Fell bundles, Dixmier-Douady theory and higher twists

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28th January 2021

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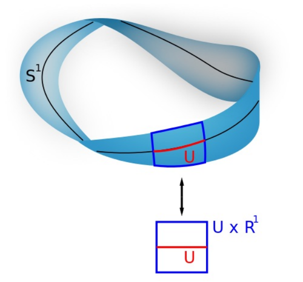
What is a bundle? - a topological perspective

When topologists hear the word “bundle”, they think of something like this...



- locally a product (ie. locally trivial),
- but (possibly) non-trivial globally,
- transitions are mediated by a structure group

Homotopy classification of fibre bundles

Theorem

*Let G be a topological group. Let $\mathcal{B}un_G(X)$ be the set of isomorphism classes of fibre bundles with structure group G. There exists a topological space BG (called **classifying space** of G) and a bundle*

$$EG \to BG$$

with structure group G that has the following property:

There is a natural $1:1$-correspondence induced by pullback of EG

$$\mathcal{B}un_G(X) \leftrightarrow [X, BG]$$

Examples of classifying spaces: Hermitian line bundles have structure group $U(1)$ and $[X, BU(1)] \cong H^2(X, \mathbb{Z})$.

Bundles with fibre $\mathbb{K}$ - Dixmier-Douady theory

Definition

Let B be a C^*-algebra. A **bundle of C^*-algebras** $\mathcal{A} \to X$ with fibre B is a fibre bundle with structure group $\mathrm{Aut}(B)$ (equipped with the point-norm topology).

Let $\mathbb{K}$ be compact operators on an ∞-dim. sep. Hilbert space. All $*$-automorphisms of $\mathbb{K}$ are inner, ie.

$$\mathrm{Aut}(\mathbb{K}) \cong PU(H) := U(H)/U(1) \ .$$

By long exact sequence of π_k and since $U(H)$ is contractible

$$\pi_k(B\mathrm{Aut}(\mathbb{K})) \cong \begin{cases} \mathbb{Z} & k = 3 \\ 0 & \text{else} \end{cases}$$

Corollary (Dixmier-Douady)

There is a natural group isomorphism:

$$\delta\colon \mathcal{B}un_{\mathbb{K}}(X) \to [X, BPU(H)] \cong H^3(X, \mathbb{Z})$$

called the ***Dixmier-Douady class****. This isomorphism is multiplicative in the sense that*

$$\delta(\mathcal{K}_1 \otimes \mathcal{K}_2) = \delta(\mathcal{K}_1) + \delta(\mathcal{K}_2) \ .$$

Let $\mathcal{K}^{op}$ be the bundle of compacts with reversed multiplication, then

$$\delta(\mathcal{K}^{op}) = -\delta(\mathcal{K}) \ .$$

Why does this work and how can we generalise it?

Note that we have an isomorphism

$$\mathbb{K} \otimes \mathbb{K} \to \mathbb{K}$$

$\mathcal{B}un_{\mathbb{K}}(X)$ inherits a *monoid structure* via fibrewise tensor product.

Definition (Toms-Winter)

A separable, unital C^*-algebra D is called **strongly self-absorbing** if $\exists$ an isomorphism $\psi\colon D \to D \otimes D$ and a path $u\colon [0,1) \to U(D \otimes D)$ with

$$\lim_{t\to 1} \|\psi(d) - u_t(d \otimes 1_D)u_t^*\| \to 0$$

Some consequences of this definition:

- $\mathrm{Aut}(D)$ is contractible (and so is $B\mathrm{Aut}(D)$),
- $K_0(D)$ is a ring (and $K_1(D) = 0$ if D satisfies the UCT),

Why does this work and how can we generalise it?

