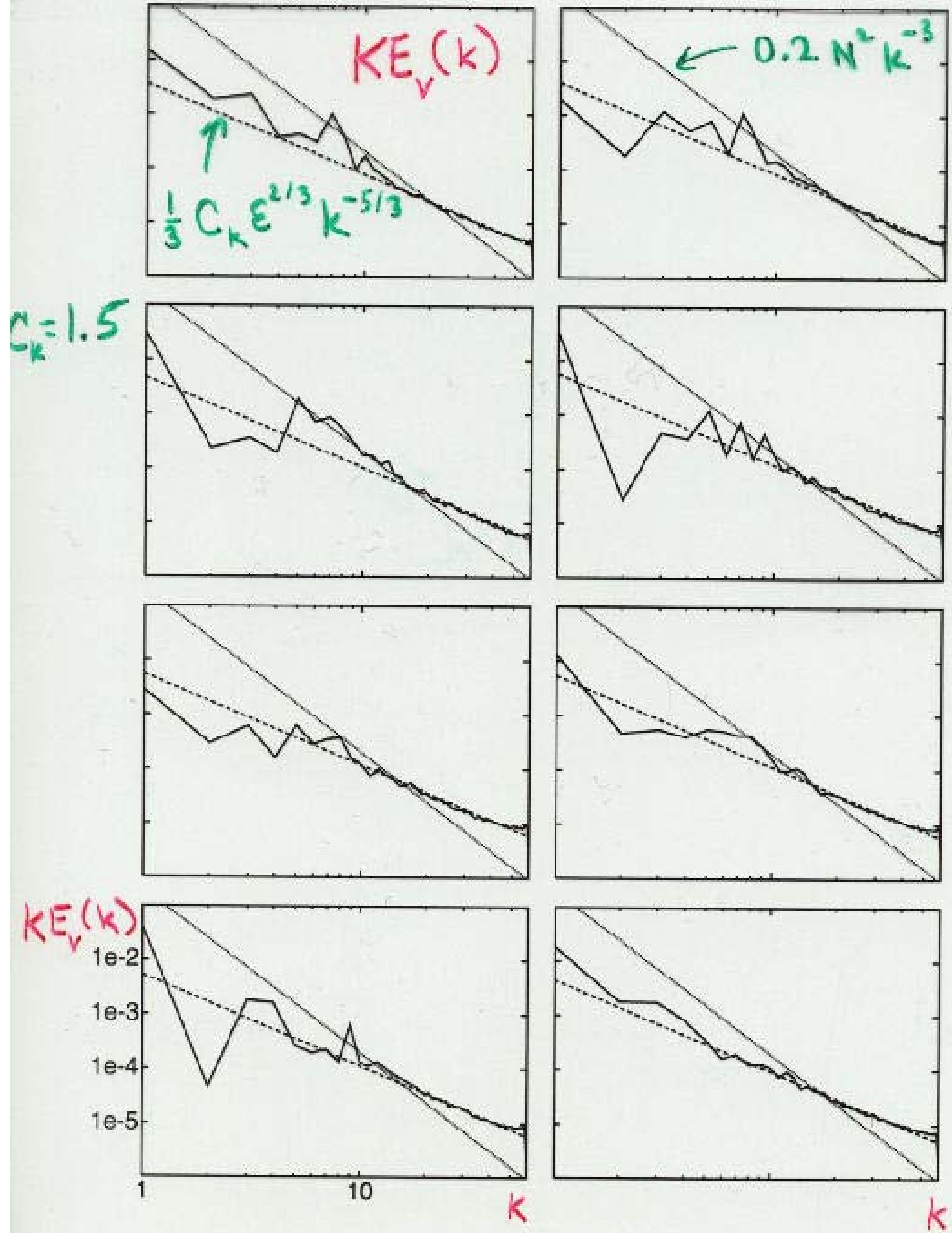
$$\vec{k} = (\pm 1, 0, \pm 1)$$

$$\omega = \pm N \frac{k_H}{k} = \pm \frac{N}{\sqrt{2}}$$

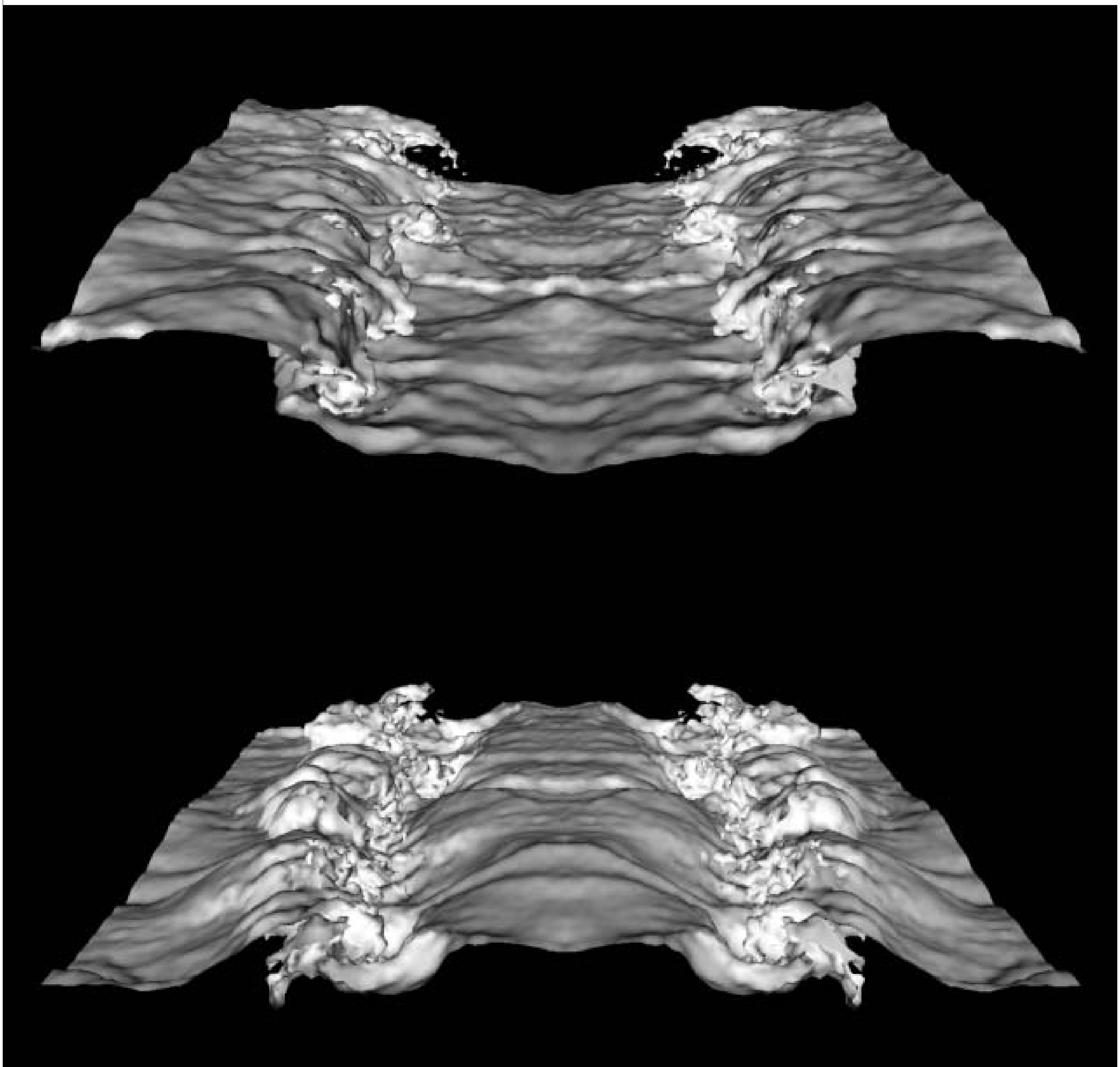
12

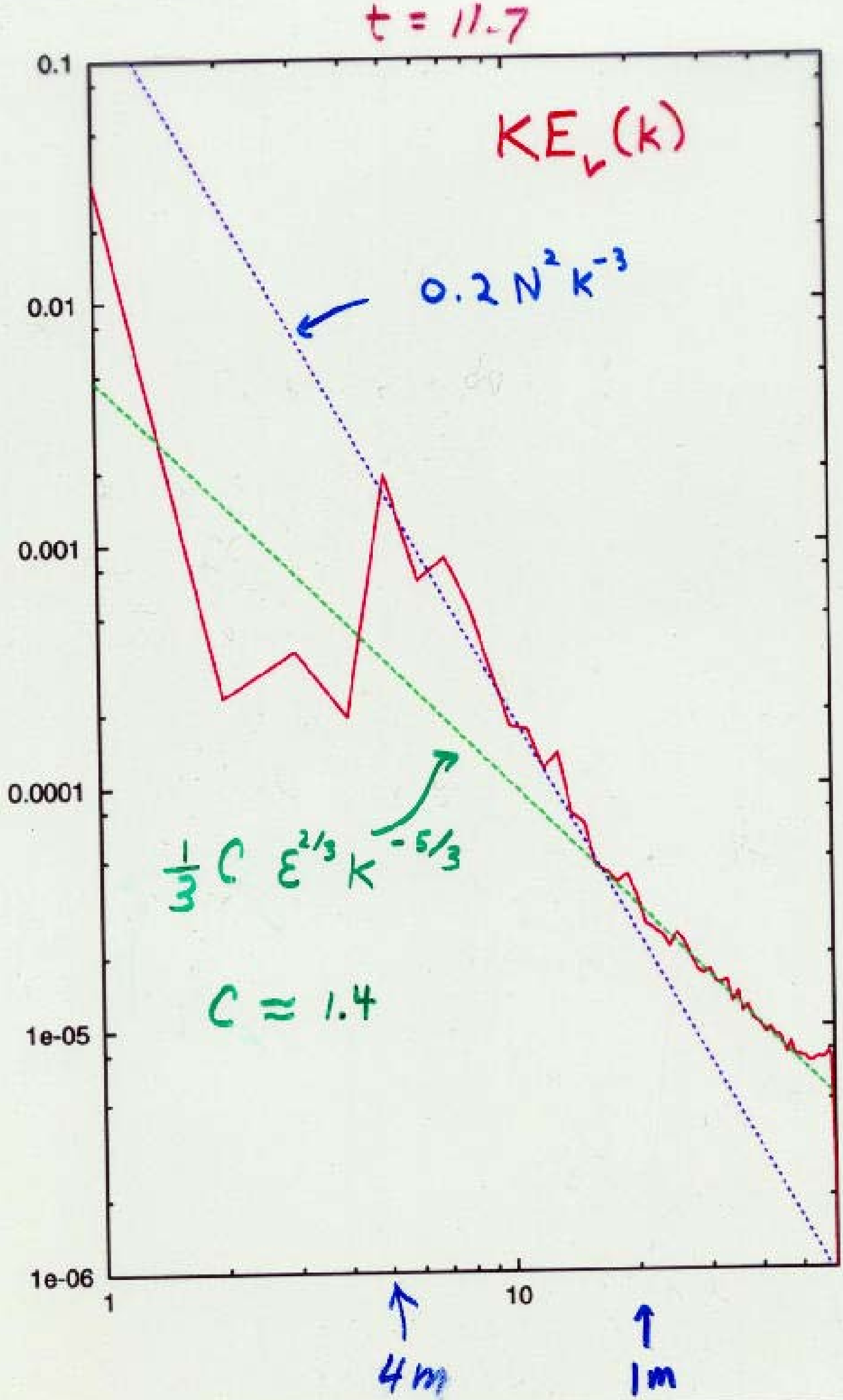

ENSTROPHY
'nenstot'
10
9
8
7
6
5
4
3
2
1
0
0
2
4
6
8
10
12
14
16
18
t
MOVIE 1
MOVIE 2
ρ0
ρ0
ρ1
ρ1



