Classifying small fusion categories

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IPAM WORKSHOP

Actions of $\otimes$-categories on C^*-algebras

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WHY FUSION CATEGORIES?

- FUSION CATEGORIES HAVE CONNECTIONS WITH MANY AREAS OF MATH & PHYSICS.

- NATURAL HOSTS FOR QUANTUM SYMMETRIES

Why classification?

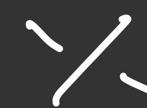
Useful for : - finding new examples,
↳ n-Ising.

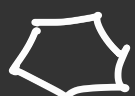
- inspiring new constructions,
↳ gauging, testing

- understanding invariants & properties.
↳ Cauchy tensor.

DEF.: $\mathcal{C}$ is a **fusion category** over $\mathbb{C}$ if abelian $\mathbb{C}$-linear, $(\mathcal{C}, \otimes, a, \mathbb{1}, l, r)$ monoidal, rigid $(\exists\, X^*, \cap, \cup)$, semisimple, finite, $\mathbb{1}$ simple.

$\mathcal{C}$ is a **braided** fusion cat. if $\exists\, \sigma_{X,Y} : X \otimes Y \xrightarrow{\sim} Y \otimes X$



+  axioms.

$\mathcal{C}$ is a **modular** fusion cat. if σ is non-deg.
+ nice trace...

EXAMPLES:

1) Rep G, G finite group.

2) Vec_G^{ω}, G finite group, ω 3-cocycle

3) subfactors, $N \subseteq M$ II_1-subfactor finite depth, fin. index
   N-Mod-N

4) Fib: $\mathbb{1}, X$ $\quad X \otimes X = \mathbb{1} \oplus X$

SOME INVARIANTS OF FUSION CATEGORIES:

- RANK($\mathcal{C}$) = # isom. classes of simples $< \infty$
- FUSION RULES $\quad \mathrm{Irr}(\mathcal{C}) = \{X_0 = \mathbb{1}, X_1, \dots, X_{r-1}\}$ $= \{\text{simples}\}/\sim$

$$X_i \otimes X_j = \bigoplus_{k=0}^{r-1} N_{ij}^k X_k \quad \in \mathbb{Z} \quad \rightsquigarrow L_{X_i} = X_i \otimes - \quad \rightsquigarrow L_{X_i} = (N_{ij}^k)_{j,k}$$

- FPdim X_i := FP eigenvalue of $L_{X_i} \in \mathbb{R}_{>0}$

- FPdim $\mathcal{C}$:= $\sum_{i=0}^{r-1} (\mathrm{FPdim}\, X_i)^2$

$$\mathrm{REP}(G) \longrightarrow \begin{array}{l} \mathrm{FPdim}(\mathrm{Rep}\, G) = |G| \\ \mathrm{FPdim}(V, \rho) = \dim_{\mathbb{C}}(V) \end{array}$$

FIB: FPdim (X) GOLDEN RATIO.

STATUS of THE Classification of small fusion cat.

- By RANK: [Ostrik] Rank 2, pivotal Rank 3.

If MODULAR: [Bruillard, Ng, Rowell, Stong, Wang, others]
Rank ≤ 5, weakly integral: Rank ≤ 7, odd dim: Rank ≤ 15

- By FPdim $\mathcal{C}$: [Etingof, Gelaki, Nikshych, Ostrik, others]
p, pq, pqr, p^n, $p^a q^b$.

If MODULAR: [Bruillard, Dong, Natale, P, Rowell, others]
FPdim $\mathcal{C} = p^n m$, m $\square$-free.

- By FPdim X, st X $\otimes$-gen. of $\mathcal{C}$: [Afzaly, Morrison, Penneys, Edie-Michell, others].
$\leq 5\frac{1}{4}$

- By cd$(\mathcal{C})$: [Natale, P., others]
$\hookrightarrow$ characterize.

- BOUND MULTIPLICITIES: X $\otimes$-gen. of $\mathcal{C}$, $X \otimes X = \mathbb{1} \oplus A \oplus B$.
[Morrison, Peters, Snyder]

WEAKLY INTEGRAL FUSION CATEGORIES

↳ $\mathrm{FPdim}\, \mathcal{C} \in \mathbb{Z}$.

- $\mathrm{FPdim}\, X \in \mathbb{Z}\sqrt{n}$ [GN], $n \in \mathbb{Z}$.

CONJECTURE [ENO 2008]: IF $\mathcal{C}$ WEAKLY INTEGRAL THEN $\mathcal{C}$ IS WEAKLY GROUP-THEORETICAL.

⟶ CAN BE CONSTRUCTED USING GROUPS.

CASE: FPdim $\mathcal{C}$ ODD. $\qquad \underbrace{X \simeq X^* \Rightarrow X \simeq \mathbb{1}}_{\text{MNSD}}$

- RANK($\mathcal{C}$) ODD $\overset{\text{FPdim } \mathcal{C} \in \mathbb{Z}}{\Longleftrightarrow}$ FPdim $\mathcal{C}$ ODD $\Leftrightarrow$ $\mathcal{C}$ MNSD

THEOR. [Bruillard, Rowell]: If $\mathcal{C}$ modular, FPdim $\mathcal{C}$ odd & rank ≤ 11 $\to 15$ $\Rightarrow$ $\mathcal{C} \simeq \mathrm{Vec}_G^\omega$

REMARK: They found an example in rank 25 not of this form Vec_G^ω.

THEOR. [Czenky, P]: If $\mathcal{C}$ modular, FPdim $\mathcal{C}$ odd & rank ≤ 23 $\Rightarrow$ $\mathcal{C} \simeq \mathrm{Vec}_G^\omega$ or $G(\mathcal{C}) = \mathbb{1}$.

THEOR. [Natale, P]: If FPdim $\mathcal{C}$ odd & $\mathcal{C}$ weakly G.T. then $\mathcal{C}$ solvable

PROP. [ENO]: $\mathcal{C}$ solvable $\Rightarrow$ $G(\mathcal{C}) \neq \mathbb{1}$.

CONCLUSION: $\mathcal{C}$ modular, FPdim odd, rank $\mathcal{C} \leq 23$
$\Rightarrow$ $\mathcal{C} \simeq \mathrm{Vec}_G^\omega$ or $\mathcal{C}$ non w.G.T.

QUESTIONS

COMMENTS

THANK

YOU!