Classification of group actions on C^*-algebras

Actions of Tensor Categories on C*-algebras, IPAM

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Objects of interest: C^*-dynamical systems (A, α, G), where

- A is a C^*-algebra
- G is a locally compact group
- $\alpha : G \curvearrowright A$ is a continuous action.

Overarching goal: Exploit invariants to classify up to cocycle conjugacy.

Definition

Let $\alpha : G \curvearrowright A$ be an action.

- An α-cocycle is a strictly continuous map $\mathbb{u} : G \to \mathcal{U}(\mathcal{M}(A))$ with $\mathbb{u}_{gh} = \mathbb{u}_g \alpha_g(\mathbb{u}_h)$ for all $g, h \in G$. In this case, $\alpha^{\mathbb{u}}_{\bullet} := \mathrm{Ad}(\mathbb{u}_{\bullet}) \circ \alpha_{\bullet}$ is another action.
- α is said to be (cocycle) conjugate to $\beta : G \curvearrowright B$, if there is an isomorphism $\varphi : A \to B$ (and an α-cocycle $\mathbb{u}$) such that
$$\alpha^{\mathbb{u}}_g = \varphi^{-1} \circ \beta_g \circ \varphi, \quad g \in G.$$

In this talk C^*-algebras shall be unital and groups discrete. (convenience!)

The "overarching goal" is meant as an extension of the Elliott program, i.e., the Elliott program should correspond to $G = \{1\}$.

In particular A is often simple amenable $\mathcal{Z}$-stable...

In order to introduce you to this subject, I would like to preview the important slogan (or meta-idea) that I choose to focus on.

> When classifying a class of C^*-dynamics, first understand how to classify the underlying C^*-algebras. Then find a way to reduce *dynamical classification* to *non-dynamical classification* by means of an *averaging process* that exploits *amenability*.

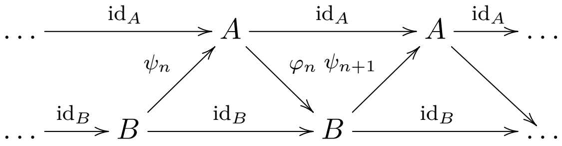
In a bit, we will discuss the classification of finite group actions with the Rokhlin property, where this theme can be nicely demonstrated with not too involved arguments.

Before looking at C^*-dynamics, first we need to go through some basics.

Theorem (Elliott intertwining)

Let A and B be two separable C^*-algebras. Suppose there are $*$-homomorphisms $\varphi : A \to B$ and $\psi : B \to A$ with $\psi \circ \varphi \approx_{\mathrm{u}} \mathrm{id}_A$ and $\varphi \circ \psi \approx_{\mathrm{u}} \mathrm{id}_B$. Then φ and ψ are approximately unitarily equivalent to mutually inverse isomorphisms.

Idea: Inductively pick unitaries $u_n \in A$, $v_n \in B$ so that with $\varphi = \mathrm{Ad}(v_n) \circ \varphi$ and $\psi_n = \mathrm{Ad}(u_n) \circ \psi$, the diagram



approximately commutes as one goes further to the right, with 1-summable speed of convergence. Then $\Phi = \lim_{n\to\infty} \varphi_n$ is an isomorphism with inverse $\Psi = \lim_{n\to\infty} \psi_n$. □

Fortunately for us there is an easy dynamical analog when G is finite.

Definition

Let two actions $\alpha : G \curvearrowright A$ and $\beta : G \curvearrowright B$ be given. Two equivariant $*$-homomorphisms $\varphi, \psi : (A, \alpha) \to (B, \beta)$ are approximately G-unitarily equivalent, $\varphi \approx_{\mathrm{u},G} \psi$, if we find unitaries $v_n \in \mathcal{U}(B^\beta)$ such that $\psi = \lim_{n\to\infty} \mathrm{Ad}(v_n) \circ \varphi$.

By copying the non-dynamical proof almost verbatim, one gets:

Theorem (dynamical Elliott intertwining for finite groups)

Let G be a finite group, and let $\alpha : G \curvearrowright A$ and $\beta : G \curvearrowright B$ be two actions on separable C^*-algebras. Suppose there are equivariant $*$-homomorphisms $\varphi : (A, \alpha) \to (B, \beta)$ and $\psi : (B, \beta) \to (A, \alpha)$ with $\psi \circ \varphi \approx_{\mathrm{u},G} \mathrm{id}_A$ and $\varphi \circ \psi \approx_{\mathrm{u},G} \mathrm{id}_B$. Then φ and ψ are approximately G-unitarily equivalent to mutually inverse conjugacies.