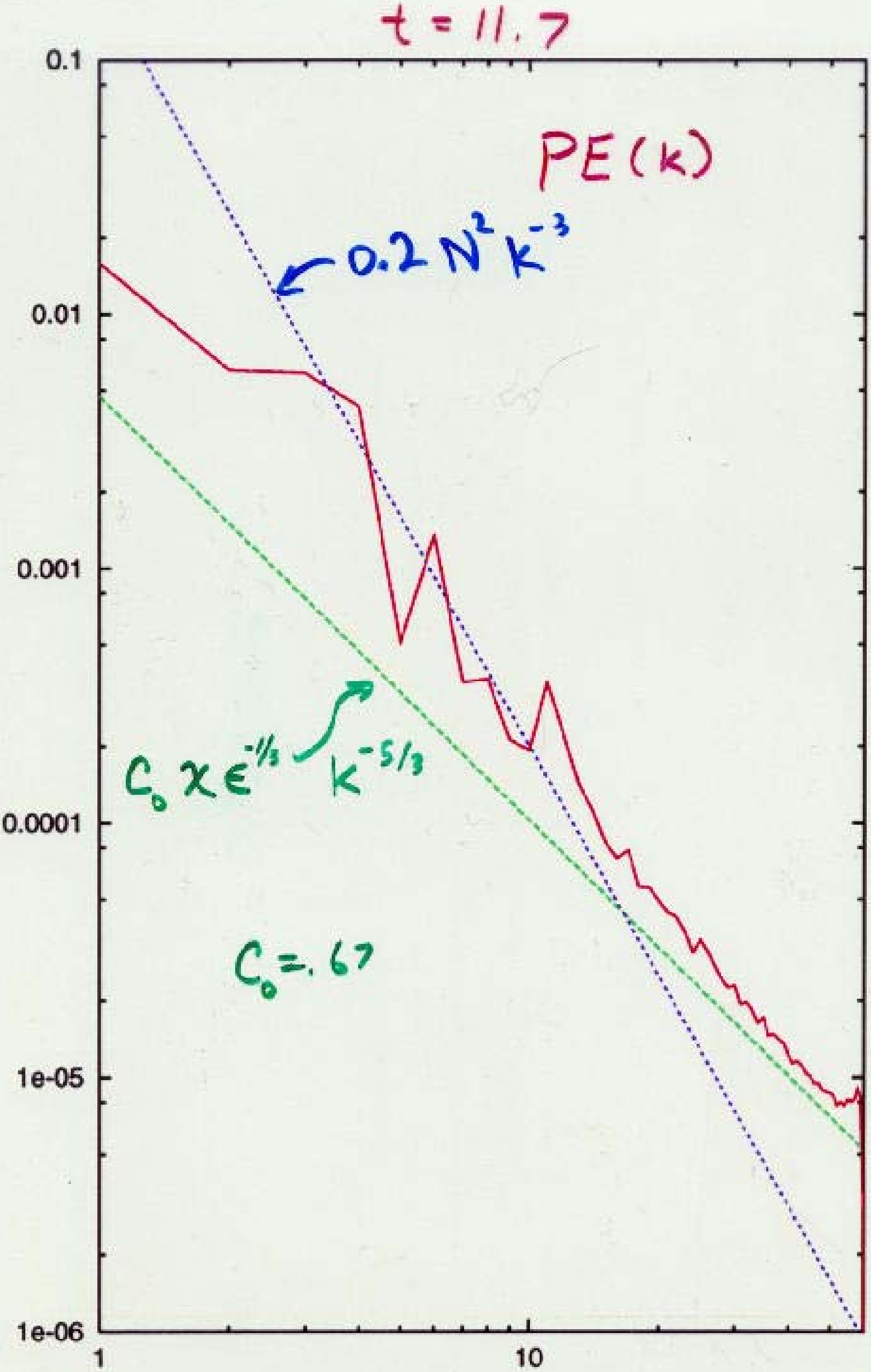


time span $= T_{FORCE}$



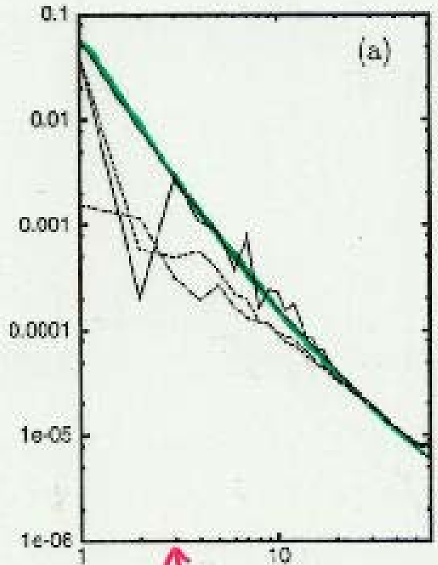


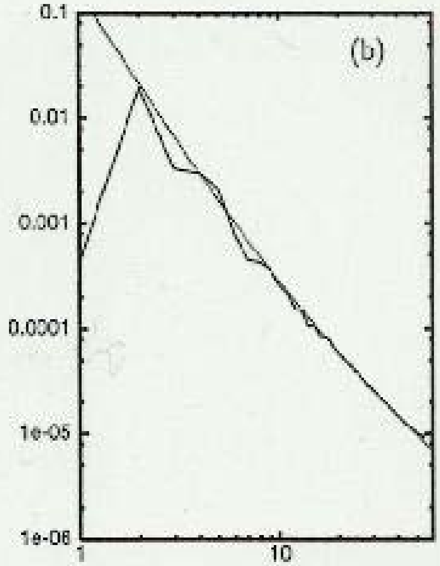
t = 11.7
0.1
0.01
0.001
0.0001
1e-05
1e-06
1
10
KE_v(k)
0.2 N^2 k^-3
1/3 C ε^2/3 k^-5/3
C ≈ 1.4
4m
1m


t = 11.7
PE(k)
0.2 N² k⁻³
C₀ χ ε^(-1/3) k^(-5/3)
C₀ = .67
0.1
0.01
0.001
0.0001
1e-05
1e-06
1
10

t = 16.8

KE(k)





PE(k)

↑ 7 m

Lumley 1964, Holloway 81, Weinstock 85

$$KE(k) = C_k \epsilon^{2/3} k^{-5/3} + C_u N^2 k^{-3}$$

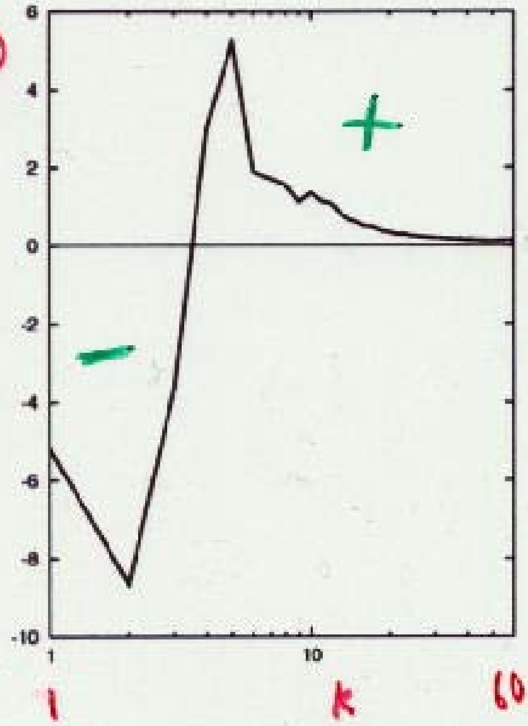
$$PE(k) = C_0 \epsilon_{pe} \epsilon^{-1/3} k^{-5/3} + C_\theta N^2 k^{-3}$$

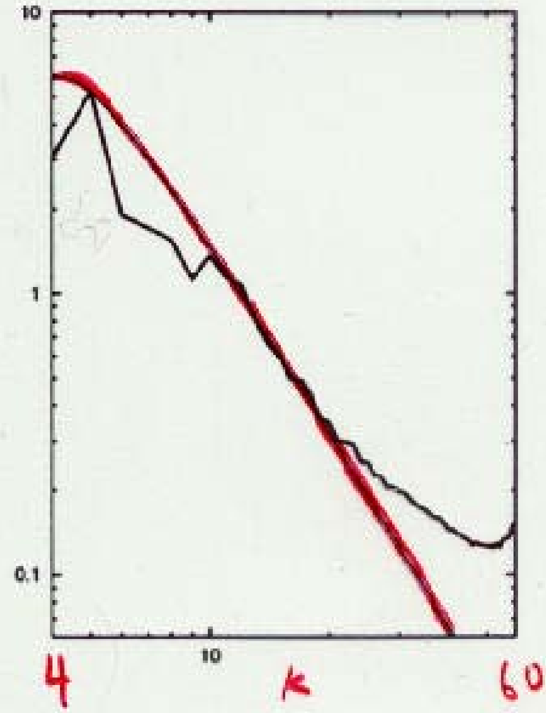
$C_k = 1.44$ $C_u = 0.16$

$C_0 = 0.73$ $C_\theta = 0.16$

$$-g\,\mathrm{Re}\langle w_k^* \rho_k' \rangle/\rho_0$$

BF(k)





$$BF(k) =$$

$$-2D\frac{\epsilon_0}{k_b}\sqrt{1+D(k/k_b)^{-4/3}}\left(k/k_b\right)^{-7/3}$$

Lumley-Shur-Weinstock (1962-1985)

Holloway (1986)

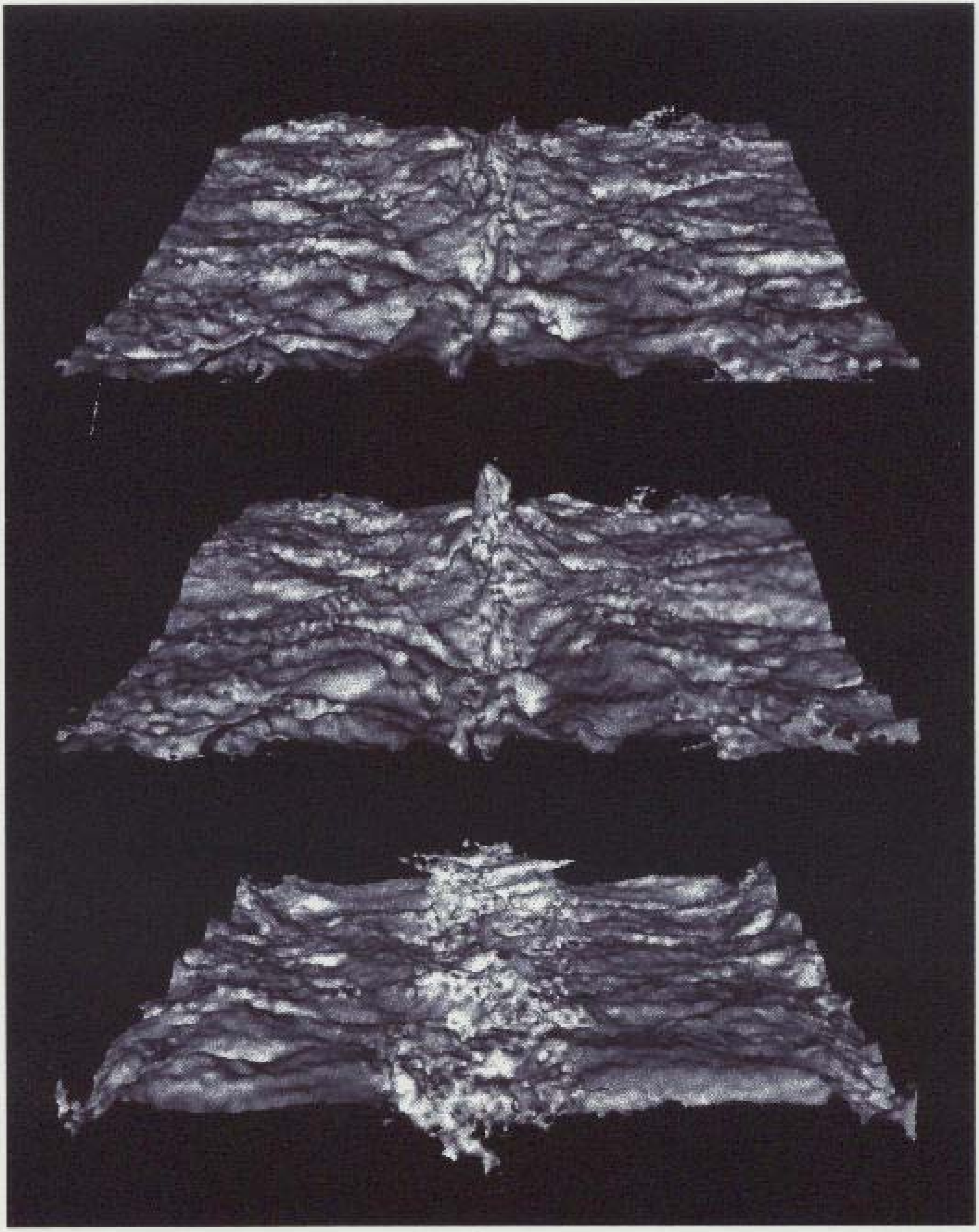
Alford + Pinkel (2000)

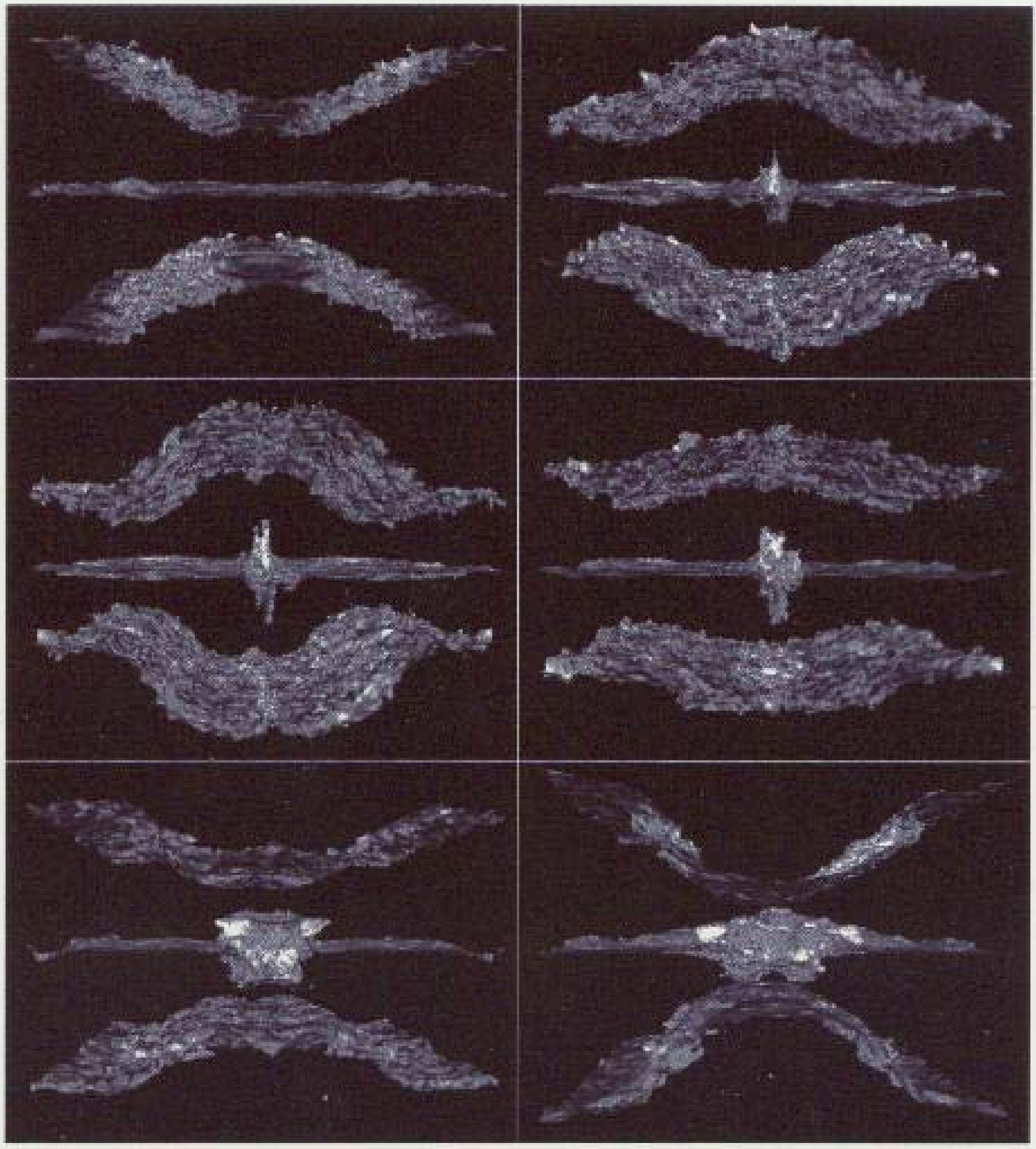
$$\epsilon \approx 4 \times 10^{-9} \ W/kg$$

