

# Non-Stable extension theory and the classification of $C^*$ -algebras.

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## The Classification Theorem (many):

Suppose  $A, B$  are separable, simple, unital, nuclear,  $\mathbb{Z}$ -stable  $C^*$ -algebras in the UCT class.

$$(1) A \cong B \iff \underline{KT}_u(A) \cong \underline{KT}_u(B),$$



$$\underline{KT}_u := (K_*, T, [1]_{K_0}, K_0 \times T \rightarrow \mathbb{R})$$

$$(2) \{A \hookrightarrow B\} / \sim_u \cong \{ \underline{KT}_u(A) \rightarrow \underline{KT}_u(B) \},$$

$$\underline{KT}_u := \left( (K_*(-; \mathbb{Z}/n))_{n=0}^{\infty}, \bar{K}_1^{alg}, T, \text{compatibility} \right)$$



We'll assume  $A, B$  have unique trace.

Set up: Enough to describe approx morphisms  $A \xrightarrow{\approx} B$   
 or equivalently, classify  $A \hookrightarrow B_\omega$  up to  $\approx$   
 $\mathcal{L} \|\cdot\|$ -ultrapower

There is an ideal  $\mathfrak{I}$  in  $B_\omega$ ,

$$\begin{aligned} \mathfrak{I} &= \{ b \in B_\omega \mid \tau(b^*b) = 0 \} \\ &\cong \underbrace{\{ b \in \ell^\infty B \mid \|b\|_2 \rightarrow 0 \}}_{\{ b \in \ell^\infty B \mid \|b\| \rightarrow 0 \}} \end{aligned}$$

Further,

$$B_\omega / \mathfrak{I} \cong (\pi_\tau(B)^\#)^\omega \cong \mathbb{R}^\omega$$

$\mathcal{L} \|\cdot\|_2$ -ultrapower

So we have the trace-kernel extension

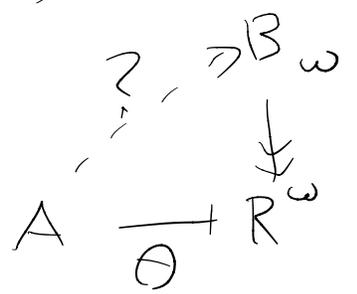
$$0 \rightarrow \mathfrak{I} \rightarrow B_\omega \xrightarrow{g_B} \mathbb{R}^\omega \rightarrow 0$$

[Matus-Svetitsky 12]

Strategy:

(1) Classify  $A \rightarrow R^w$

(2) Classify lifts of a fixed  $\theta: A \rightarrow R^w$



(via KK/Ext-theory)

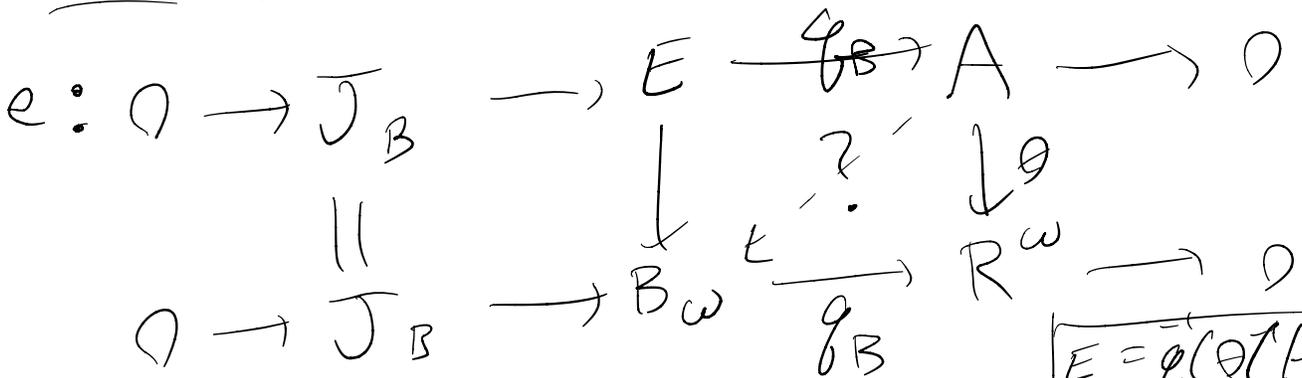
(3) Compute the invariant in (2) [uses UCT]

(1):  $h^1$ 'isms  $A \rightarrow R^w$  are in bijection with  $h^1$ 'isms  $\tilde{e}(A) \rightarrow R^w$

$\cong$   
 $R$

So  $\exists \theta: A \rightarrow R^w$ , unique up to  $\sim$ .

(2): Fix  $\theta: A \rightarrow R^w$



$$E = \tilde{e}(\theta(A))$$

(2a): ( $\exists$  of lift)

Have an obstruction  $\langle e \rangle \in \text{Ext}(A, J_B)$

$\ominus$  lifts  $\Rightarrow \langle e \rangle = 0$ .

Suppose  $\langle e \rangle = 0$ .

Then  $\exists$

A

$$\downarrow \begin{pmatrix} \theta & 0 \\ 0 & \pi \end{pmatrix}$$

$$0 \rightarrow M_2 J_B \rightarrow \begin{pmatrix} B_w & * \\ * & * \end{pmatrix} \rightarrow \begin{pmatrix} R^w & * \\ * & * \end{pmatrix} \rightarrow 0$$

So that  $\begin{pmatrix} \theta & 0 \\ 0 & \pi \end{pmatrix}$  and  $\pi$  both lift to  $*$ -lines

$\otimes$  Suitable version of Verdulescu's theorem  $\begin{pmatrix} \theta & 0 \\ 0 & \pi \end{pmatrix} \sim \theta$  is some sense,

so  $\theta$  lifts,

So  $\exists \psi: A \rightarrow B_w \Leftrightarrow \langle e \rangle \in \text{Ext}(A, J_B)$  vanishes.

(2b): Given  $\gamma: A \rightarrow B_w$ ,

$$\{ \varphi: A \rightarrow B_w \} / \sim_u \cong \text{KK}(A, J_B)$$

$\varphi \longmapsto \text{"}\varphi - \gamma\text{"}$

(3): Compute  $\text{KK}(A, J_B)$ ,  $\text{Ext}(A, J_B)$ .