Heisenberg Pairs on Hilbert $C^*$-modules

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Actions of Tensor Categories on $C^*$-algebras @ IPAM

January 27, 2021
Heisenberg pairs on Hilbert spaces

Goal:
Classify pairs $(A, B)$ of (possibly unbounded) self-adjoint operators with domains $D_A, D_B$, respectively, in a Hilbert space $H$ that satisfy $D_A \cap D_B$ contains an $(A, B)$-invariant dense subspace $K \subseteq H$ and $[A, B]_h = ih$ for all $h \in K$. (Heisenberg Commutation Relation)

Let's call $(A, B)$ a Heisenberg pair on $K$.

Example (Schrödinger Pair)
$Q = Mx$ and $P = -id_\mathbb{R}$ on $S(\mathbb{R}) \subseteq L^2(\mathbb{R})$ is such a pair.

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von Neumann (1931): No $\star$. 

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**Theorem (Stone-von Neumann, 1931)**

If \((S, R)\) is a Heisenberg representation of a l.c.a. \(G\), then

\[(S, R) \sim_u \oplus (U, V)\.

**Corollary**

If \((A, B)\) is a Heisenberg pair on \(K \subseteq \mathcal{H}\) that integrates to a Heisenberg representation on \(\mathcal{H}\), then \((A, B) \sim_u \oplus (P, Q)\).
Applications of the Stone-von Neumann Theorem

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- Rellich, Fuglede, Jorgensen, Moore, Muhly, Schmüdgen, Dorfmeister, ...
Let $T$ be a linear operator on a Banach space $X$. 

Theorem (Nelson, 1960) Suppose $T$ is a closed symmetric operator on a Hilbert space $H$. Then $T$ is self-adjoint if and only if $A_T$ is dense in $H$. 

Theorem (Flato-Simon-Snellman-Sternheimer (1972), Huang (2017)) Let $(A, B)$ be a Heisenberg pair on $K \subseteq H$. If $K$ consists of analytic vectors for both $A$ and $B$, then $\star (A, B) \sim u \oplus (P, Q)$.
Let $T$ be a linear operator on a Banach space $X$.

- $x \in X$ is **analytic for** $T$ if $x \in D_{T^n}$ for all $n \in \mathbb{N}$ and $\exists t > 0$ such that
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Analytic Vectors and Integrability

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(1) Stone-von Neumann Theorem for Hilbert $C^*$-modules

Theorem (Huang-I., 2020)

Every $(G, \mathcal{K}(\mathcal{H}), \alpha)$-Heisenberg representation on a Hilbert $\mathcal{K}(\mathcal{H})$-module is unitarily equivalent to a direct sum of copies of the $(G, \mathcal{K}(\mathcal{H}), \alpha)$-Schrödinger representation on $L^2(G, \mathcal{K}(\mathcal{H}), \alpha)$.
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- The domain of the adjoint of $T$ is

$$\mathcal{D}_{T^*} = \{ \xi \in X : \exists \nu \text{ s.t. } \forall \eta \in \mathcal{D}_T, \langle T\eta, \xi \rangle = \langle \eta, \nu \rangle \}$$
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Theorem (Woronowicz-Napiorkowski, 1992) Every regular self-adjoint operator on a Hilbert $C^*$-modules has a functional calculus.
Self-adjoint operators on Hilbert $C^*$-modules

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Major theorems about self-adjoint operators
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<table>
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<tr>
<th>Hilbert space</th>
<th>Hilbert (C^*)-module</th>
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Let $(A, B)$ be a Heisenberg pair on a dense submodule $K$ of a Hilbert $C^*$-module. If $K$ consists of analytic vectors for both $A$ and $B$, then $(A, B)$ integrates to a Heisenberg representation.
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**Corollary**

Let $X$ be a Hilbert $K(H)$-module, and suppose $(A, B)$ is a Heisenberg pair on $K$. If $K$ contains a dense set of analytic vectors for both $A$ and $B$, then

$$(A, B) \sim_u \bigoplus (P, Q).$$
Thank you!