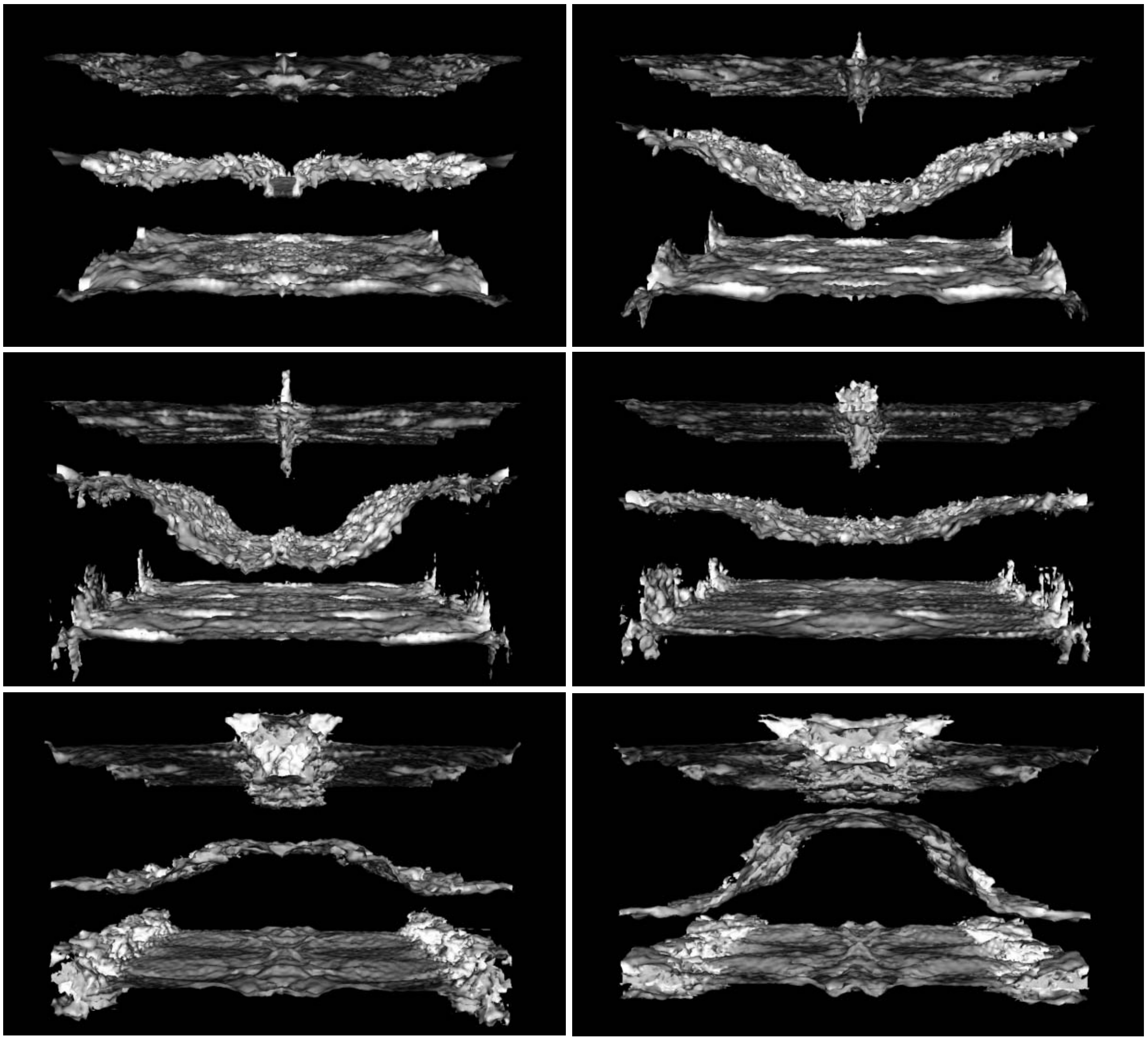
for $N = 3$ cph

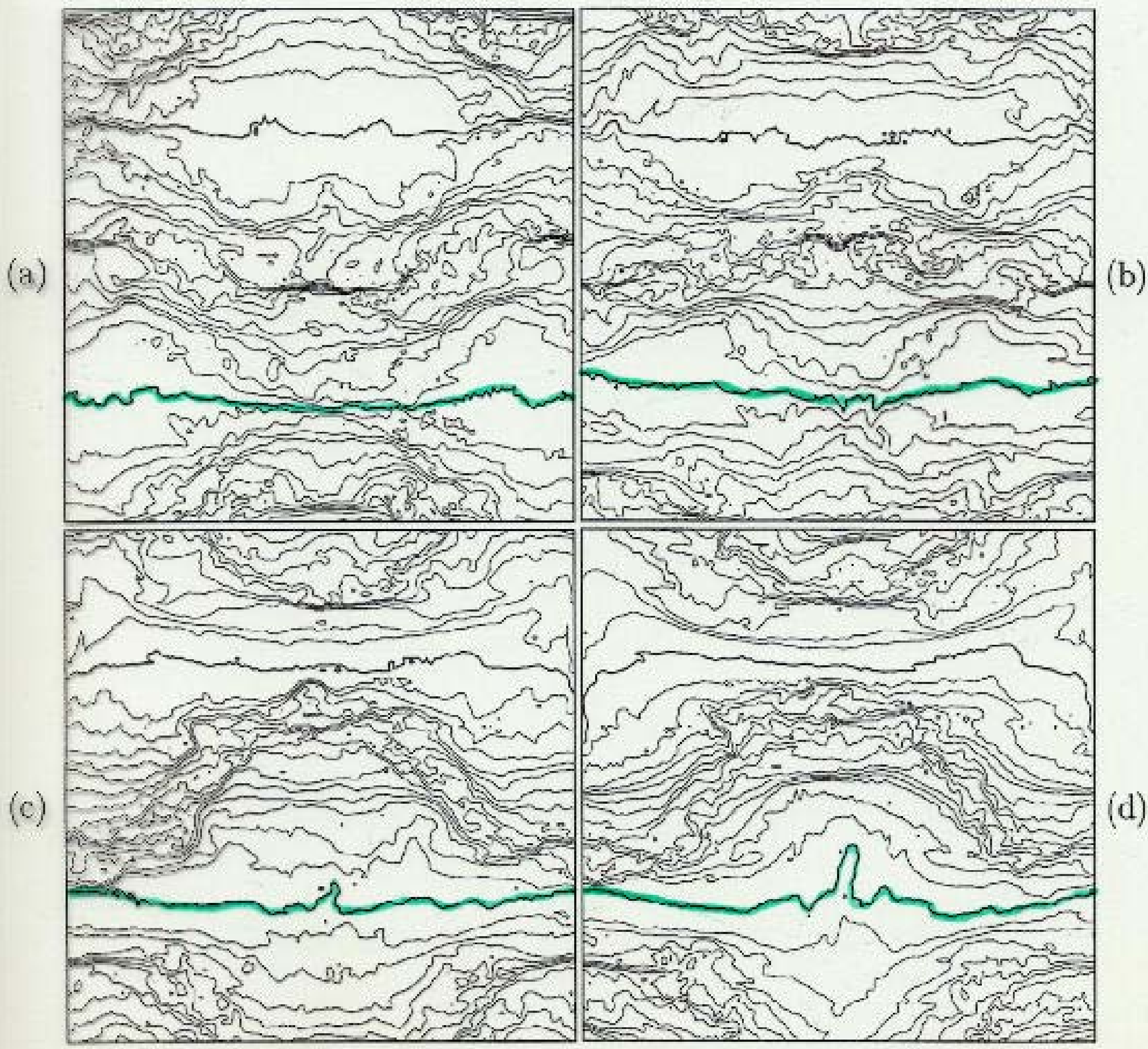
this simulation

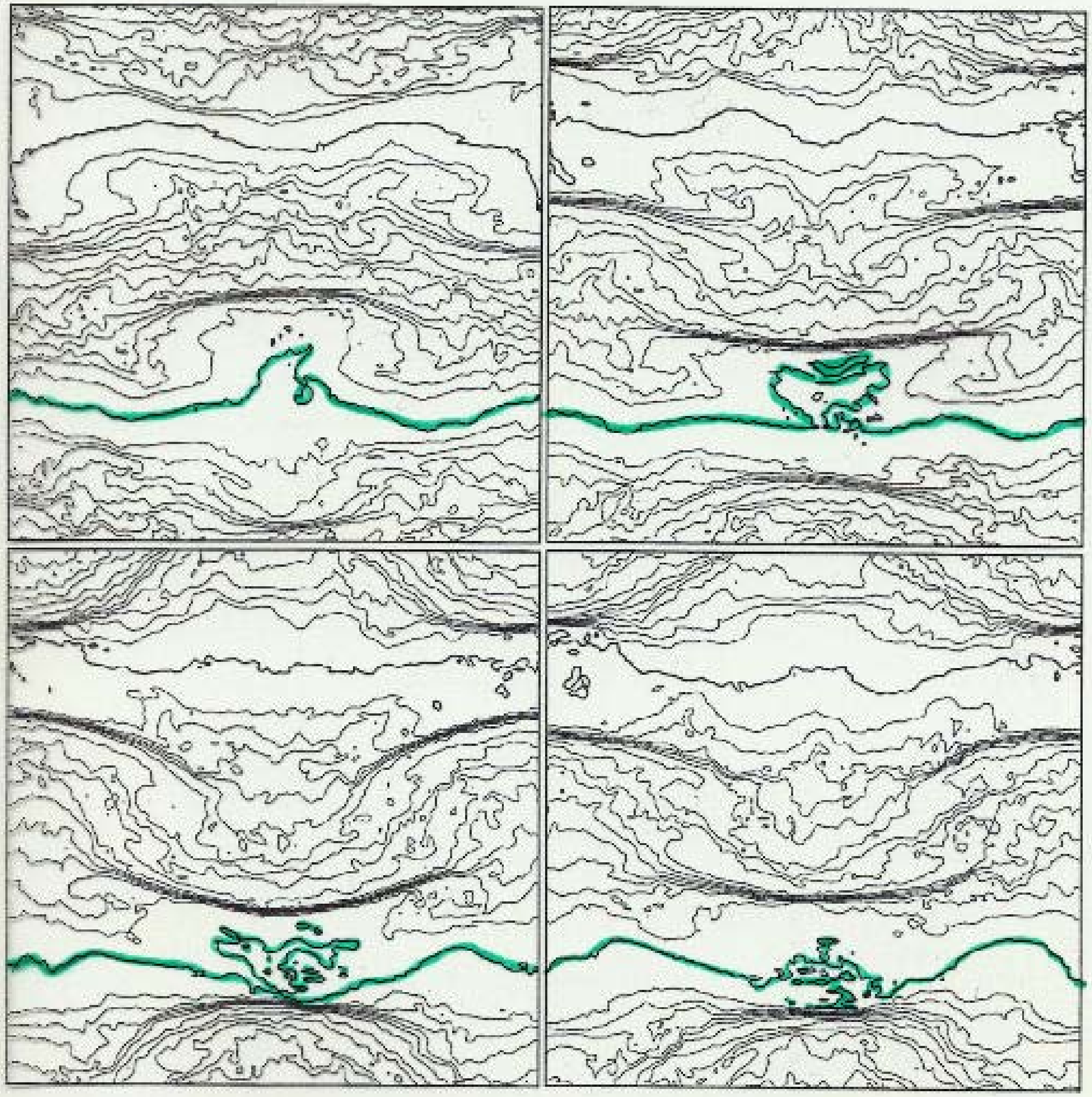
$$\epsilon = 1.3 \times 10^{-9} \ W/kg$$