For the applications yet to come, I use as a black box the so-called existence/uniqueness theorems underpinning the modern Elliott program, which were set up in earlier talks. Let $\mathfrak{E}$ denote the class of separable unital simple amenable $\mathcal{Z}$-stable C^*-algebras satisfying the UCT.

Theorem (many hands, in this form CGSTW)

Let $A, B \in \mathfrak{E}$ and let $\varphi, \psi : A \to B$ be two unital $*$-homomorphisms. Then $\varphi \approx_{\mathrm{u}} \psi$ if and only if $\underline{\mathrm{K}}\mathrm{T}_{\mathrm{u}}(\varphi) = \underline{\mathrm{K}}\mathrm{T}_{\mathrm{u}}(\psi)$.[1]

Theorem (many hands, in this form CGSTW)

Let $A, B \in \mathfrak{E}$. For any morphism $\zeta : \underline{\mathrm{K}}\mathrm{T}_{\mathrm{u}}(A) \to \underline{\mathrm{K}}\mathrm{T}_{\mathrm{u}}(B)$, there exists a unital $*$-homomorphism $\varphi : A \to B$ with $\underline{\mathrm{K}}\mathrm{T}_{\mathrm{u}}(\varphi) = \zeta$.

[1] Remember: the total invariant is finer than the ordinary Elliott invariant, but contains the same information about isomorphism classes.

Now let us finally look at the classification of Rokhlin actions!

Definition (Izumi)

Let G be a finite group and A a separable unital C^*-algebra. An action $\alpha : G \curvearrowright A$ is said to have the Rokhlin property, if there exists a sequence of projections $e_n \in A$ such that

- $\|[a, e_n]\| \to 0$ for all $a \in A$
- $\sum_{g \in G} \alpha_g(e_n) \to \mathbf{1}_A$.

$(A, \alpha) \approx (A \otimes \mathcal{C}(G), \alpha \otimes \text{shift})$

Although there exist plenty of example of such actions, the Rokhlin property is quite restrictive. (In contrast to von Neumann algebras!) However, as shown in the work of Izumi, Rokhlin actions can be very effectively classified.

Example (Prototypical one)

Let G be a finite group with its left-regular representation $\lambda : G \to \mathcal{B}(\ell^2(G)) = M_{|G|}$. Then $\gamma = \mathrm{Ad}(\lambda)^{\otimes\infty} : G \curvearrowright M_{|G|^\infty}$ has the Rokhlin property.

Going forward, I wish to convince you that for Rokhlin actions, the previous existence/uniqueness theorems imply their own equivariant versions, which will ultimately give us equivariant classification.

We shall start with the following reduction principle regarding the uniqueness of $*$-homomorphisms.

Theorem

Let G be a finite group. Let $\alpha : G \curvearrowright A$ and $\beta : G \curvearrowright B$ be actions on separable unital C^*-algebras, and assume β has the Rokhlin property. For two unital equivariant $*$-homomorphisms $\varphi, \psi : (A, \alpha) \to (B, \beta)$, we have $\varphi \approx_{\mathrm{u},G} \psi$ if and only if $\varphi \approx_{\mathrm{u}} \psi$.

Theorem (continued)

Let G be a finite group. Let $\alpha : G \curvearrowright A$ and $\beta : G \curvearrowright B$ be actions on separable unital C^*-algebras, and assume β has the Rokhlin property. For two unital equivariant $*$-homomorphisms $\varphi, \psi : (A, \alpha) \to (B, \beta)$, we have $\varphi \approx_{\mathrm{u},G} \psi$ if and only if $\varphi \approx_{\mathrm{u}} \psi$.

Sketch of proof: Suppose $v_n \in \mathcal{U}(B)$ satisfies $\psi = \lim_{n\to\infty} \mathrm{Ad}(v_n) \circ \varphi$. Note that since φ and ψ were equivariant, one also has

$$\lim_{n\to\infty} \mathrm{Ad}(\beta_g(v_n)) \circ \varphi = \lim_{n\to\infty} \beta_g \circ \mathrm{Ad}(v_n) \circ \varphi \circ \alpha_g^{-1} = \beta_g \circ \psi \circ \alpha_g^{-1} = \psi.$$

Let $e_n \in B$ be a sequence of projections as required by the Rokhlin property. Without loss of generality we may assume $\|[e_n, v_n]\| \to 0$.

