$\mathcal{Z}$

The Jiang—Su algebra

After several successful classification results were obtained by lifting maps from the level of K-theory and tracial states, George Elliott conjectured that it might be possible to classify all simple, separable, unital, nuclear C*-algebras by the “Elliott Invariant”:

$$\mathrm{Ell}(A) := (K_0(A), K_0(A)_+, [1_A], K_1(A), T(A), \rho : T(A) \to S(K_0(A), [1_a]))$$

Karen Strung, Institute of Mathematics, Czech Academy of Sciences

If $A \cong \mathbb{C}$, a perfectly nice simple, separable, unital, nuclear C*-algebra, then we have the following:

$$\mathbf{Ell}(\mathbb{C}) := (\mathbb{Z}, \mathbb{Z}_{>0}, [1], 0, \{\tau = \mathbf{id}\}, \rho(\tau)(n) = n))$$

Jiang and Su set out to answer the following question:

satisfactory classification of purely infinite separable nuclear C^*-algebras (cf. Kirchberg [33], Phillips [38]). Note that, if $A \neq 0$ is a unital simple nuclear C^*-algebra and $A \cong A \otimes \mathcal{O}_\infty$, then A must be purely infinite [33]. In light of these developments, it has become desirable to find an analog of $\mathcal{O}_\infty$ for a broader class of C^*-algebras that includes some stably finite C^*-algebras. As a test case, we wish to include the class of AF algebras (of Bratteli [7]), which was the first class of C^*-algebras classified using K-theory (cf. Elliott [19]). Searching for an analog of $\mathcal{O}_\infty$ for this class leads quickly to the following modified version of Question 1:

Question 2. Is there a unital simple nuclear C^*-algebra A, infinite-dimensional and having a unique tracial state, such that $K_0(A) \cong K_0(\mathbb{C})$ as scaled ordered groups and $K_1(A) \cong K_1(\mathbb{C})(\cong 0)$?

Since $\mathbb{C}$ is certainly *not* infinite dimensional, this would imply that Elliott's conjecture would obviously need to be reformulated so that one only considers simple, separable, unital, nuclear, *infinite dimensional* C*-algebras.

NBD, right?

But even for infinite-dimensional C*-algebras, such a C*-algebra $\mathscr{Z}$ would have further interesting implications.

For example, the Künneth theorem tells us that

$$K_0(A \otimes \mathcal{Z}) \cong K_0(A) \otimes K_0(\mathcal{Z}) \oplus K_1(A) \otimes K_1(\mathcal{Z}) \cong K_0(A),$$

and

$$K_1(A \otimes \mathcal{Z}) \cong K_0(A) \otimes K_1(\mathcal{Z}) \oplus K_1(A) \otimes K_0(\mathcal{Z}) \cong K_1(A).$$

We also have that the map

$$T(A) \to T(A \otimes \mathcal{Z}), \qquad \tau \mapsto \tau \otimes \tau_{\mathcal{Z}},$$

where $\tau_{\mathcal{Z}}$ is the unique tracial state of $\mathcal{Z}$, gives us an affine homeomorphism $T(A) \cong T(A \otimes \mathcal{Z})$

Jiang and Su's construction

$\mathcal{Z}$ can be constructed as an inductive limit of *dimension-drop algebras.*

15.1.1. Definition. Let $p, q, d \in \mathbb{N}\backslash\{0\}$ with both p and q dividing d. The *dimension-drop algebra* $I(p, d, q)$ is defined to be

$$I(p, d, q) := \{f \in C([0,1], M_d) \mid f(0) \in M_p \otimes 1_{d/p}, f(1) \in 1_{d/q} \otimes M_d\}.$$

If p and q are relatively prime, then we call $I(p, pq, q)$ a *prime dimension-drop algebra*. In this case, we sometimes simply write $I(p, q)$, since it is understood that $d = pq$.

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If p and q are relatively prime, then we call $I(p,pq,q)$ a *prime dimension-drop algebra*. In this case, we sometimes simply write $I(p,q)$, since it is understood that $d = pq$.