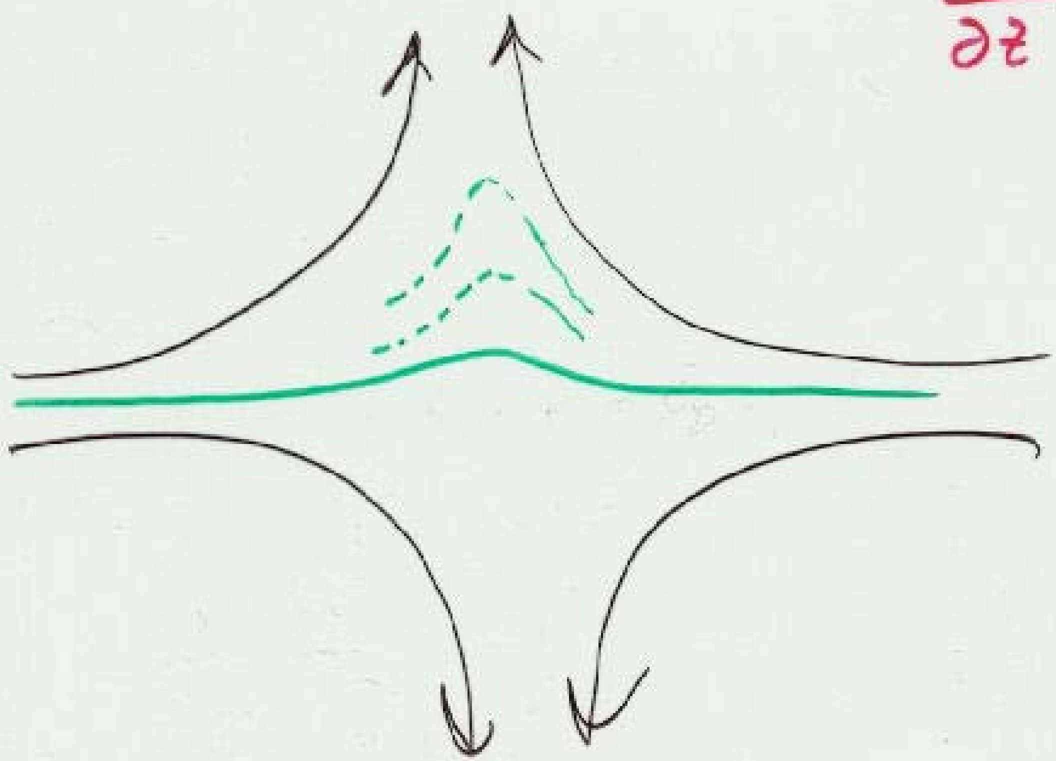




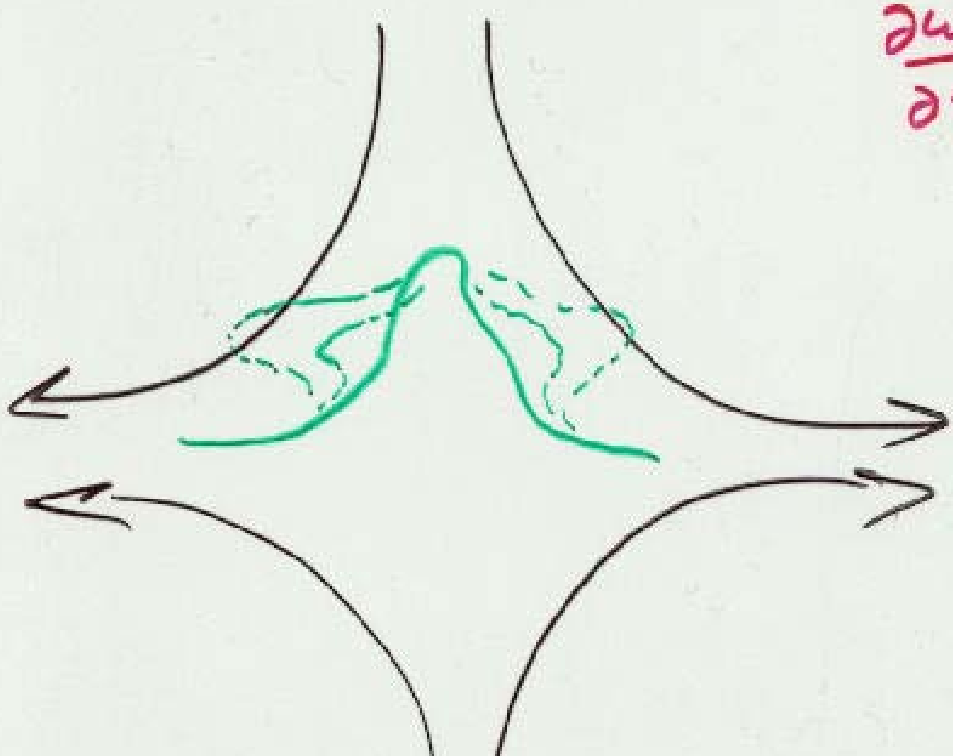

(a)
(b)
(c)
(d)


a)
(b)
:)
(d)

$$\frac{\partial w}{\partial z} > 0$$



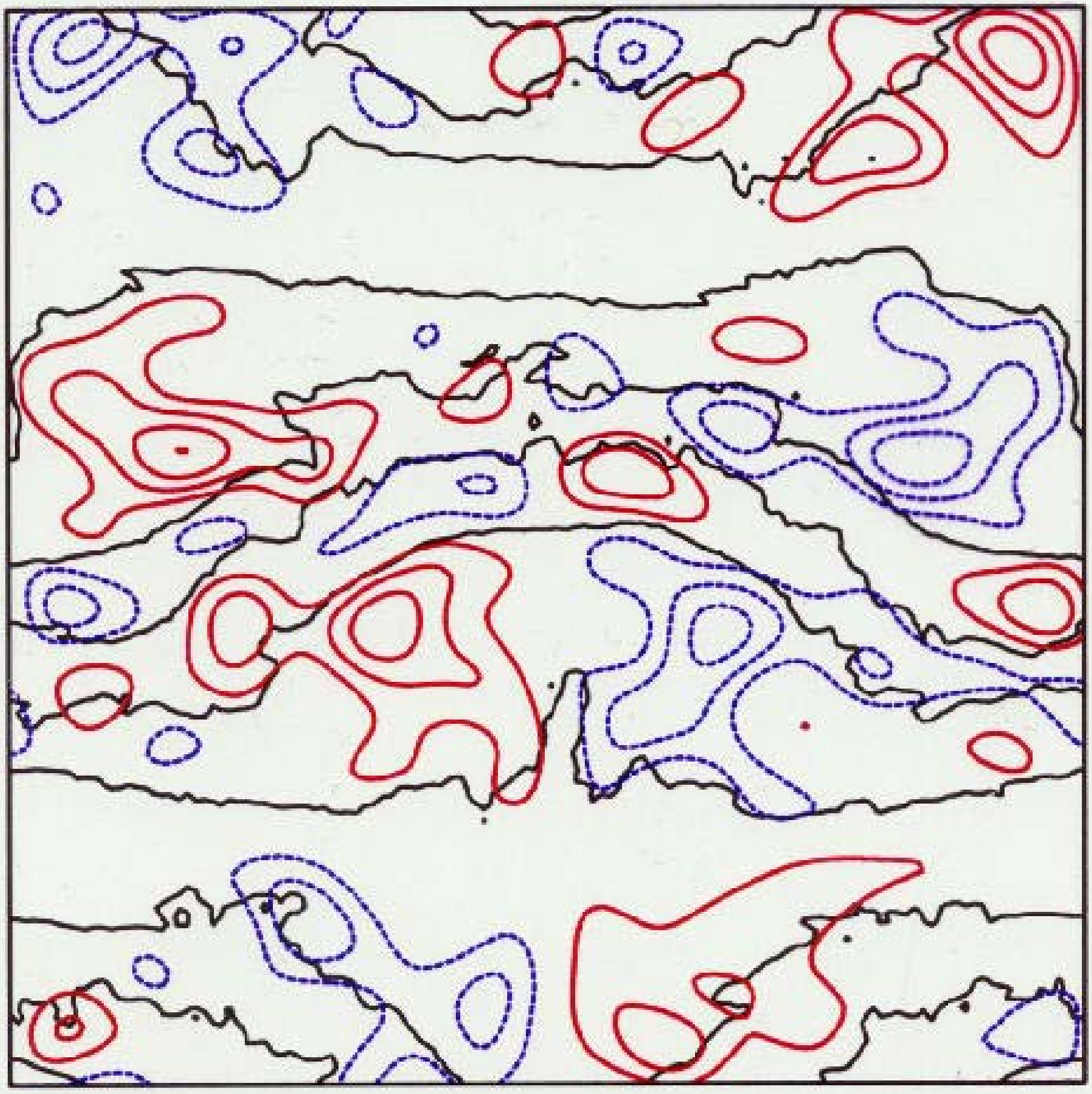
$$\frac{\partial w}{\partial z} < 0$$



Vorticity filtered at 2 m

phase lag 0.3 π

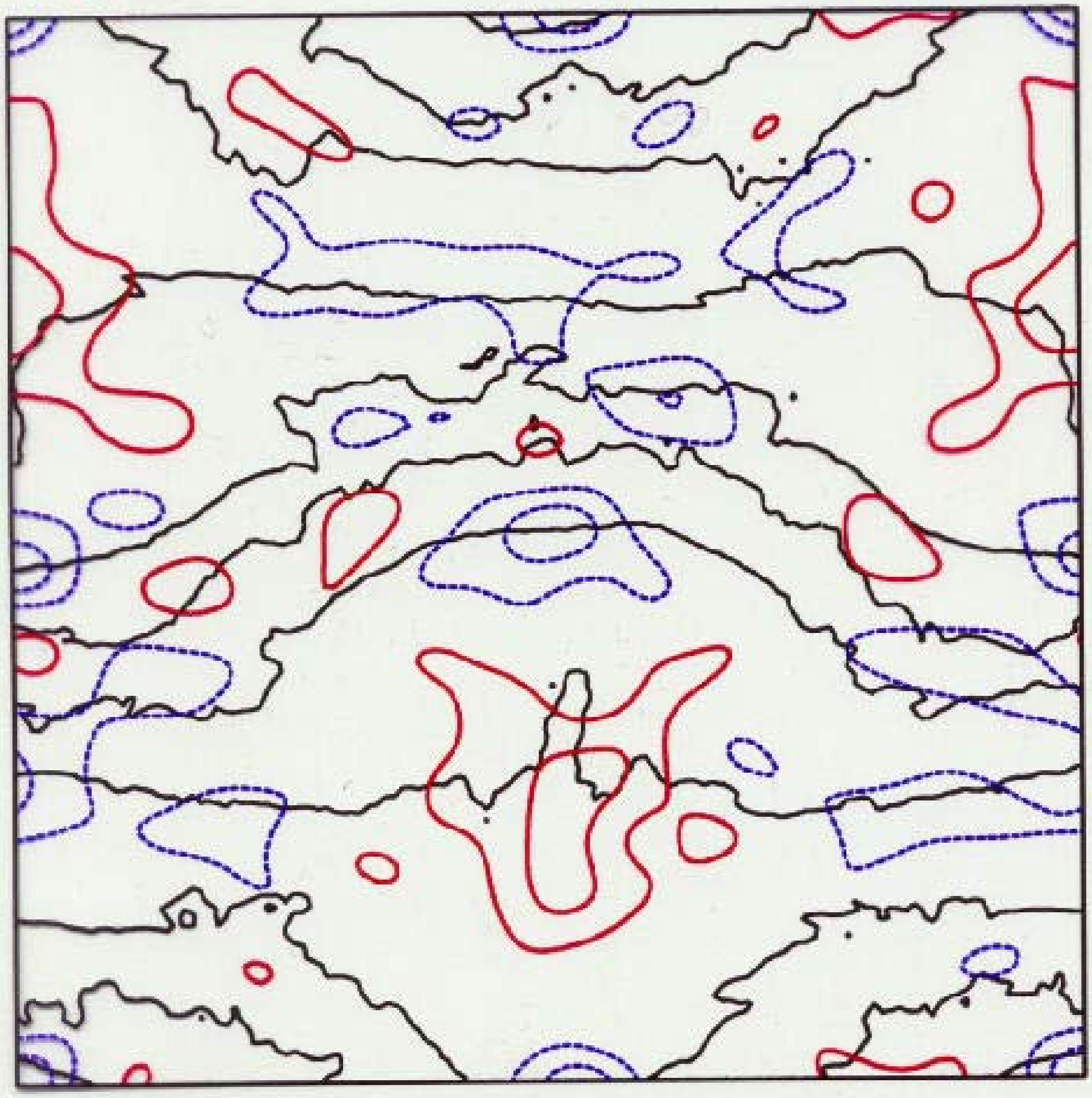
red +   blue -



$\frac{\partial w}{\partial z}$ filtered at 2m

phase lag 0.3π

red +   blue −



Wave Packet

$$\rho' = \mathrm{Re} \int G(\vec{k}-\vec{k}_0)\, e^{i(\vec{k}\cdot\vec{r}-\sigma t)}\, d^3k$$