Then we find a sequence of unitaries $\mathcal{U}(B^\beta) \ni u_n \approx \sum_{g\in G} \beta_g(e_n v_n)$.

Then:
$$\mathrm{Ad}(u_n) \circ \varphi \approx \sum_{g\in G} \beta_g(e_n) \cdot \underbrace{\mathrm{Ad}(\beta_g(v_n)) \circ \varphi}_{\approx \psi} \approx \psi.$$

Thus the sequence u_n witnesses $\varphi \approx_{\mathrm{u},G} \psi$. □

Next we discuss the reduction principle regarding existence.

Theorem (Gardella–Santiago)

Let G be a finite group. Let $\alpha : G \curvearrowright A$ and $\beta : G \curvearrowright B$ be actions on separable unital C^*-algebras, and assume β has the Rokhlin property. Suppose $\varphi : A \to B$ is a unital $*$-homomorphism with $\varphi \circ \alpha_g \approx_{\mathrm{u}} \beta_g \circ \varphi$ for all $g \in G$. Then there exists a unital equivariant $*$-homomorphism $\psi : (A, \alpha) \to (B, \beta)$ with $\varphi \approx_{\mathrm{u}} \psi$.

Sketch of proof: For each $h \in G$ let $w_h \in \mathcal{U}(B)$ be some unitaries such that

$$\beta_h \circ \varphi \circ \alpha_h^{-1} \approx \mathrm{Ad}(w_h) \circ \varphi, \quad h \in G.$$

Let $e \in B$ be a *good enough* projection as required by the Rokhlin property. Then we find a unitary $\mathcal{U}(B) \ni v \approx \sum_{h \in G} \beta_h(e) w_h$.

Set $\varphi_1 = \mathrm{Ad}(v) \circ \varphi$.

Sketch of proof: (continued)

We find a unitary $\mathcal{U}(B) \ni v \approx \sum_{h\in G} \beta_h(e) w_h$ and set $\varphi_1 = \mathrm{Ad}(v) \circ \varphi$.

We observe for all $g \in G$:

$$\begin{aligned} \beta_g \circ \varphi_1 &\approx \sum_{h\in G} \beta_{gh}(e) \cdot \beta_g \circ \underbrace{\mathrm{Ad}(w_h) \circ \varphi}_{\approx \beta_h \circ \varphi \circ \alpha_h^{-1}} \\ &\approx \sum_{h\in G} \beta_{gh}(e) \cdot \beta_{gh} \circ \varphi \circ \alpha_h^{-1} \\ &= \sum_{h\in G} \beta_h(e) \cdot \underbrace{\beta_h \circ \varphi \circ \alpha_h^{-1}}_{\approx \mathrm{Ad}(w_h) \circ \varphi} \circ \alpha_g \\ &\approx \varphi_1 \circ \alpha_g. \end{aligned}$$

Repeat this inductively and get a sequence of maps $\varphi_1, \varphi_2, \varphi_3, \dots$ for which these approximations hold better and better. If one does this carefully, one can arrange the maps (φ_n) to be Cauchy in point-norm, which allows us to get the desired map as $\psi = \lim_{n\to\infty} \varphi_n$. □

As a consequence of all of this, we get the following classification result:

Theorem

Let G be a finite group. Let $\alpha : G \curvearrowright A$ and $\beta : G \curvearrowright B$ be two Rokhlin actions on classifiable C^*-algebras. Then α and β are conjugate if and only if

$$\underline{\mathrm{K}}\mathrm{T}_{\mathrm{u}}(\alpha) : G \curvearrowright \underline{\mathrm{K}}\mathrm{T}_{\mathrm{u}}(A) \quad \textit{and} \quad \underline{\mathrm{K}}\mathrm{T}_{\mathrm{u}}(\beta) : G \curvearrowright \underline{\mathrm{K}}\mathrm{T}_{\mathrm{u}}(B)$$

are conjugate.