Observation: $PU(H) \simeq BU(1)$

object	classifying space
hermitian line bundle L	$PU(H) \simeq BU(1)$
bundle of compact operators $\mathcal{A}$	$BPU(H) \simeq BBU(1)$

$$U(1) \text{ is an abelian group} \quad \Rightarrow \quad BU(1), BBU(1), \dots \text{ exist}$$

Observation: Line bundles form a subgroup in $GL_1(K^0(X))$.
Idea: Extend the above table to all of $GL_1(K^0(X))$!

object	classifying space
virtual line bundles	$GL_1(KU)$
?	$BGL_1(KU)$

$$GL_1(KU) \text{ is an infinite loop space} \quad \Rightarrow \quad BGL_1(KU) \text{ exists}$$

Generalized Dixmier-Douady Theory

Theorem (Dadarlat-P.)

Let D be a strongly self-absorbing C^*-algebra and let KU^D be the ring spectrum representing K-theory with coefficients $K_i(D)$. Then:

$$\mathcal{B}un_{D\otimes\mathcal{O}_\infty\otimes\mathbb{K}}(X) \cong [X, BGL_1(KU^D)]\ ,$$
$$\mathcal{B}un_{\mathcal{O}_\infty\otimes\mathbb{K}}(X) \cong [X, BGL_1(KU)]\ ,$$
$$\mathcal{B}un_{M_{n^\infty}\otimes\mathcal{O}_\infty\otimes\mathbb{K}}(X) \cong [X, BGL_1(KU[\tfrac{1}{n}])]\ .$$

Remarks:

- We also determined the homotopy type of $B\mathrm{Aut}(D\otimes\mathbb{K})$ for all strongly self-absorbing C^*-algebras D,
- The group $[X, BGL_1(KU)]$ and its variants can be determined via the Atiyah-Hirzebruch spectral sequence.

The Verlinde Ring and Twisted K-theory

Theorem (Freed, Hopkins, Teleman)

Let G be a simply connected compact Lie group and let LG be the free loop group on G. Then we have

$$^{\tau(m)}K_G^{\mathbf{dim}(G)}(G^{adj}) \cong R_m(LG) \; .$$

- $R_m(LG)$ - formal differences of isomorphism classes of positive energy representations of LG at a fixed level $m \in \mathbb{Z}$.
- Given m there is a G-equivariant bundle of compact operators

$$\mathcal{K}_m \to G$$

and $^{\tau(m)}K_G^i(G^{\mathrm{adj}}) \cong K_i^G(C(G, \mathcal{K}_m))$

Fell bundles

Let $\mathcal{G}$ be a topological groupoid.

Definition (roughly)

A *saturated Fell bundle* $\pi\colon \mathcal{E} \to \mathcal{G}$ is an upper-semicontinuous Banach bundle together with an associative bilinear multiplication

$$\mathcal{E}^{(2)} \to \mathcal{E} \quad , \quad (e_1, e_2) \mapsto e_1 \cdot e_2$$

covering the groupoid multiplication such that $\overline{\mathcal{E}_{g_1} \cdot \mathcal{E}_{g_2}} = \mathcal{E}_{g_1 g_2}$ and an involution $*\colon \mathcal{E} \to \mathcal{E}$ covering the groupoid inversion.

- fibres over units $\mathcal{G}^{(0)} \subset \mathcal{G}$ are C^*-algebras,
- for $g \in \mathcal{G}$ the fibre $\mathcal{E}_g$ is a $\mathcal{E}_{s(g)}$-$\mathcal{E}_{t(g)}$-Morita equivalence.

Caveat: Fell bundles need **not** be locally trivial!

A Higher Twist over $SU(n)$ (jt. with D. Evans)

Input: Exponential functor F on $\mathcal{V}ect_{\mathbb{C}}^{iso}$, $G = SU(n)$.

Output:

- groupoid $\mathcal{G}$ with G-action and a G-map $\mathcal{G} \to G$,
- G-equivariant saturated Fell bundle $\mathcal{E} \to \mathcal{G}$,
- ...with $\mathcal{E}|_{\mathcal{G}^{(0)}} = \mathcal{G}^{(0)} \times D$, where

$$D = \mathrm{End}(F(\mathbb{C}^n))^{\otimes\infty} .$$

Theorem (Evans-P.)

$$C^*(\mathcal{E}) \otimes \mathbb{K} \cong C(G, \mathcal{A}) ,$$

where $\mathcal{A} \to G$ is a bundle with fibre $D \otimes \mathbb{K}$.