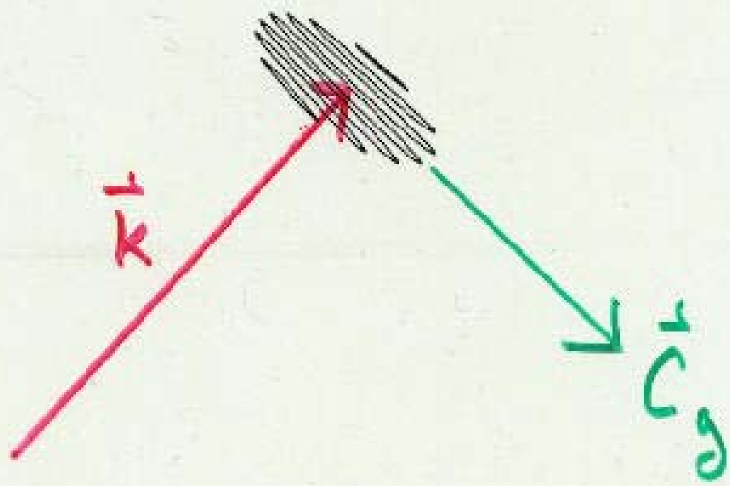
$$G(\vec{p}) =$$

$$\alpha \exp\left\{-\frac{a^2}{2}p_x^2 - \frac{b^2}{2}p_y^2 - \frac{c^2}{2}p_z^2\right\}$$

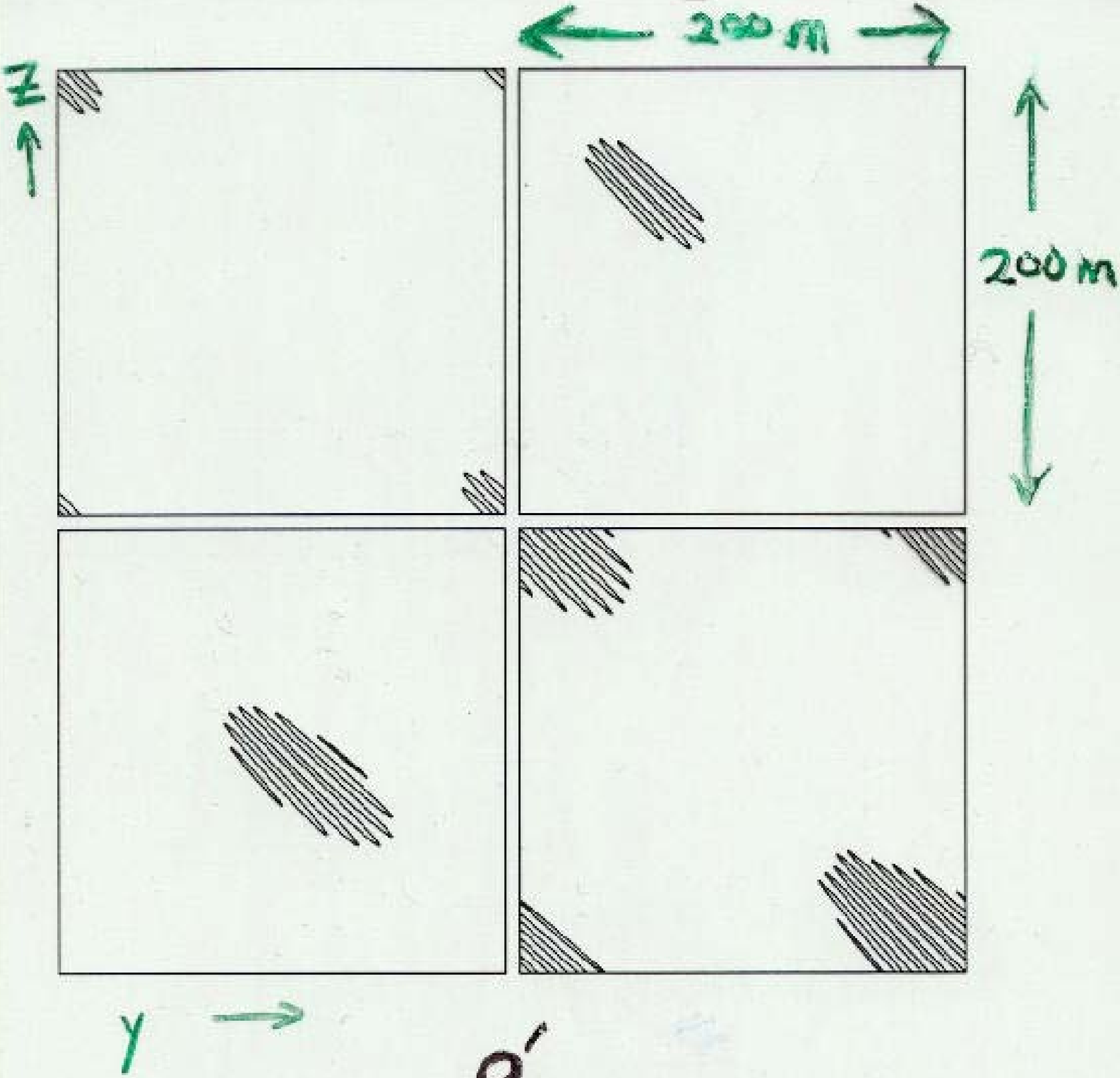
Group velocity

$$\mathbf{C_g} = \nabla_{\mathbf{k}}\omega_{\mathbf{k}} = \omega_{\mathbf{k}}\frac{k_z^2}{k^2 k_h^2}\left(k_x, k_y, \frac{-k_h^2}{k_z}\right)$$

$$\mathbf{k}\cdot\mathbf{C_g} = 0$$



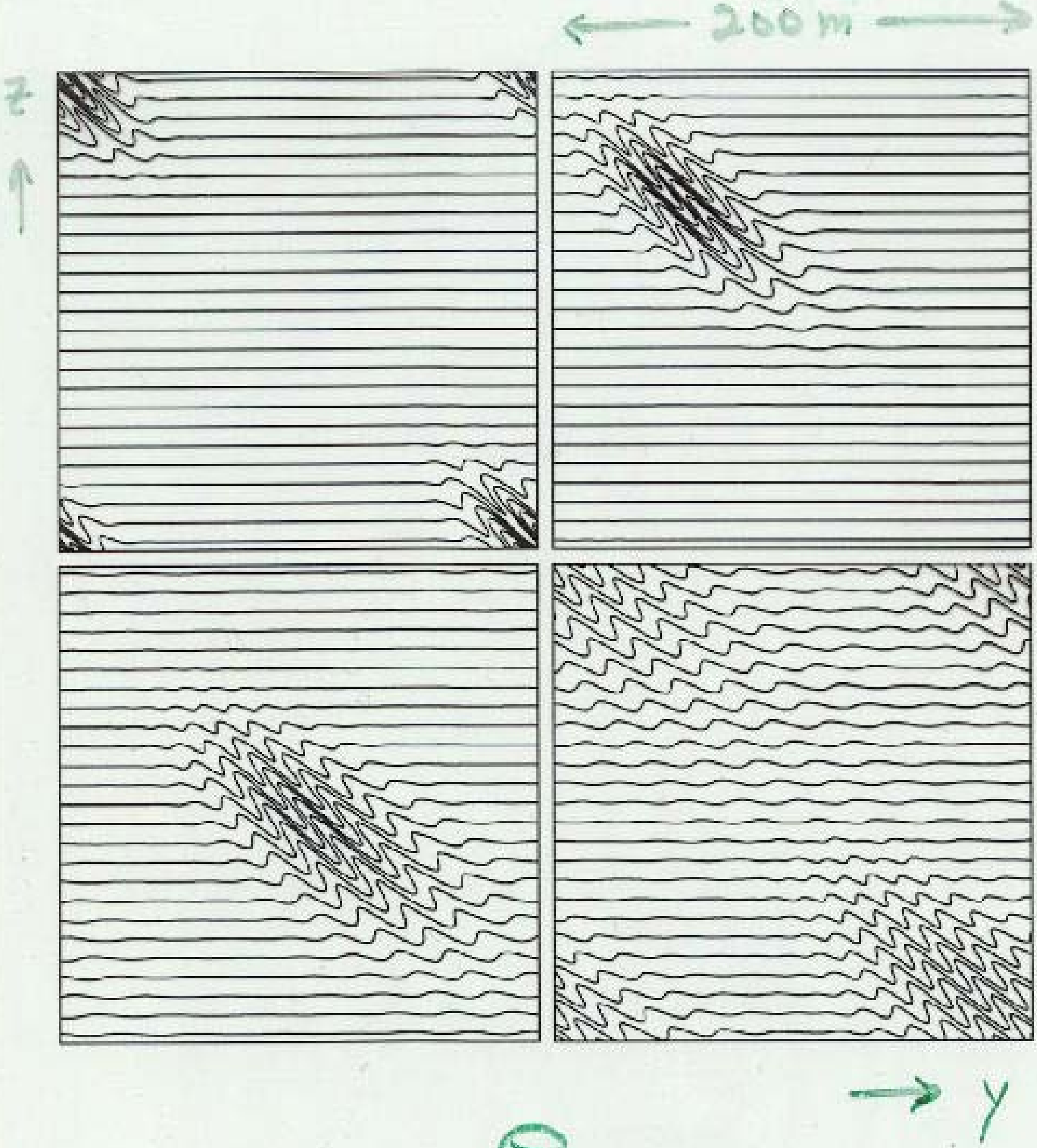
$$\sigma_k = \omega_k$$



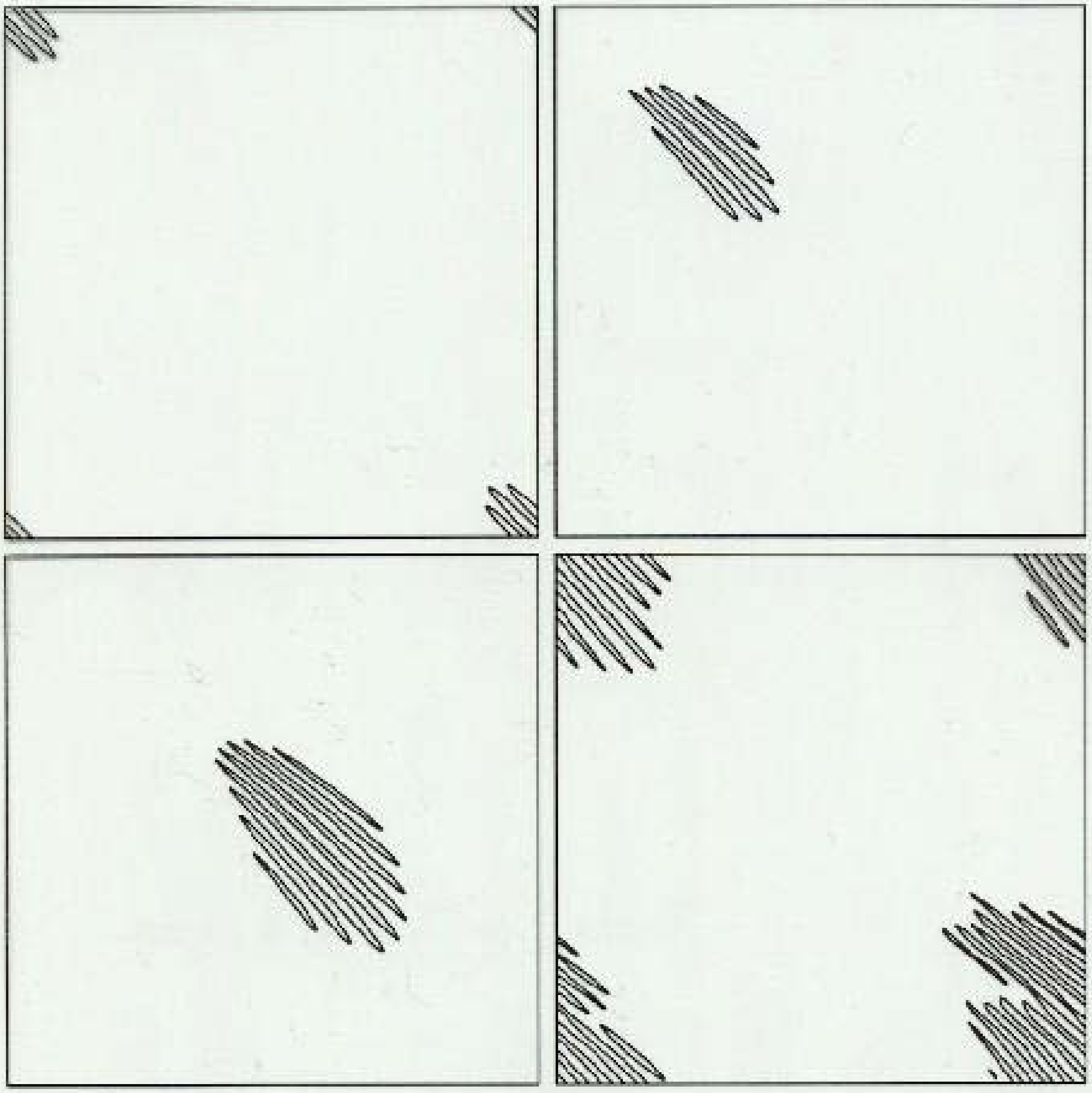
ρ'

