L^2 cohomology and maximal rigid subalgebras of s-malleable deformations

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Group von Neumann Algebras

G discrete countable group $\rightsquigarrow L(G)$ group von Neumann algebra

- $\lambda : G \hookrightarrow \mathcal{U}(\ell^2(G))$ by $\lambda_g(\eta)(x) := \eta(g^{-1}x) \quad \forall \eta \in \ell^2(G),\ g, x \in G.$
- $L(G) := \overline{C[G]}^{WOT} = \{\lambda_g\}''_{g \in G} \subseteq B(\ell^2(G))$
- $\tau : L(G) \to \mathbb{C}$ by $\tau(x) = \langle x\delta_e, \delta_e \rangle$.

Rigidity Phenomena

Given G (initial data) and some group H(target) with $L(G) \cong L(H)$. What properties of G transfer to H? If G has a group theoretic property, does $L(G)$ has an analogous property?
Eg:

- W*-superrigidity: If $L(G) \cong L(H)$, then $G \cong H$.
- Primeness: If $G \neq G_1 \times G_2$, then $L(G) \ncong M \bar{\otimes} N$ for any M, N type II_1.
- Product Rigidity: If $L(G_1 \times G_2) \cong L(H)$, then $H \cong H_1 \times H_2$.
- Product Rigidity': If $L(G) \cong M \bar{\otimes} N$ then $G \cong G_1 \times G_2$

Lack Rigidity

- $L(G) \cong L^\infty(\mathbb{T})$ if G infinite abelian.
- If G, H amenable ICC, $L(G) \cong L(H) \cong \mathcal{R}$, the unique hyperfinite II_1 factor (Connes '76).
  - Ex: $G = S_\infty$, $(\mathbb{Z}/2\mathbb{Z}) \wr \mathbb{Z}$, or $\mathbb{Z} \wr \mathbb{Z}$.
- $G_1, \dots, G_k$ infinite amenable, then $L(G_1 * \cdots * G_k) \cong L(\mathbb{F}_k)$ (Dykema '93).
- $L(\mathbb{F}_2) \neq L(\mathbb{F}_3)$?
- Conj: If G ICC has prop (T), $L(G) \cong L(H)$ implies $G \cong H$.

Rigidity (Cont)

- $L(\mathbb{F}_2) \ncong L(S_\infty)$ (Murray and von Neumann 42).
- There exist $\{G_\alpha\}_{\alpha\in[0,1]}$ s.t. $L(G_\alpha) \ncong L(G_\beta)$ if $\alpha \neq \beta$ (McDuff 69).
- If G is ICC hyperbolic then $L(G) \ncong L(H_1 \times H_2)$ for any infinite ICC groups H_i (Ozawa 02).

Using Popa's deformation/rigidity theory:

- If G is icc with $\beta_1^{(2)}(G) > 0$. then $L(G) \ncong L(H_1 \times H_2)$ for any infinite ICC groups H_i (Peterson 06).
- There exist groups G so that if $L(G) \cong L(H)$, then $G \cong H$ (Ioana-Popa-Vaes '12), (Berbec-Vaes 12), (Chifan-Ioana '17).
- $G = G_1 \times G_2$, G_i hyperbolic ICC. If $L(G) \cong L(H)$, then $H = H_1 \times H_2$ (Chifan-dS-Sinclair 15).

On a Conjecture of Peterson and Thom

Conjecture (Peterson-Thom 11)

If $A, B \leq L(\mathbb{F}_2)$ are amenable and $A \cap B$ is diffuse, then $A \vee B$ is amenable. In particular $A \vee B \neq L(\mathbb{F}_2)$

Proposition (Peterson-Thom 11)

If G is a discrete group and $H_1, H_2 < G$ with $|H_1 \cap H_2| = \infty$, $H_1 \vee H_2$ has vanishing first reduced ℓ^2-cohomology if both H_1 and H_2 do. In particular, $\mathbb{F}_2 \neq H_1 \vee H_2$ with H_i amenable and $|H_1 \cap H_2| = \infty$

Alternate Approach to PT Conjecture: *A Random Matrix Approach to the Peterson-Thom Conjecture* by B. Hayes.

