Non-Stable extension theory and the classification of C*-algebras.

Christopher Schafhauser
University of Nebraska-Lincoln
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j/w: J. Carrión, J. Gabe, A. Tikuisis, S. White.

The Classification Theorem (many):

Suppose A, B are separable, simple, unital, nuclear, $\mathcal{Z}$-stable C^*-algebras in the UCT class.

(1) $A \cong B \iff KT_u(A) \cong KT_u(B)$,

$$KT_u := (K_*, T, [1]_{K_0}, K_0 \times T \to \mathbb{R})$$

(2) $\{A \hookrightarrow B\}/\approx_u \cong \{\underline{K}T_u(A) \longrightarrow \underline{K}T_u(B)\}$,

$$\underline{K}T_u := \left(\left(K_*(-;\mathbb{Z}/n)\right)_{n=0}^{\infty}, \overline{K}_1^{alg}, T, \text{compatibility} \right)$$

We'll assume A, B have unique trace.

<u>Set up</u>: Enough to describe approx morphisms $A \xrightarrow{\approx} B$

classifying

or equivalently, $A \hookrightarrow B_\omega$ up to $\sim_{ue}$

$\tau \|\cdot\|$-ultrapower

There is an ideal in B_ω,

$$J_B = \{ b \in B_\omega \mid \tau(b^* b) = 0 \}$$
$$\cong \frac{\{ b \in \ell^\infty B \mid \|b_n\|_2 \to 0 \}}{\{ b \in \ell^\infty B \mid \|b_n\| \to 0 \}}$$

Further,

$$B_\omega / J_B \cong \left(\pi_\tau(B)'' \right)^\omega \cong R^\omega$$

$\tau \|\cdot\|_2$-ultrapower

So we have the <u>trace-kernel extension</u>

$$0 \to J_B \longrightarrow B_\omega \xrightarrow[q_B]{} R^\omega \longrightarrow 0$$

[~ Matui–Sato '12]

Strategy:

(1) Classify $A \longrightarrow R^\omega$

(2) Classify lifts of a fixed $\theta : A \longrightarrow R^\omega$

$$
\begin{tikzcd}
 & B_\omega \arrow[d] \\
A \arrow[ur, dashed, "?"] \arrow[r, "\theta"'] & R^\omega
\end{tikzcd}
$$

(via KK/Ext-theory)

(3) Compute the invariant in (2) [uses UCT]

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(1): h'isms $A \longrightarrow R^\omega$ are in bijection with h'isms $\pi_\tau(A)'' \longrightarrow R^\omega$

$$\pi_\tau(A)'' \cong R$$

So $\exists\, \theta : A \to R^\omega$, unique up to $\sim_u$.

(2): Fix $\theta : A \longrightarrow R^\omega$

$$
\begin{tikzcd}
e: 0 \arrow[r] & J_B \arrow[r] \arrow[d, equal] & E \arrow[r, "q_B"] \arrow[d] & A \arrow[r] \arrow[d, "\theta"] \arrow[dl, dashed, "?"'] & 0 \\
0 \arrow[r] & J_B \arrow[r] & B_\omega \arrow[r, "q_B"'] & R^\omega \arrow[r] & 0
\end{tikzcd}
$$

$$\boxed{E = q^{-1}(\theta(A))}$$

(2a): ($\exists$ of lift)

Have an obstruction $\langle e\rangle \in Ext(A, J_B)$

θ lifts $\Longrightarrow \langle e\rangle = 0$.

Suppose $\langle e\rangle = 0$.

Then $\exists$

$$\begin{tikzcd}
 & & & A \arrow[d, "\begin{pmatrix}\theta & 0\\ 0 & \pi\end{pmatrix}"] & \\
0 \arrow[r] & M_2 J_B \arrow[r] & \begin{pmatrix} B_\omega & * \\ * & * \end{pmatrix} \arrow[r] & \begin{pmatrix} R^\omega & * \\ * & * \end{pmatrix} \arrow[r] & 0
\end{tikzcd}$$

So that $\begin{pmatrix}\theta & 0\\ 0 & \pi\end{pmatrix}$ and π both lift to $*$-hom's

($*$) Suitable version of Voiculescu's theorem $\begin{pmatrix}\theta & 0\\ 0 & \pi\end{pmatrix} \sim \theta$ in some sense.

So θ lifts.

So $\exists\, \psi : A \longrightarrow B_\omega \iff \langle e\rangle \in Ext(A, J_B)$ vanishes.

(2b): Given $\psi : A \longrightarrow B_\omega$,

$$\{\varphi : A \longrightarrow B_\omega\}/\sim_u \;\cong\; KK(A, J_B)$$

$$\varphi \longmapsto \text{"}\varphi - \psi\text{"}$$

(3): Compute $KK(A, J_B)$, $Ext(A, J_B)$.