LINEAR DYNAMICS

$t = 0 \rightarrow 10.5\,h$



LINEAR DYNAMICS



NONLINEAR DYNAMICS

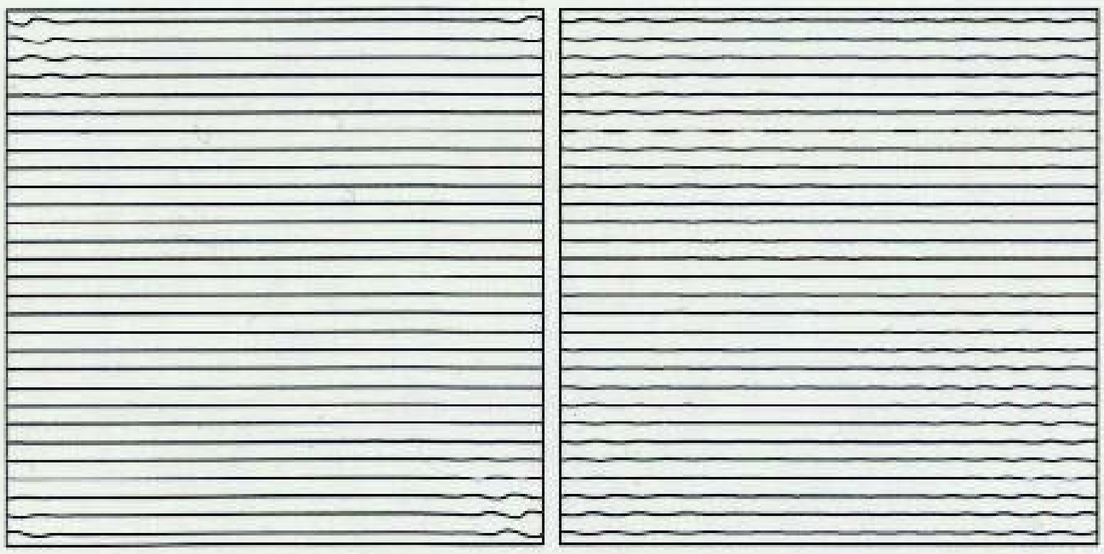
ρ'