An Easier Result

Remark

Fix G, nonamenable, such that: $\beta_1^{(2)}(G) > 0$; $H, K \leq G$ with property (T); and $H \cap K$ infinite. Then $H \vee K \neq G$.

Proof.

$\beta_1^{(1)}(G) > 0$, there exists an unbounded (non-trivial) 1-cocyle $c : \Gamma \to \ell^2(G)$, a map

$$c(gh) = c(g) + \lambda_g(c(h)).$$

Prop (T) implies $c|_H$, and $c|_K$ are bounded/inner/trivial.

$$c|_H(h) = \xi - \lambda_h(\xi) \qquad \forall\, h \in H;$$

$$c|_K(k) = \eta - \lambda_k(\eta) \qquad \forall\, k \in K.$$

$$\xi - \eta = \lambda_g(\xi - \eta) \qquad \forall\, g \in H \cap K.$$

An Easier Result (Cont)

Remark

Fix G, nonamenable, such that: $\beta_1^{(2)}(G) > 0$; $H, K \leq G$ with property (T); and $H \cap K$ infinite. Then $H \vee K \neq G$.

Proof.

$$\xi - \eta = \lambda_g(\xi - \eta) \qquad \forall\, g \in H \cap K.$$

$|H \cap K| = \infty$ and λ a *mixing rep* $\implies \xi = \eta$.
c is inner/bounded/trivial on $H \vee K$. □

Remark

Substitute λ with any mixing unitary rep admitting, unbounded cocyle, $H, K < G$ with $|H \cap K| = \infty$.

Conjecture

Fix G, nonamenable, such that: $\beta_1^{(2)}(G) > 0$; $A, B \leq L(\Gamma)$ with Property (T); and $A \cap B$ diffuse, then $A \vee B \neq L(\Gamma)$.

An Easier Result (Cont)

Remark

Fix G, nonamenable, such that: $\beta_1^{(2)}(G) > 0$; $H, K \leq G$ with property (T); and $H \cap K$ infinite. Then $H \vee K \neq G$.

Proof.

$$\xi - \eta = \lambda_g(\xi - \eta) \qquad \forall g \in H \cap K.$$

$|H \cap K| = \infty$ and λ a *mixing rep* $\implies \xi = \eta$.
c is inner/bounded/trivial on $H \vee K$. □

Remark

Substitute λ with any mixing unitary rep admitting, unbounded cocyle, bounded on $H, K < G$ with $|H \cap K| = \infty$.

Theorem (dS-Hayes-Hoff-Sinclair)

Fix G, nonamenable, such that: $\beta_1^{(2)}(G) > 0$; $A, B \leq L(\Gamma)$ with Property (T); and $A \cap B$ diffuse, then $A \vee B \neq L(\Gamma)$.

s-Malleable Deformations

Definition (Popa)

Let (M, τ) be a finite von Neumann algebra. An s-malleable deformation of M is a tuple $(\alpha_t, \beta, \widetilde{M}, \widetilde{\tau})$ s.t.

- $M \leq \widetilde{M}$ in a trace preserving manner,
- α_t is a 1-parameter group in Aut $\widetilde{M}$ $\|\alpha_t(x) - x\|_2 \to 0$ as $t \to 0$,
- $\beta \in \mathrm{Aut}(\widetilde{M})$ s.t. $\beta|_M = \mathrm{id}_M$, and $\beta\alpha_t = \alpha_{-t}\beta, \beta^2 = \mathrm{id}_{\tilde{M}}$.