Prime dimension-drop algebras contain only the trivial projections 0 and 1.

In fact,

LEMMA 2.3. *Let* $A = \mathbf{I}[m_0, m, m_1]$. *Then:*

(1) $(K_0(A), K_0^+(A), [1_A]) \cong (\mathbb{Z}, \mathbb{N}, r)$, *where* $r = (m_0, m_1)$ *denotes the greatest common divisor of* m_0 *and* m_1;

(2) $K_1(A) \cong \mathbb{Z}_p$, *where* $p = mr/(m_0 m_1)$.

In particular, prime dimension-drop algebras have vanishing K_1.

Since K-theory is continuous with respect to inductive limits, an inductive limit of prime dimension-drop algebras will already give us the K-theory we're after.

To get something simple and monotracial, one has to make sure the matrix sizes grow to infinity and that the connecting maps "shrink" the spectrum.

PROPOSITION 2.5. *There exists an inductive sequence* $A_1 \xrightarrow{\phi_1} A_2 \xrightarrow{\phi_2} A_3 \xrightarrow{\phi_3} \cdots$, *where each* $A_n = \mathbf{I}[p_n, d_n, q_n]$ *is a prime dimension drop algebra, such that each connecting map* $\phi_{m,n} = \phi_{n-1} \circ \cdots \circ \phi_{m+1} \circ \phi_m \colon A_m \to A_n$ *is an injective morphism of the form:*

$$(2.1) \qquad \phi_{m,n}(f) = u^* \begin{bmatrix} f \circ \xi_1 & 0 & \cdots & 0 \\ 0 & f \circ \xi_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & f \circ \xi_k \end{bmatrix} u, \quad \forall f \in A_m,$$

where u *is a continuous path in* $\mathbf{U}_{d_n}$, *and* $\{\xi_i\}$ *is a sequence of continuous paths in* $[0, 1]$, *each of which satisfies the following:*

$$(2.2) \qquad |\xi_i(x) - \xi_i(y)| \leq (1/2)^{n-m}, \quad \forall x, y \in [0, 1].$$

That's enough to show the following:

PROPOSITION 2.8. *Let* (A_m, ϕ_m) *be any sequence as in Proposition 2.5, then its limit,* $\varinjlim (A_m, \phi_m)$*, is a unital simple* C^**-algebra with a unique tracial state.*

From which it follows that

THEOREM 2.9. *There exists an infinite-dimensional unital simple limit* $\mathcal{Z}$ *of dimension drop algebras, such that* $\mathcal{Z}$ *has a unique tracial state,* $K_0(\mathcal{Z}) \cong K_0(\mathbb{C})$ *as scaled ordered groups, and* $K_1(\mathcal{Z}) = K_1(\mathbb{C}) = 0$.

The $\mathscr{Z}$ of Theorem 2.9 is what we now call the Jiang—Su algebra!

A C*-algebra A is said to be *$\mathscr{Z}$-stable* or *$\mathscr{Z}$-absorbing* if $A \cong A \otimes \mathscr{Z}$.

Jiang and Su showed that

- if A is purely infinite, unital, simple and nuclear, then A is $\mathscr{Z}$-stable
- if A is unital simple approximately finite (AF) then A is $\mathscr{Z}$-stable
- $\mathscr{Z}$ is $\mathscr{Z}$-stable

In fact, $\mathscr{Z}$ is *strongly self-absorbing*.

A unital C*-algebra $\mathscr{D}$ is *strongly self-absorbing* if there exists a *-isomorphism

$$\varphi : \mathscr{D} \to \mathscr{D} \otimes \mathscr{D}$$

that is approximately unitarily equivalent to the first factor embedding:

there exist unitaries $(u_n)_{n\in\mathbb{N}} \subset \mathscr{D} \otimes \mathscr{D}$ s.t.

$$\|u_n\varphi(a)u_n^* - a \otimes 1_{\mathscr{D}}\| \to 0, \quad n \to \infty\,.$$

A strongly self-absorbing C*-algebra is automatically nuclear and simple. The only other strongly self-absorbing C*-algebras in the UCT class are $M_{n^\infty}, \mathscr{O}_\infty, \mathscr{O}_\infty \otimes M_{n^\infty},$ and $\mathscr{O}_2$.