Small amplitude

$$\left|\frac{\partial w}{\partial z}\right|_{max} \approx 0.4\,N$$

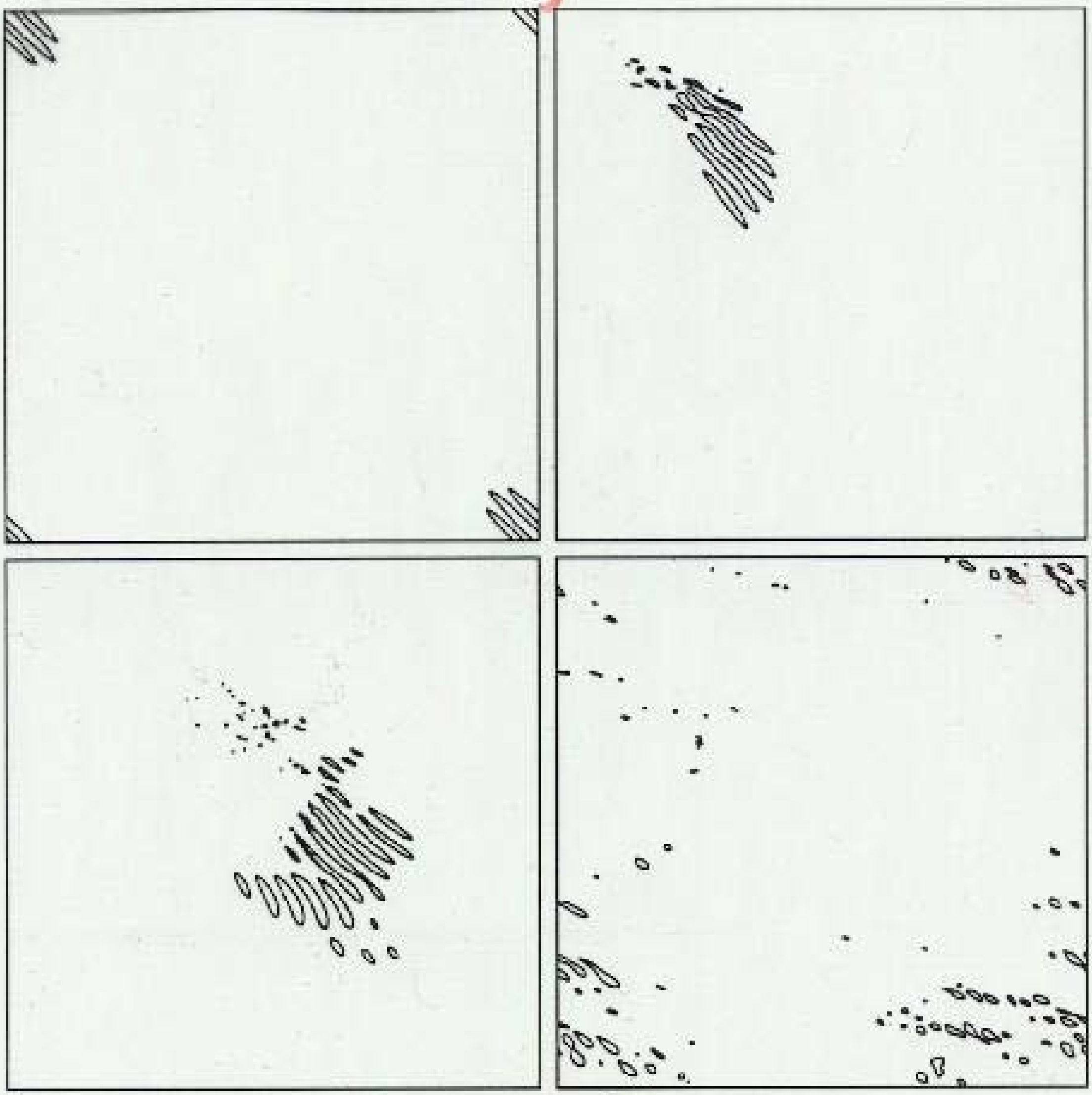
small amplitude

$t = 0$ $\qquad$ $t = 10.5\,h$



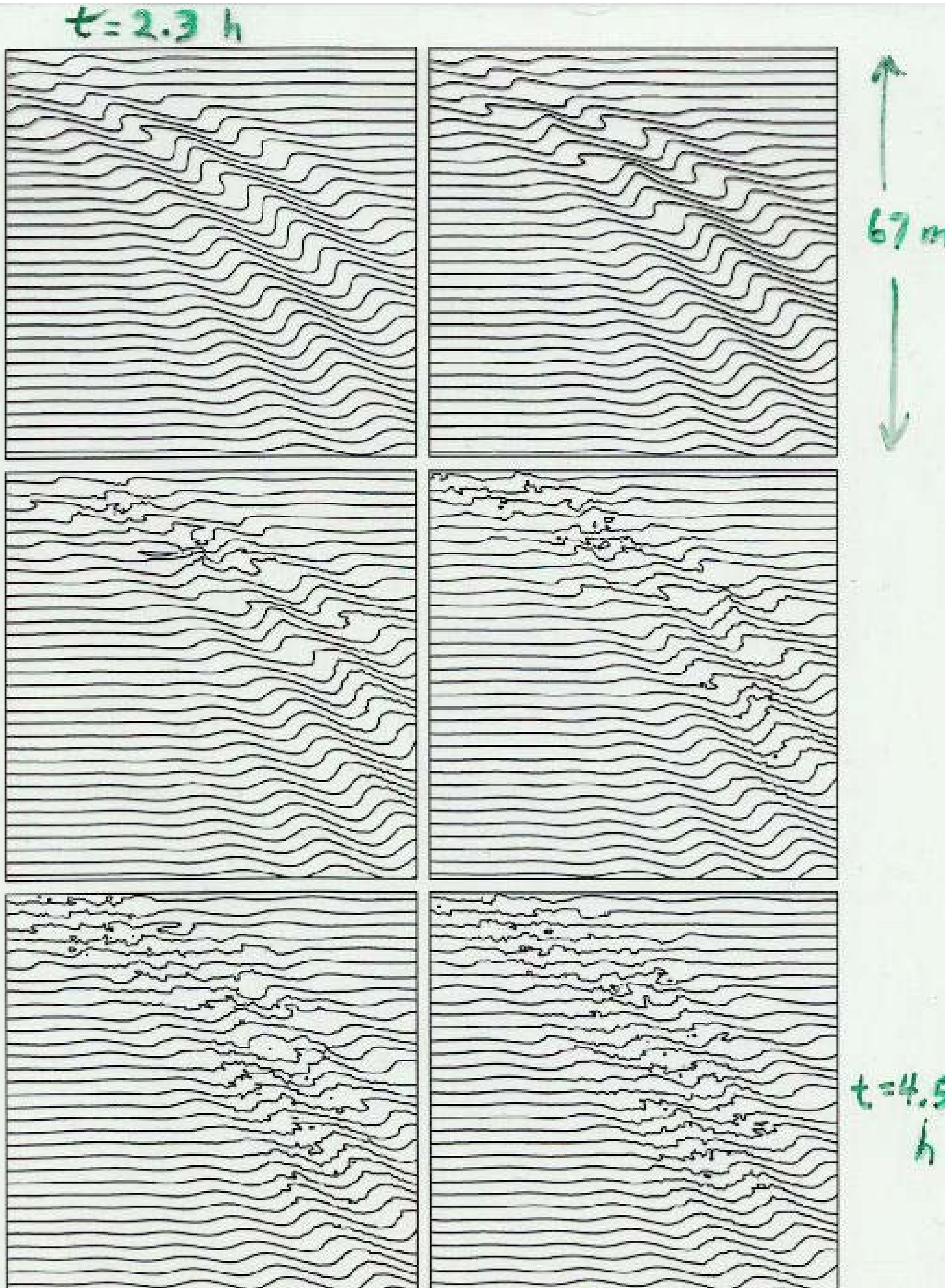
ρ

$$\left|\frac{\partial w}{\partial z}\right|_{MAX} \approx 0.4 N$$



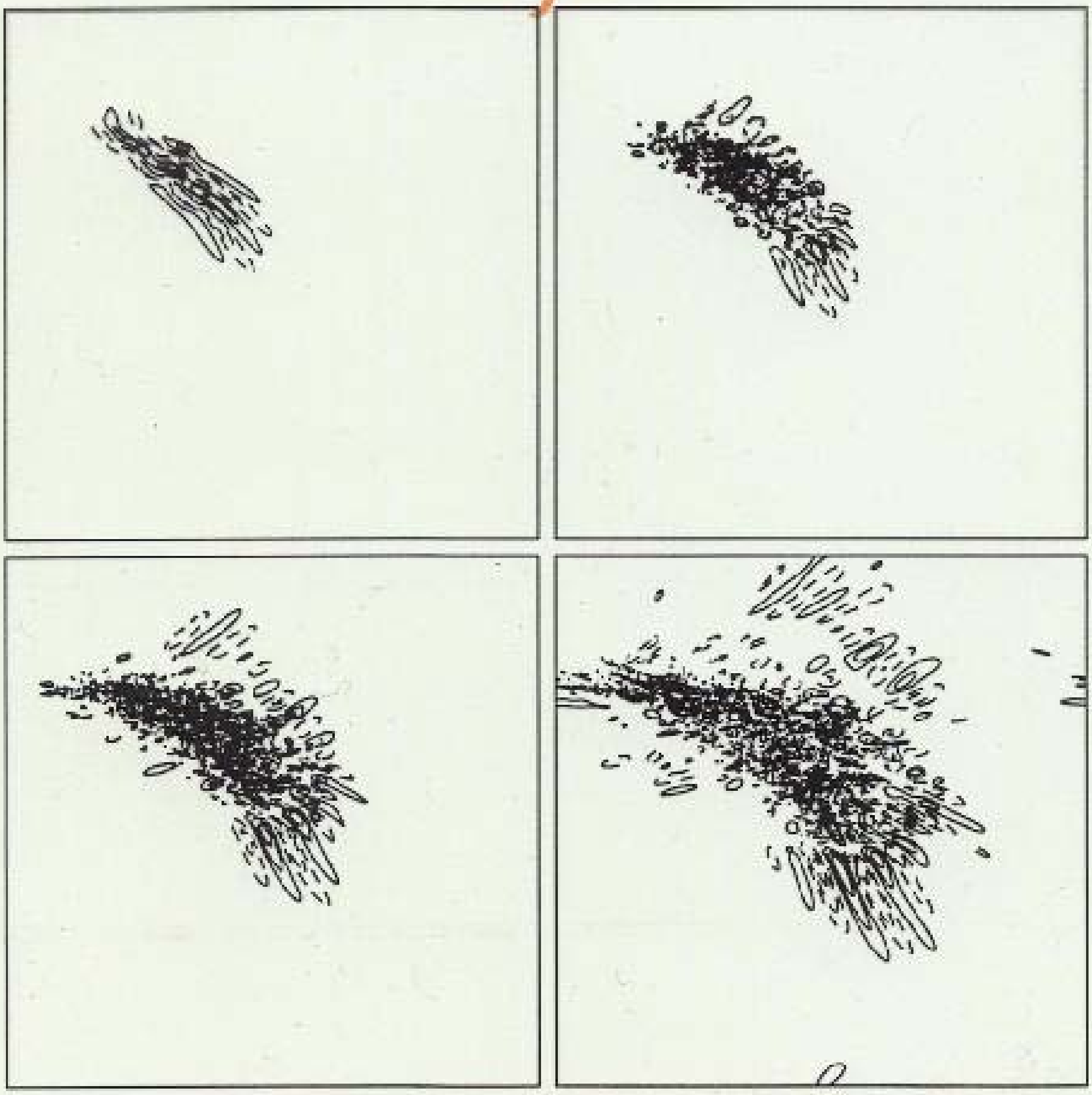
medium amplitude

$$\left|\frac{\partial w}{\partial z}\right|_{max} \approx 0.8\ N$$



MEDIUM AMP

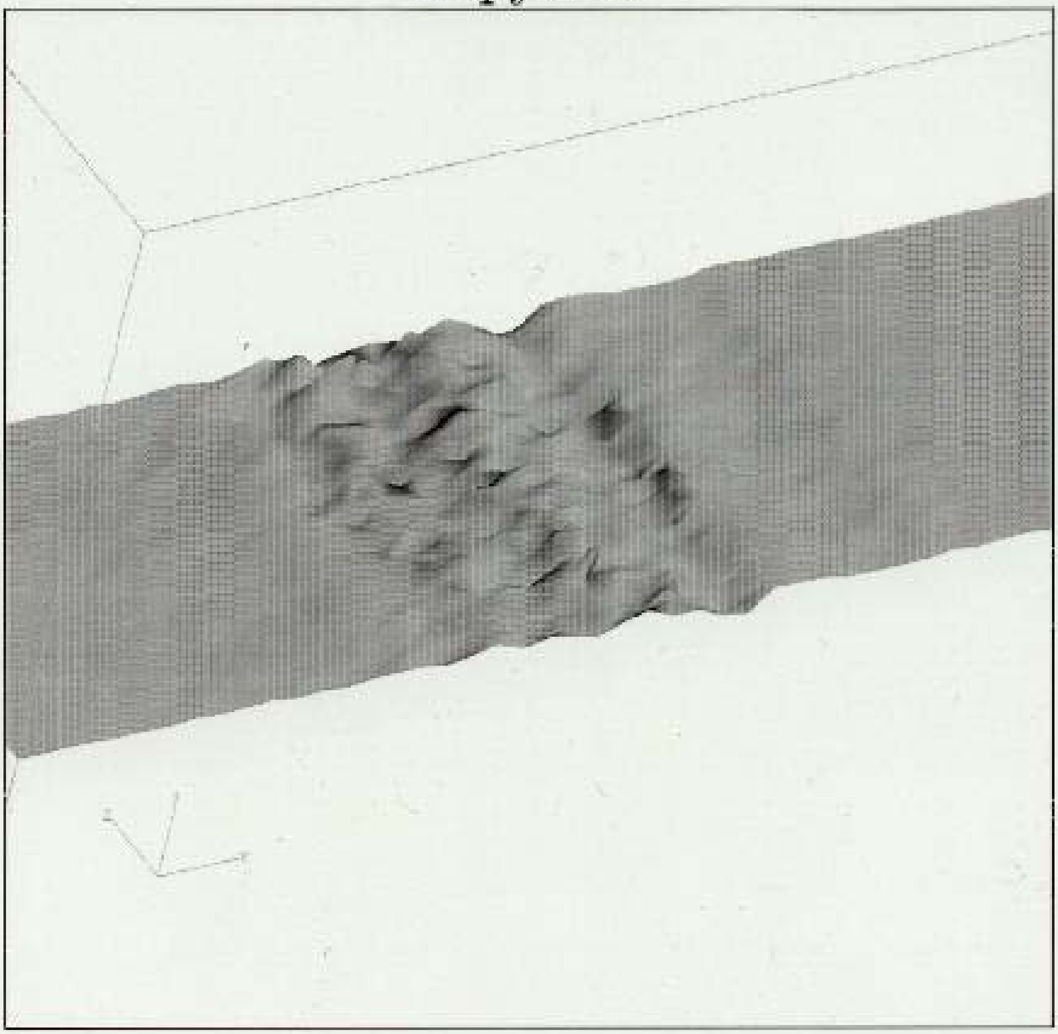
Close up



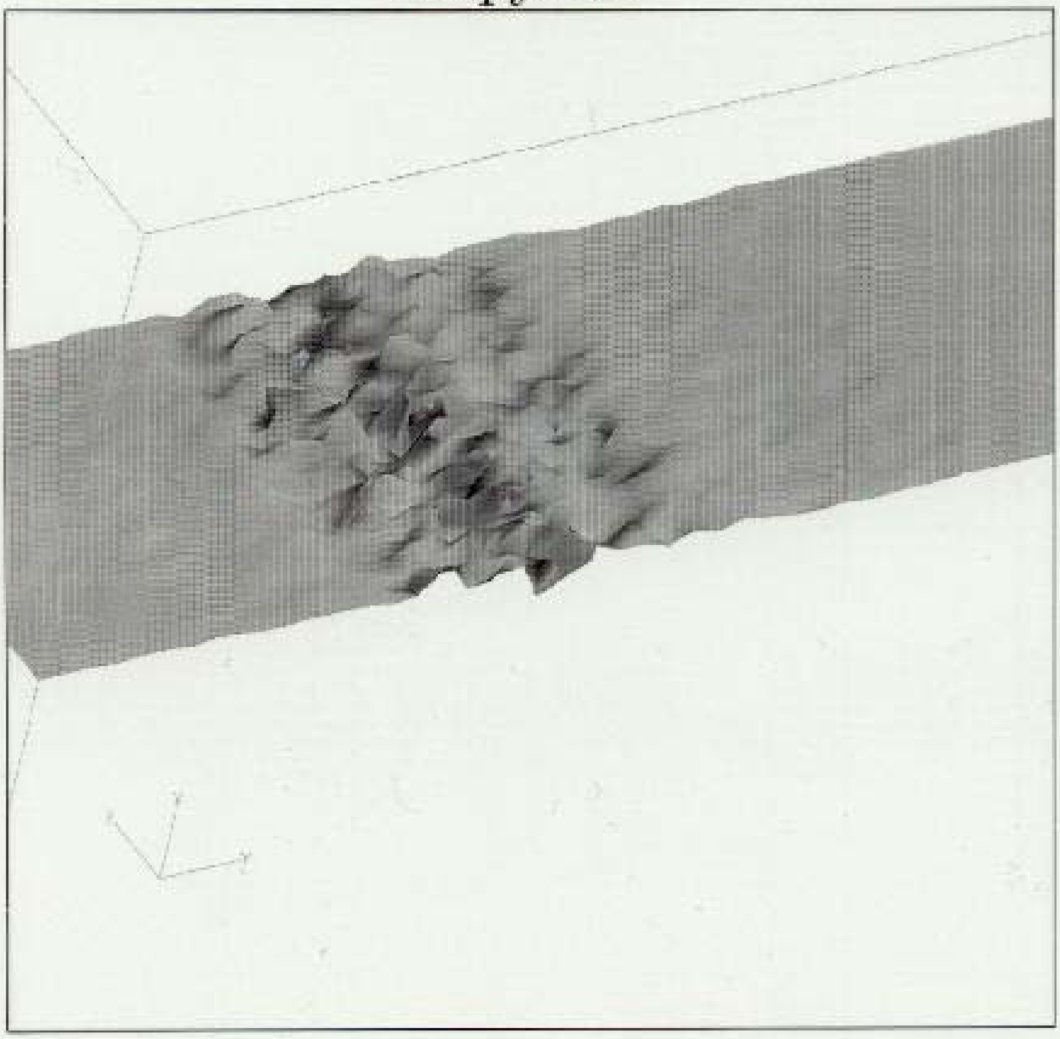
High Amp

$$\left|\frac{\partial w}{\partial z}\right|_{MAX} \approx 1.3\ N$$

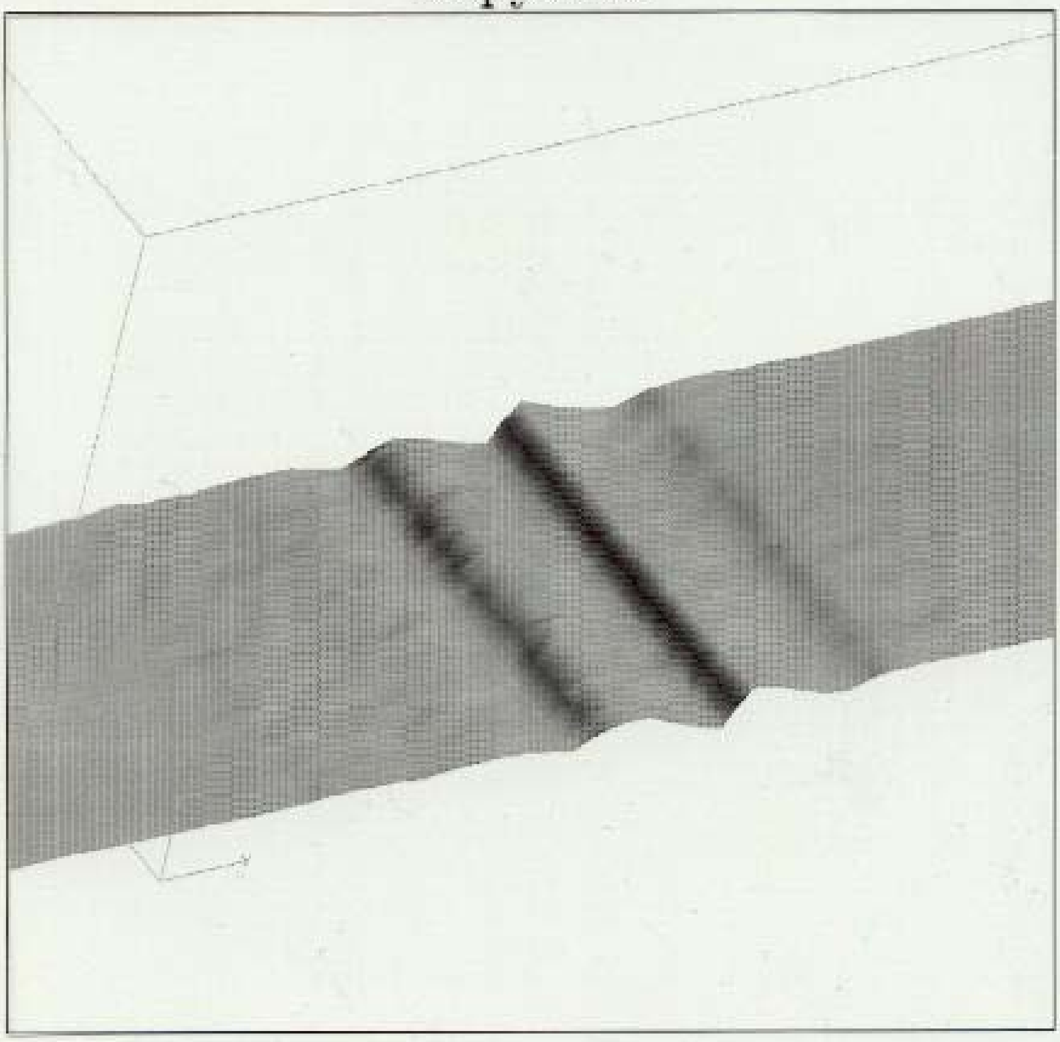
Isopycnal

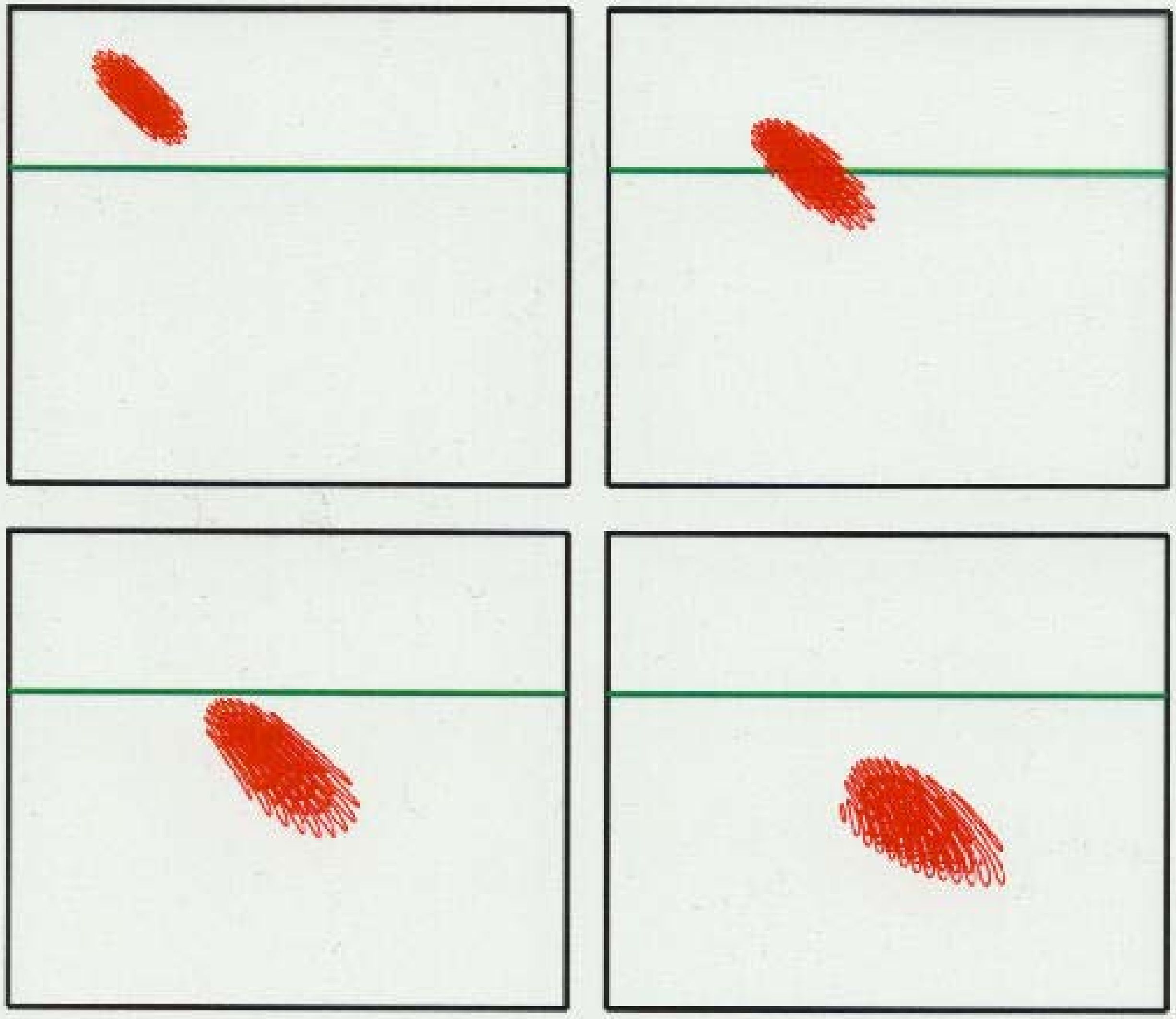


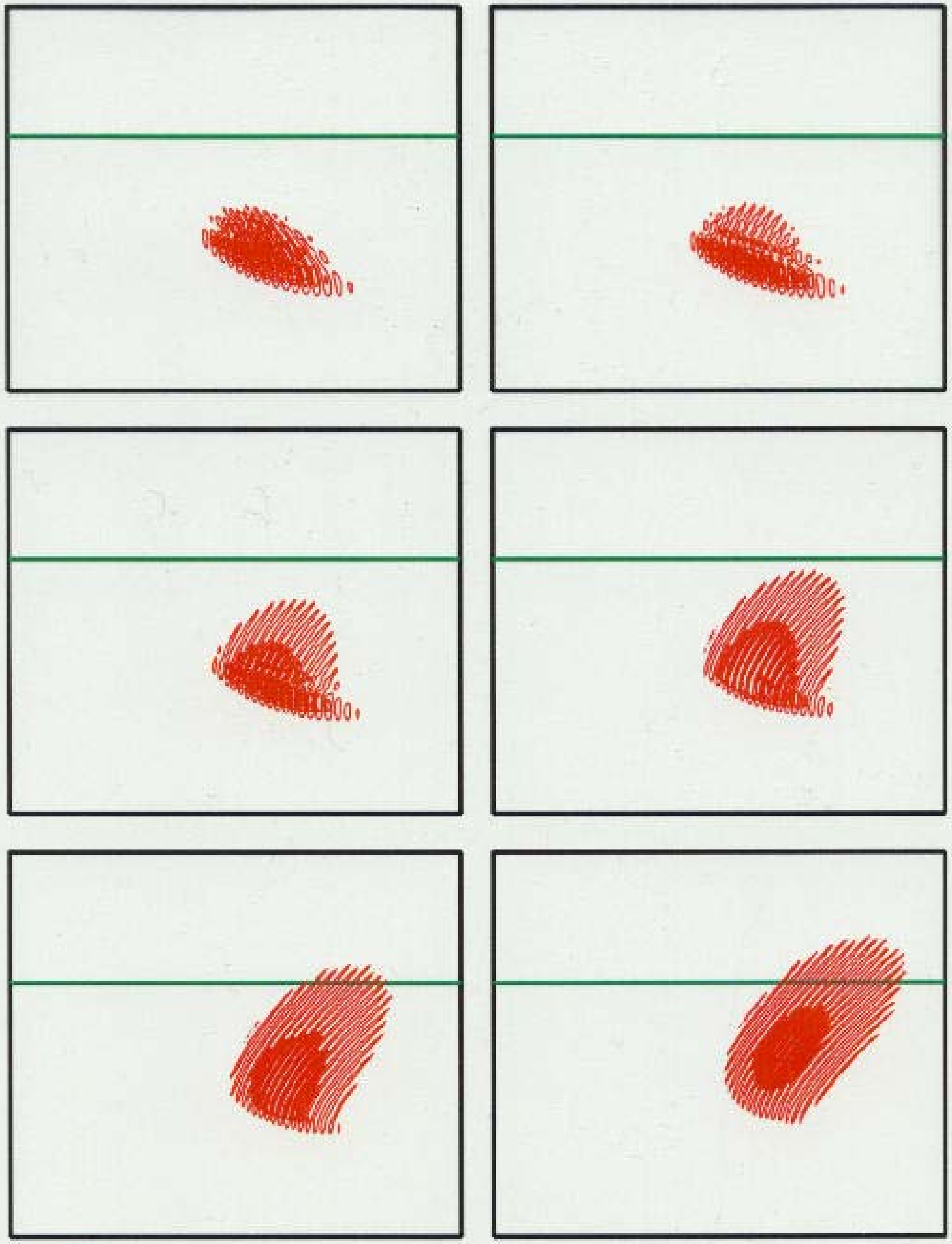
Isopycnal



Isopycnal







EDQNM

$$\frac{\partial U_{\vec{k}}}{\partial t} = \int d^3p\, d^3q\, \delta^3(\vec{k}+\vec{p}+\vec{q})\, \Theta_{\vec{k}\vec{p}\vec{q}}\, a_{\vec{k}\vec{p}\vec{q}} (U_{\vec{p}} U_{\vec{q}} - U_{\vec{q}} U$$

$$+ 2\nu k^2 U_{\vec{k}}$$

$$U_{\vec{k}} = \langle |v_{\vec{k}}|^2 \rangle$$

$$\Theta_{\vec{k}\vec{p}\vec{q}} = \frac{1}{\mu_{\vec{k}} + \mu_{\vec{p}} + \mu_{\vec{q}}}$$

$$E(k) = 2\pi k^2 U_{\vec{k}} \propto k^{-5/3}$$

$\Longrightarrow$ Cusp Model

EDQNM

$$\frac{\partial y_i}{\partial t} = \sum_{jk} A_{ijk} y_j y_k$$

MOMENT HIERARCHY

$$\frac{\partial}{\partial t} \langle yy \rangle = A \langle yyy \rangle$$

$$\frac{\partial}{\partial t} \langle yyy \rangle = A \langle yyyy \rangle$$

$$\vdots$$

CUMULANT EXPANSION

$$\frac{\partial}{\partial t} \langle yyy \rangle = A \langle yy \rangle \langle yy \rangle + A \langle yyyy \rangle_c$$

EDDY DAMPING

$$A \langle yyyy \rangle_c \longrightarrow -\mu \langle yyy \rangle$$

$t \gg 1/\mu$

$$\frac{\partial}{\partial t}\langle yy\rangle = \mu^{-1} AA \langle yy\rangle\langle yy\rangle$$

$\mu^{-1} = \Theta$

H-theorem

$$S = -\int P \ln P \prod_j dy_j$$

$$\rightarrow \frac{1}{2} \ln |\langle yy\rangle|$$

$$\frac{\partial S}{\partial t} \geq 0$$

$$\frac{\partial S}{\partial t} = 0 \iff \langle yy\rangle_{eq}$$

Given Invariants Byy

Rossby Waves

$$\frac{\partial \zeta_k}{\partial t} = -i\omega_k \zeta_k + \int d\vec{p}\, d\vec{q}\, A_{kpq}\, \zeta_p \zeta_q \times \delta^2(\vec{k}-\vec{p}-\vec{q})$$

EDQNM