$Q \leq M$ is rigid wrt α if

$$\epsilon_t(Q) := \sup_{x \in (Q)_1} \|\alpha_t(x) - x\|_2 \quad \text{has} \quad \epsilon_t(Q) \to 0 \quad \text{as} \quad t \to 0.$$

We write $Q \in \mathrm{Rig}(\alpha)$

Deformations from Cocycles

Let $\beta_1^{(2)}(G) > 0$:
1-cocycle c gives rise to *Gaussian dilation* (Sinclair 11) $L(G) \subseteq L^\infty(X) \rtimes G = \tilde{M}$ and s-malleable deformation:

- $\alpha_t, \beta \in \mathrm{Aut}(\tilde{M})$, $t \in \mathbb{R}$, with $\beta\alpha_t\beta = \alpha_{-t}$ and $\beta|_{L(G)} = id_{L(G)}$;
- $\|\alpha_t(x) - x\|_2 \to 0$ as $t \to 0$ for all $x \in (\tilde{M})_1$.

c unbounded $\implies \epsilon_t(L(G)) \not\to 0$.
$P \leq L(G)$ with Prop (T) $\implies \epsilon_t(P) \to 0$, i.e. Prop (T) $\implies P \in \mathrm{Rig}(\alpha)$.

Analysis of s-malleable deformations

Theorem (ds-Hayes-Hoff-Sinclair 20)

Let (α, β) be an s-malleable deformation of $M \leq \widetilde{M}$. Then for any $P, Q \in \mathrm{Rig}(\alpha)$ with $(P \cap Q)' \cap \widetilde{M} \subseteq M$, we have

$$\epsilon_{2t}(P \vee Q) \leq 24\epsilon_t(P)$$

Remark

- $N' \cap \widetilde{M} \subseteq M$ is automatic for all diffuse N when ${}_M L^2(\widetilde{M}) \ominus L^2(M)_M$ is *mixing*.
- ${}_M L^2(\widetilde{M}) \ominus L^2(M)_M$ is mixing when $M = L(\Gamma)$, $\widetilde{M} = L^\infty(X) \rtimes \Gamma$ is the Gaussian Dilation.
- If $\beta_1^{(2)}(\Gamma) > 0$, $P, Q \leq L(\Gamma)$, P, Q have Prop (T), $P \cap Q$ diffuse then $P \vee Q \neq L(\Gamma)$.

Back To Peterson-Thom

Conjecture (Peterson-Thom)

If $Q_1, Q_2 \leq L(\mathbb{F}_2)$ are amenable and $Q_1 \cap Q_2$ is diffuse, then $Q_1 \vee Q_2$ is amenable.

Unfortunately, this does not settle the PT conjecture. We need an approximate version that handles MULTIPLE deformations.

Conjecture (ds-Hayes-Hoff-Sinclair 20)

Let M be a II_1 factor and $Q_1, Q_2 \leq M$ such that $Q_i \leq M$ is approximately L^2-rigid for $i = 1, 2$. If $Q_1 \cap Q_2$ is diffuse, then $Q_1 \vee Q_2 \leq M$ is approximately L^2-rigid.

Closing Thoughts

- Group von Neumann algebras are challenging.
- Progress due to Popa's Deformation/Rigidity Theory is ongoing.
  - Chifan-DiazArias-Drimbe 20: New Examples of W^* and C^* superrigid groups,
  - Drimbe 21: Product rigidity in von Neumann and C^*-algebras via s-malleable deformations.
- Full power of Deformation/Rigidity Theory has yet to be unlocked.

On Mathematics

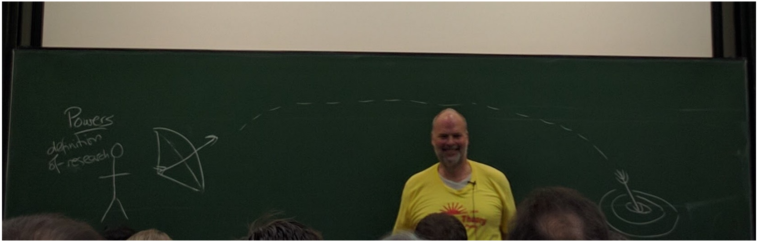

Figure: Vaughan on Power's definition of research