For any strongly self-absorbing $\mathscr{D}$, we have $\mathscr{Z} \otimes \mathscr{D} \cong \mathscr{D}$ while $\mathscr{O}_2 \otimes \mathscr{D} \cong \mathscr{O}_2$.

One can similarly talk about $\mathscr{D}$-stability with respect to these other strongly self-absorbing C*-algebras. For example, if A is a UCT unital purely infinite simple C*-algebra then $A \otimes \mathscr{O}_\infty \cong \mathscr{O}_\infty$.

Indeed, finding a stably finite analogue to $\mathscr{O}_\infty$ was one of Jiang and Su's motivations.

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satisfactory classification of purely infinite separable nuclear C^*-algebras (cf. Kirchberg [33], Phillips [38]). Note that, if $A \neq 0$ is a unital simple nuclear C^*-algebra and $A \cong A \otimes \mathcal{O}_\infty$, then A must be purely infinite [33]. In light of these developments, it has become desirable to find an analog of $\mathcal{O}_\infty$ for a broader class of C^*-algebras that includes some stably finite C^*-algebras. As a test case, We noted earlier that A and $A \otimes \mathcal{Z}$ have the same K_0-group, K_1-group and traces. But the Künneth theorem doesn't say anything about the order structure on K_0.

An ordered abelian group (G, G_+) is *weakly unperforated* if whenever $x \in G$ and there is an integer $n > 0$ such that $nx \in G_+$, then we must have $x > 0$.

This provides an obstruction to $\mathcal{Z}$-stability [Gong, Jiang, Su, '00]:

Theorem 1 *Let A be a unital simple C^*-algebra. Then:*

(a) $K_0(A \otimes \mathcal{Z})$ *is weakly unperforated;*
(b) $\iota_* : K_0(A) \to K_0(A \otimes \mathcal{Z})$ *is an isomorphism of pre-ordered groups if and only if* $K_0(A)$ *is weakly unperforated.*

It follows that Elliott's original conjecture predicts $\mathscr{Z}$-stability for any simple, separable, unital, nuclear, infinite-dimensional C*-algebra A such that $((K_0(A), K_0(A)_+)$ is weakly unperforated.

In the presence of weak unperfoation, the pairing map $\rho : T(A) \to S(K_0(A), [1_A])$ contains the same information as the order structure on $K_0(A)$.

Some notable examples of "wild" simple, separable, unital and nuclear C*-algebras, eg. Villadsen algebras, do not have weakly unperforated K_0 .

However, weak unperforation in K_0 is not enough to guarantee $\mathscr{Z}$-stability.

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On the classification problem for nuclear C^*-algebras

By Andrew S. Toms

Abstract

We exhibit a counterexample to Elliott's classification conjecture for simple, separable, and nuclear C*-algebras whose construction is elementary, and demonstrate the necessity of extremely fine invariants in distinguishing both approximate unitary equivalence classes of automorphisms of such algebras and isomorphism classes of the algebras themselves. The consequences for the program to classify nuclear C*-algebras are far-reaching: one has, among other things, that existing results on the classification of simple, unital AH algebras via the Elliott invariant of K-theoretic data are the best possible, and that these cannot be improved by the addition of continuous homotopy invariant functors to the Elliott invariant.

Counterexamples that can’t easily be excluded.

Refine the invariant

Cuntz semigroup

Restrict to $\mathscr{Z}$-stable C*-algebras;
try to characterize $\mathscr{Z}$-stability

Toms—Winter conjecture

Let A be a C*-algebra and $a, b \in A_+$. We say that a is Cuntz subequivalent to b, written $a \precsim b$, if there exists a sequence $(r_n)_{n\in\mathbb{N}} \subset A$ such that $\|r_n^* b r_n - b\| \to 0, \quad n \to \infty$.

