Heisenberg Pairs on Hilbert C^*-modules

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Actions of Tensor Categories on C^*-algebras @ IPAM

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$Q = M_x$ and $P = -i\frac{d}{dx}$ on $\mathcal{S}(\mathbb{R}) \subseteq L^2(\mathbb{R})$ is such a pair.

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Unitary representations and integrability

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Theorem (Stone-von Neumann, 1931)

If (S, R) is a Heisenberg representation of a l.c.a. G, then

$$(S, R) \sim_u \oplus(U, V).$$

Corollary

If (A, B) is a Heisenberg pair on $K \subseteq \mathcal{H}$ that integrates to a Heisenberg representation on $\mathcal{H}$, then $(A, B) \sim_u \oplus(P, Q)$.

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Applications of the Stone-von Neumann Theorem

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- Rellich, Fuglede, Jorgensen, Moore, Muhly, Schmüdgen, Dorfmeister,...

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Suppose T is a closed symmetric operator on a Hilbert space $\mathcal{H}$. Then T is self-adjoint if and only if $\mathcal{A}_T$ is dense in $\mathcal{H}$.

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Theorem (Flato-Simon-Snellman-Sternheimer (1972), Huang (2017))

Let (A, B) be a Heisenberg pair on $K \subseteq \mathcal{H}$. If K consists of analytic vectors for both A and B, then* $(A, B) \sim_u \oplus(P, Q)$.

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(2) Integrability criterion for Heisenberg pairs on Hilbert C^*-modules

(1) Stone-von Neumann Theorem for Hilbert C^*-modules

Theorem (Huang-I., 2020)

Every $(G, \mathcal{K}(\mathcal{H}), \alpha)$-Heisenberg representation on a Hilbert $\mathcal{K}(\mathcal{H})$-module is unitarily equivalent to a direct sum of copies of the $(G, \mathcal{K}(\mathcal{H}), \alpha)$-Schrödinger representation on $L^2(G, \mathcal{K}(\mathcal{H}), \alpha)$.

(G, A, α)-Heisenberg rep.	(G, A, α)-Schrödinger rep.

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Theorem (Woronowicz-Napiorkowski, 1992)

Every regular self-adjoint operator on a Hilbert C^*-modules has a functional calculus.

Major theorems about self-adjoint operators

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Hilbert space	Hilbert C^*-module

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Corollary

Let X be a Hilbert $\mathcal{K}(\mathcal{H})$-module, and suppose (A, B) is a Heisenberg pair on K. If K contains a dense set of analytic vectors for both A and B, then

$$(A, B) \sim_u \oplus(P, Q).$$

Thank you!