Proof: Assume that $\zeta : \underline{\mathrm{K}}\mathrm{T}_{\mathrm{u}}(A) \to \underline{\mathrm{K}}\mathrm{T}_{\mathrm{u}}(B)$ is an equivariant isomorphism. By the black box, we find $*$-homomorphisms $\varphi_0 : A \to B$ and $\psi_0 : B \to A$ lifting ζ and ζ^{-1}, respectively. Since ζ is equivariant, it follows from the black box that these maps are equivariant modulo $\approx_{\mathrm{u}}$. By the reduction trick, we may find equivariant $*$-homomorphisms $\varphi : (A, \alpha) \to (B, \beta)$ and $\psi : (B, \beta) \to (A, \alpha)$ lifting ζ and ζ^{-1}. Using again the black box and the other reduction trick, we see $\psi \circ \varphi \approx_{\mathrm{u},G} \mathrm{id}_A$ and $\varphi \circ \psi \approx_{\mathrm{u},G} \mathrm{id}_B$. The dynamical Elliott intertwining takes care of the rest.

With a bit more work one can actually obtain a more satisfactory version of this result, but this involves pure homological algebra.

Theorem (Izumi; published for A Kirchberg or TAF)

Let G be a finite group. Let $\alpha : G \curvearrowright A$ and $\beta : G \curvearrowright B$ be two Rokhlin actions on classifiable C^*-algebras. Then α and β are conjugate if and only if

$$\mathrm{KT_u}(\alpha) : G \curvearrowright \mathrm{KT_u}(A) \quad \textit{and} \quad \mathrm{KT_u}(\beta) : G \curvearrowright \mathrm{KT_u}(B)$$

are conjugate.

Example

For any finite group G, there is a unique Rokhlin action $G \curvearrowright \mathcal{O}_2$. For example, the two actions

$$\alpha : \mathbb{Z}_2 \curvearrowright \mathcal{O}_2 = \mathrm{C}^*(s_1, s_2), \quad \alpha(s_j) = (-1)^j s_j$$

and

$$\beta : \mathbb{Z}_2 \curvearrowright \mathcal{O}_2 \otimes \mathcal{O}_2 \cong \mathcal{O}_2, \quad \beta(x_1 \otimes x_2) = x_2 \otimes x_1$$

are conjugate.

To round off this *(mini-)mini-course* I would like to say a few words about the issues surrounding the classification of more general C^*-dynamics.

Warning 1: The theory of Rokhlin actions might lead you to believe that ultimately, nice actions on classifiable C^*-algebras are determined by how they act on the Elliott invariant. Although the analogous statement is true for injective factors, this expectation fails quite spectacularly in the general C^*-context.

Example (Izumi)

Let $\mathcal{O}_\infty^{\mathrm{st}} \subset \mathcal{O}_\infty$ be the corner spanned by an inclusion $\mathcal{O}_2 \subset \mathcal{O}_\infty$. For some $q \in \mathcal{O}_\infty^{\mathrm{st}}$ that is the range projection of an isometry in $\mathcal{O}_\infty$, we consider the order 2 automorphism

$$\gamma = \bigotimes_{\mathbb{N}} \mathrm{Ad}(2q - \mathbf{1}) : \mathbb{Z}_2 \curvearrowright \bigotimes_{\mathbb{N}} \mathcal{O}_\infty^{\mathrm{st}} \cong \mathcal{O}_2.$$

Then $\mathcal{O}_2 \rtimes_\alpha \mathbb{Z}_2 \cong \mathcal{O}_\infty^{\mathrm{st}} \otimes M_{2^\infty}$ and $\mathcal{O}_2^\gamma \cong \mathcal{O}_\infty \otimes M_{2^\infty}$.

Example

Given a unital Kirchberg algebra A, we can induce an action

$$\alpha = \mathrm{id}_A \otimes \gamma : \mathbb{Z}_2 \curvearrowright A \otimes \mathcal{O}_2 \cong \mathcal{O}_2.$$

Since the crossed products of such actions have many possible K-groups (all pairs of uniquely 2-divisible abelian groups), this yields uncountably many outer actions $\mathbb{Z}_2 \curvearrowright \mathcal{O}_2$ that are pairwise non-cocycle conjugate.

Warning 2: I started the presentation talking about *cocycle* conjugacy, after which no cocycles were to be seen. It just so happens that cocycles can always be trivialized for Rokhlin actions, which is special. The cocycles are **very** important in general.