Two positive elements $a, b \in A$ are *Cuntz equivalent*, written $a \sim b$, if $a \precsim b$ and $b \precsim a$.

The *Cuntz semigroup* of A is CU$(A) := (A \otimes \mathscr{K})/\sim$, where addition is $[a] + [b] = [\text{diag}\left(a, b\right)]$. Cuntz subequivalent makes this into an ordered semigroup.

One thinks of the Cuntz semigroup as a "Murray—von Neumann semigroup" for positive elements, rather than just projections.

To the von Neumann algebraists in the crowd: remember, a C*-algebra need not have any projections at all!

Indeed, $\mathscr{Z}$ contains only trivial projections $0,1$.

The Cuntz semigroup is the only invariant that can distinguish Toms' examples.

The sequel clarifies the nature of the information not captured by the Elliott invariant. We exhibit a pair of simple, separable, nuclear, and nonisomorphic C*-algebras which agree not only on Ell($\bullet$), but also on a host of other invariants including the real rank and continuous (with respect to inductive sequences) homotopy invariant functors. The Cuntz semigroup, employed to distinguish our algebras, is thus the minimum quantity by which the Elliott invariant must be enlarged in order to obtain a complete invariant, but we shall see that the question of range for this semigroup is out of reach. Any

The Cuntz semigroup of a C*-algebra A can be pretty wild.

However, A is $\mathcal{Z}$-stable the Cuntz semigroup is *almost unperforated.*

A partially ordered abelian semigroup P is *almost unperforated* if $x, y, \in P$ and there are $n, m \in \mathbb{N}$ with $n > m$ and $nx \leq my$, then $x \leq y$.

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THE STABLE AND THE REAL RANK OF $\mathcal{Z}$-ABSORBING C*-ALGEBRAS

MIKAEL RØRDAM

https://doi.org/10.1142/S0129167X04002661

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Abstract

Suppose that A is a C*-algebra for which $A \cong A \otimes \mathcal{Z}$, where $\mathcal{Z}$ is the Jiang–Su algebra: a unital, simple, stably finite, separable, nuclear, infinite-dimensional C*-algebra with the same Elliott invariant as the complex numbers. We show that:

(i) The Cuntz semigroup W(A) of equivalence classes of positive elements in matrix algebras over A is almost unperforated.

When A is $\mathscr{Z}$-stable, then the Cuntz semigroup contains the same information as the Murray—von Neumann semigroup and the tracial state space. So adding this to the invariant doesn't give us anything new here.

Almost unperforation of the Cuntz semigroup of A implies that A has *strict comparison (of positive elements)*:

For any tracial state $\tau \in T(A)$, define the *dimension function* d_τ

$$d_\tau(a) = \lim_{n\to\infty} \tau(a^{1/n}), \quad a \in A_+ .$$

We say that A has strict comparison if, whenever $d_\tau(a) < d_\tau(b)$ for every $\tau \in T(A)$, then $a \precsim b$.

This brings us to the Toms—Winter conjecture:

A.S. Toms, W. Winter / Journal of Functional Analysis 256 (2009) 1311–1340

Theorem 3.4. *Let A be a simple $\mathcal{VI}$ algebra admitting a standard decomposition with seed space a finite-dimensional CW complex. The following are equivalent:*

(i) *A is $\mathcal{Z}$-stable;*
(ii) *A has strict comparison of positive elements;*
(iii) *A has finite decomposition rank;*
(iv) *A has slow dimension growth (as an AH algebra);*
(v) *A has bounded dimension growth (as an AH algebra);*
(vi) *A is approximately divisible.*

- Conditions (i)–(iii) should remain equivalent in much larger classes of simple, separable, nuclear, and stably finite C^*-algebras. Conditions (iv), (v), and (vi) cannot be expected to hold in general (conditions (iv) and (v) exclude non-AH algebras, and (vi) excludes projection-

Decomposition rank is a precursor to nuclear dimension. They are the same for stably finite C*-algebras. However a purely infinite C*-algebra can have finite nuclear dimension, but will always have infinite decomposition rank.