$$\frac{\partial U_k}{\partial t} = \int d^2p\, d^2q\, \delta^2(\vec{k}+\vec{p}+\vec{q})\, \Theta_{kpq} \left(a_{kpq} U_p U_q + b_{kpq} U_q U_k \right)$$

$$\Theta_{kpq} = \frac{\mu_k + \mu_p + \mu_q}{(\mu_k + \mu_p + \mu_q)^2 + (\Omega_k + \Omega_p + \Omega_q)^2}$$

$$\Omega_k = \omega_k + NL$$

CONSERVATION LAWS

$$E = \int \zeta_k^2 / k^2 \quad d^2k$$

$$Z = \int \zeta_k^2 \quad d^2k$$

ENTROPY

$$S = \int \ln U_k \; d^2k$$

$$\frac{dS}{dt} \propto \int d^2k\, d^2p\, d^2q\; \theta_{kpq} \left(\frac{p \times q}{kpq} \right)^2 \delta^2(\vec{k} + \vec{p} + \vec{q})$$
$$\times \left[\frac{p^2 - q^2}{U_k} + \frac{q^2 - k^2}{U_p} + \frac{k^2 - p^2}{U_q} \right]^2$$

$$U_k \rightarrow \frac{1}{a + b k^2}$$

RI $\quad \mu \to 0$

$$\Theta_{kpq} \to \pi\, \delta(\omega_k + \omega_p + \omega_q)$$

$$n_k = U_k / \omega_k$$

$$\frac{\partial n_k}{\partial t} = \int d^2p\, d^2q\, D_{kpq}\, \delta^2(k+p+q)\, \delta(\omega_k + \omega_p + \omega_q)$$
$$[n_p n_q + n_k n_p + n_k n_q]$$

Conservation Laws

$$E = \int n_k \omega_k = \int U_k$$

$$Z = -\int n_k k_x = +\int K^2 U_k$$

$$P_y = \int n_k k_y = -\int \frac{k_y U_k K^2}{k_x}$$

RI

$$S = \int \ln n_k \, d^2k$$

$$\frac{\partial S}{\partial t} \propto \int d^2k\, d^2p\, d^2q \;\; \delta^2(k+p+q)\, \delta(\omega_k + \omega_p + \omega_q)$$

$$\left(\frac{D_{kpq}}{kpq}\right)^2 U_k U_p U_q \left(n_k^{-1} + n_p^{-1} + n_q^{-1}\right)^2$$

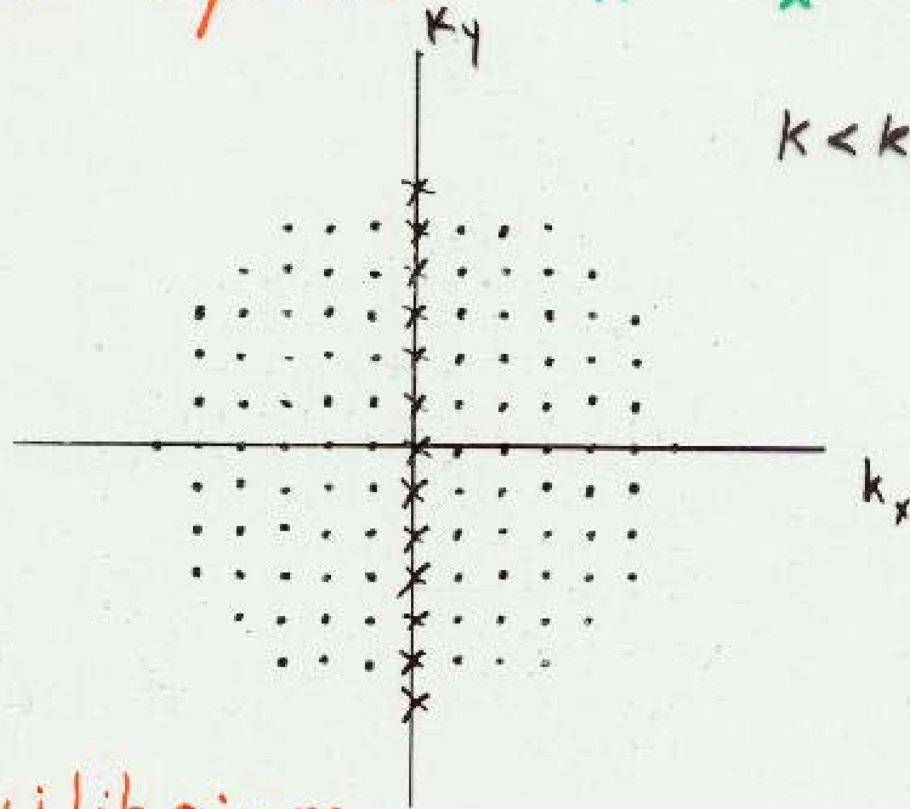
$$n_k \longrightarrow (a\,\omega_k + b\,k_x + c\,k_y)^{-1}$$

$$U_k = \frac{1}{a + bk^2 + \frac{ck^2 k_y}{k_x}}$$

Truncated System

NO $k_x = 0$ modes

$k < k_{MAX}$



RI Equilibrium

$$U_k = \frac{1}{a + b k^2 + c \, \underline{k_y k^2 / k_x}}$$

$$a + b\,k^2 + c\,k_y k^2/k_x \;>\; 0$$

Quasi-homogeneous

$$U(x_1, x_2) = \langle \vec{u}(x_1) \cdot \vec{u}(x_2) \rangle$$

$$= U(x; X)$$

$$x = x_2 - x_1$$

$$X = \tfrac{1}{2}(x_1 + x_2)$$

$$\frac{\partial U_k}{\partial T} + \frac{\partial \Omega_{\vec{k}}}{\partial \vec{k}} \cdot \frac{\partial U_k}{\partial \vec{X}} - \frac{\partial \Omega_{\vec{k}}}{\partial \vec{X}} \cdot \frac{\partial U_k}{\partial k}$$

$$= \int d^2p \, d^2q \, \delta^2(k+p+q) \, \theta_{kpq}$$

$$\left[a_{kpq} U_p U_q + b_{kpq} U_k U_q \right]$$

$$+ \cdots$$

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