Define equivariant K-theory of G with twist $\mathcal{E}$ by

$${}^{\mathcal{E}}K^i_G(G^{\mathrm{adj}}) := K^G_i(C^*(\mathcal{E})) .$$

Exponential functors

- $\mathcal{V}ect_{\mathbb{C}}^{fin}$ - fin. dim. complex inner product spaces and linear maps
- $\mathcal{V}ect_{\mathbb{C}}^{iso}$ - same objects but with unitary isomorphisms

Definition

An exponential functor on $\mathcal{V}ect_{\mathbb{C}}^{fin}$ consists of a triple (F, κ, ι), where

- $F \colon \mathcal{V}ect_{\mathbb{C}}^{fin} \to \mathcal{V}ect_{\mathbb{C}}^{fin}$ is a unitary functor,
- $\kappa_{V,W} \colon F(V \oplus W) \to F(V) \otimes F(W)$ is a natural isomorphism,
- $\iota \colon F(0) \to \mathbb{C}$ is another natural isomorphism,

such that the obvious associativity and unitality diagrams commute.

Exponential functors

Example

- $F = \left(\bigwedge^*\right)^{\otimes m}$ for any $m \in \mathbb{N}_0$,
- $F = \left(\bigwedge^{top}\right)^{\otimes m}$ for any $m \in \mathbb{N}_0$ on $\mathcal{V}ect_{\mathbb{C}}^{iso}$,
- Fix $W \in obj(\mathcal{V}ect_{\mathbb{C}}^{fin})$, then

$$F^W(V) = \bigoplus_{k=0}^{\infty} W^{\otimes k} \otimes \bigwedge^k(V)\ .$$

- non-example: symmetric algebra
- classification of polynomial exponential functors via involutive R-matrices (based on Lechner-P-Wood)

A Higher Twist over $SU(n)$ - the nitty-gritty

Let F be an exponential functor on $\mathcal{V}ect_{\mathbb{C}}^{iso}$, let $G = SU(n)$.

$$\mathcal{G} = \{(g, z_1, z_2) \in G \times (\mathbb{T} \setminus \{1\})^2 \mid z_1, z_2 \notin \mathrm{EV}(g)\}$$

Choose an order on $\mathbb{T} \setminus \{1\}$. For $z_1 < z_2$ define

$$\mathcal{E}_{(g,z_1,z_2)} = \left(\bigotimes_{\substack{\lambda \in \mathrm{EV}(g) \\ z_1 < \lambda < z_2}} F(\mathrm{Eig}(g, \lambda)) \right) \otimes D$$

- right multiplication on D,
- left multiplication induced by

$$\mathrm{End}(F(\mathrm{Eig}(g, \lambda))) \otimes D \cong D \ ,$$

- F exponential $\rightsquigarrow$ associative choice of above isomorphism.

For $z_1 > z_2$ define $\mathcal{E}_{(g,z_1,z_2)} = \mathcal{E}^{\mathrm{op}}_{(g,z_2,z_1)}$.

A Higher Twist over $SU(n)$

Final remarks:

- For $F = \left(\bigwedge^{top}\right)^{\otimes m}$ we have $D = \mathbb{C}$ and construction boils down to the m-th tensor power of the basic gerbe a la Murray-Stevenson, which represents classical twist at level m.
- Can compute ${}^{\mathcal{E}}K^i_G(G) := K^G_i(C^*(\mathcal{E}))$ ...
  - for $G = SU(2)$ and all F,
  - for $G = SU(3)$ and all F after rationalisation.

  In all these examples ${}^{\mathcal{E}}K^{\mathbf{dim}(G)}_G(G)$ is still a ring!
- Even better: Sometimes seem to get fusion rings from loop group CFTs, e.g. for $G = SU(2)$ and $F = \left(\bigwedge^*\right)^{\otimes(2m+1)}$.

The End

Thank you!