Slogan: Decomposition rank and nuclear dimension are *noncommutative covering dimensions.*

In particular, if X is a locally compact metric space then $\mathrm{dim}_{\mathrm{nuc}}(C(X)) = \mathrm{dr}(C(X)) = \mathrm{dim}(X)$.

Both are given by refining the completely positive approximation property:

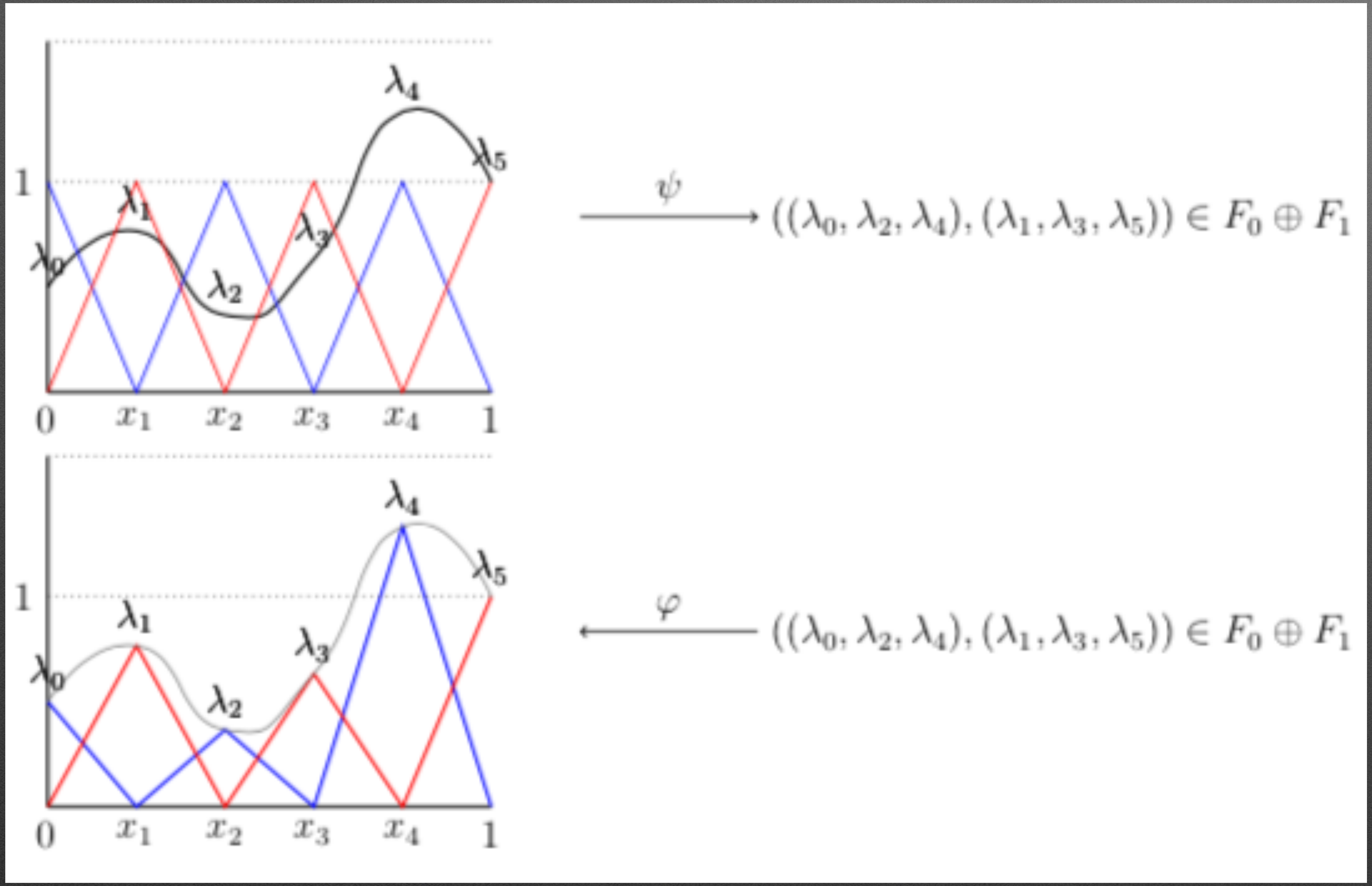
17.2.1. Definition. Let A be a separable C*-algebra. We say that A has *nuclear dimension* d, written $\dim_{\mathrm{nuc}} A = d$, if d is the least integer satisfying the following: For every finite subset $\mathcal{F} \subset A$ and every $\epsilon > 0$ there are a finite-dimensional, C*-algebra with $d+1$ ideals, $F = F_0 \oplus \cdots \oplus F_d$, and completely positive maps $\psi : A \to F$ and $\varphi : F \to A$ such that ψ is contractive, $\varphi|_{F_n}$ are completely positive contractive order zero maps and