Example (Izumi)

Let A and B be two unital Kirchberg algebras that absorb M_{2^∞}. Let $\alpha, \beta : \mathbb{Z}_2 \curvearrowright \mathcal{O}_2$ be two actions as constructed above. Suppose A and B are stably isomorphic, but not isomorphic. (E.g. $A = \mathcal{O}_\infty \otimes M_{2^\infty}$ and $B = \mathcal{O}_\infty^{\mathrm{st}} \otimes M_{2^\infty}$.) Then α and β are cocycle conjugate, but not conjugate.

Warning 3: In general, working only with genuine equivariant maps between C^*-dynamics is too restrictive.

For example, for $G = \mathbb{Z}$, one ends up classifying single automorphisms on classifiable C^*-algebras, but one cannot do it up to conjugacy.

Theorem (Evans–Kishimoto)

Let $\alpha, \beta \in \mathrm{Aut}(A)$ be two single automorphisms on an AF algebra with the Rokhlin property. If $K_0(\alpha) = K_0(\beta)$, then α and β are cocycle conjugate.

The proof involved the invention of what is now called the *Evans–Kishimoto intertwining* method. Roughly speaking, one works very hard for taking care of certain technical obstacles, after which one inductively perturbs α and β with unitary conjugates and/or cocycles in A to push them closer together. If one does this right, then the compositions of the unitary conjugates and the cocycles satisfy a certain Cauchy criterion, and one obtains a cocycle conjugacy via a limit procedure.

(This is a lot more involved than what we saw before!)

My suggested approach is to work in a category where an arrow between C^*-dynamical systems is a pair

$$(\varphi, \mathbb{u}) : (A, \alpha) \to (B, \beta),$$

where $\mathbb{u}$ is a β-cocycle and φ is a $*$-homomorphism which is equivariant with respect to α and $\beta^{\mathbb{u}}$. This can indeed be defined, and this category comes equipped with a flexible notion of (approximate) unitary equivalence

$$(\varphi, \mathbb{u}) \sim_{\mathrm{u}} (\mathrm{Ad}(v) \circ \varphi, v\mathbb{u}_\bullet \beta_\bullet(v)^*), \quad v \in \mathcal{U}(B).$$

This gives one access to an Elliott intertwining machinery with obvious candidates for existence/uniqueness theorems, which are entirely analogous to what we have seen in the first part of the talk.

$\rightsquigarrow$ *Applications of this framework are in progess*

To end this talk, let me cherry-pick a few illustrative further results:

Theorem (Nakamura)

Let $\alpha, \beta : \mathbb{Z} \curvearrowright A$ be two outer actions on a Kirchberg algebra. If $KK(\alpha) = KK(\beta)$, then α and β are cocycle conjugate.

Theorem (Matui–Sato, unpublished)

Let A be a classifiable, monotracial C^*-algebra with trace τ. Let $\alpha, \beta : \mathbb{Z} \curvearrowright A$ be two strongly outer actions, i.e., the induced actions on $\pi_\tau(A)''$ are outer. If α and β are asymptotically unitarily equivalent, then they are cocycle conjugate.

Theorem (Izumi–Matui)

Let G be a poly-$\mathbb{Z}$ group and A a stable Kirchberg algebra. To each action $\alpha : G \curvearrowright A$ one can consider its generalized mapping torus $\mathcal{M}_\alpha$, which is a locally trivial, continuous $\mathcal{C}(BG)$-C^*-algebra with fibres isomorphic to A. Then two outer actions $\alpha, \beta : G \curvearrowright A$ are cocycle conjugate if and only if $\mathcal{M}_\alpha \cong \mathcal{M}_\beta$.

Theorem (Meyer)

In the situation of Izumi–Matui's theorem, one has $\mathcal{M}_\alpha \cong \mathcal{M}_\beta$ if and only if α and β are KK^G-equivalent.

Theorem (Gabe–S, in progress)

Let G be any countable discrete group. Then amenable and outer G-actions on Kirchberg algebras are classified up to cocycle conjugacy via equivariant Kasparov theory.

Warning 4: Unlike for von Neumann algebras, there is a wealth of interesting *amenable actions* of non-amenable groups on C^*-algebras.

$\alpha : G \curvearrowright A$ is amenable

$\iff$ $\alpha : G \curvearrowright A^{**}$ is amenable

$\iff$ $\exists$ equivariant conditional expectation $\ell^\infty(G, Z(A^{**})) \to Z(A^{**})$

Thank you for your attention!