$$\|\varphi \circ \psi(a) - a\| < \epsilon \text{ for every } a \in \mathcal{F}.$$

If no such d exists, then we say $\dim_{\mathrm{nuc}} A = \infty$.

If the φ can always be chosen to be contractive, then we say that A has *decomposition rank* d, written $\mathrm{dr}\, A = d$.

$C([0,1])$ has nuclear dimension one:



Here $\mathscr{F}$ consists of a single element, and ϵ is pretty big.

The finite-dimensional C*-algebra is $F = F_0 \oplus F_1$ where both F_0 and F_1 are three copies of $\mathbb{C}$.

By now we have the following, under the assumption that the extreme boundary of tracial state space is compact:

Theorem

~~Conjecture~~ 9.3. *For a separable, simple, unital, infinite dimensional and nuclear C^*-algebra A, the following are equivalent:*

(i) *A has finite nuclear dimension.*
(ii) *A is $\mathcal{Z}$-stable.*
(iii) *A has strict comparison of positive elements.*

We saw that the hyperfinite II_1 factor $\mathscr{R}$ can also be constructed, for example, as $L^\infty(X) \rtimes \mathbb{Z}$. We also know that $\mathcal{O}_\infty$ is usually defined using generators and relations. So one might ask if the Jiang—Su algebra might appear in more “natural” settings, not just as an inductive limit.

- Rørdam—Winter: $\mathscr{Z}$ can be written as a stable inductive limit of *generalized dimension drop algebras*

$$\mathscr{Z}_{p^\infty,q^\infty} = \{f \in C([0,1], M_{p^\infty} \otimes M_{q^\infty} \mid f(0) \in 1 \otimes M_{q^\infty}, f(1) \in M_{p^\infty} \otimes 1\},$$

where the connecting map is any trace-collapsing endomorphism.

- Jacelon—Winter: $\mathscr{Z}$ is universal. $\mathscr{Z}$ can be realized as a universal C*-algebra on countably many generators and relations.

- Deeley—Putnam—S.: $\mathscr{Z}$ can be realized as the orbit-breaking equivalence relation of a minimal dynamical system.

- Li: $\mathscr{Z}$ can be realized as the C*-algebra of a principal groupoid C*-algebra with 1-dimensional unit space.

(This along with Deeley—Putnam—S. shows that $\mathscr{Z}$ has a Cartan subalgebra; in fact infinitely many Cartan subalgebras of arbitrarily high dimension.)

- Ghasemi: $\mathscr{Z}$ can be realized as a Fraïssé limit.

- (Since $\mathscr{Z}$ has an approximately inner half-flip, which follows from Jiang and Su's original paper, this gives another proof that $\mathscr{Z}$ is strongly self-absorbing.)