Weak quasi-Hopf algebras associated to Verlinde fusion categories

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Tensor category

- A ($\mathbb{C}$-linear) tensor category $\mathcal{C}$ is a category with a tensor product $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ (bifunctor), unit object ι, associativity morphisms

$$\alpha_{\rho,\sigma,\tau} : (\rho \otimes \sigma) \otimes \tau \to \rho \otimes (\sigma \otimes \tau)$$

  satisfying the pentagonal equation.
- A unitary tensor category has Banach spaces as morphism spaces, a *-involution $T \in (\rho, \sigma) \to T^* \in (\sigma, \rho)$ s.t. $\|T^* T\| = \|T\|^2$, $(S \otimes T)^* = S^* \otimes T^*$ and α is unitary

We are interested in rigid semisimple tensor categories with f.d. morphism spaces, simple tensor unit, a braiding. For the talk, I restrict to fusion categories.

Examples

Tensor categories encode quantum symmetries. We have many important sources of examples,

- Compact groups, discrete groups
- compact quantum groups $\mathrm{Rep}(G)$,
- quasi-Hopf algebras,
- weak Hopf algebras,
- AQFT
- Bimodule categories over non-commutative algebras,
- Subfactors ($\mathrm{Bim}(R)$, standard invariant)
- Quantum groups at roots of 1 $U_q(\mathfrak{g})$,
- CFT (Vertex operator algebras and conformal nets)

Modular categories

- A braiding on $\mathcal{C}$ is defined by $c(\rho, \sigma) : \rho \otimes \sigma \to \sigma \otimes \rho$ satisfying two hexagonal equations.
- When the braiding of a fusion category has a ribbon structure that satisfies certain nondegeneracy properties, the category is called modular.

Unitary modular fusion categories arise from

- Quantum groups at roots of 1 via a quotient construction by Andersen $\mathcal{C}(\mathfrak{g}, q)$ (Turaev-Wenzl, Kirillov)
- Conformal nets: $I \to \mathcal{A}(I)$, $\mathrm{Rep}(A)$ (Kawahigashi-Longo-Mueger)
- Vertex operator algebras (Kac Peterson, Huang)
- A class of examples important for my talk are the affine Lie algebras $\hat{\mathfrak{g}}$, that give rise to the affine vertex operator algebras $V_{\mathfrak{g}_k}$ at a positive integer level k (Frenkel-Zhu)

Finkelberg Theorem

Theorem (Finkelberg) Let $\mathfrak{g}$ be a complex Lie algebra and $k \in \mathbb{N}$. Then there is an equivalence of ribbon categories
$\mathcal{C}(\mathfrak{g}, q) \simeq \mathrm{Rep}(V_{\mathfrak{g}_k})$, for $q = e^{\frac{i\pi}{d(k+h^\vee)}}$.

- The history behind this theorem is very intricated, and Huang wrote in 2018 a review paper about it.
- Kazhdan and Lusztig proved a theorem relating affine Lie algebras to quantum groups at roots of 1 at levels $k \in \mathbb{C}\backslash\mathbb{Q}^+$.
- Finkelberg proof uses Kazhdan-Lusztig work on affine Lie algebras at levels $k \in -\mathbb{N}$,
- Is there a direct proof of Finkelberg theorem (for $k \in \mathbb{N}$)? (Huang, Gannon)

A reformulation of the problem

Consider a faithful linear functor $\mathcal{F} : \mathcal{C} \to \mathrm{Vec}_{f.d.}$. Then one would like

- to associate more structure to $\mathcal{F}$ and try to classify them

A paradigmatic example is Doplicher-Roberts/Deligne theorem for symmetric tensor categories with applications to AQFT.

Another example is the work by Neshveyev and Yamashita on classifications of compact quantum groups of Lie type.

- use specific structures to compare up to equivalence categories that we suspect to be equivalent

For the second problem, that is construction of specific structures for the purpose of studying equivalence, we have

Theorem (Drinfeld-Kohno-Kazhdan-Lusztig, In the formulation by Bakalov-Kirillov)

$$\mathrm{Rep}_{KZ}(U(\mathfrak{g})) \simeq \mathrm{Rep}(U_q(\mathfrak{g})), \qquad q = e^{i\pi/d\chi}, \quad \chi \in \mathbb{C}\backslash\mathbb{Q},$$

where $\mathrm{Rep}_{KZ}(U(\mathfrak{g}))$ is Drinfeld category associated to $U(\mathfrak{g})$ as a quasi-Hopf algebra, with R-matrix and associator from KZ differential equations of CFT.

- Neshveyev and Tuset gave a reformulation in terms of $\mathcal{F} = \mathrm{forgetful} : \mathcal{C} = \mathrm{Rep}_{KZ}(U(\mathfrak{g})) \to \mathrm{Vec}_{f.d.}$ and a proof for $\chi \in i\mathbb{R}$.
- For levels $k \in \mathbb{Q}^-$, BK definition does not work,

Is there a direct proof of Finkelberg theorem (for $k \in \mathbb{N}$) following BK and NT setting?

We need

- an analogue of the Hopf algebra $U_q(\mathfrak{g})$ for q generic for Andersen fusion category $\mathcal{C}(\mathfrak{g}, q)$ at roots of 1,
- an analogue of the quasi-Hopf algebra $U(\mathfrak{g})$ for $\mathrm{Rep}(V_{\mathfrak{g}_k})$ for $k \in \mathbb{N}$
- a ribbon equivalence between the corresponding their representation categories.

My talk is about these questions. To attempt to get an idea about the equivalence, we get a closer look at Drinfeld first of Drinfeld-Kohno theorem. This approach is motivated by Wenzl work on unitarity of $\mathcal{C}(\mathfrak{g}, q)$, that I recall later. Drinfeld worked over a formal variable h.

Quantum groups, the formal variable case

- Let $\mathfrak{g}$ be a simple complex Lie algebra.
- Drinfeld-Jimbo quantum group $U_h(\mathfrak{g})$ is a Hopf algebra over a formal variable h,
- There is an R-matrix $R \in U_h(\mathfrak{g})\overline{\otimes}U_h(\mathfrak{g})$, and this implies that $\mathrm{Rep}U_h(\mathfrak{g})$ is a braided tensor category with braiding $c(\rho,\sigma) = \Sigma\rho\otimes\sigma(R)$, with Σ the permutation map.

Quasi-Hopf algebras

- Drinfeld turned categorical equivalence issues into algebraic issues. He generalized the notion of Hopf algebra to quasi-Hopf algebra.
- A quasi-Hopf algebra is given by among other data by (A, Δ, R, Φ).
- A is an algebra, Δ a coproduct of A, R an R-matrix, and Φ an associator.
- $\mathrm{Rep}(A)$ is a rigid braided tensor category.
- If $F \in A \otimes A$ is invertible then we have a new quasi-Hopf algebra where some of the data are twisted $(A, \Delta_F, R_F, \Phi_F)$.
- $\mathrm{Rep}(A) \simeq \mathrm{Rep}(A_F)$.

- Consider $U(\mathfrak{g})[[h]]$ deformation of $U(\mathfrak{g})$, $\Delta(x) = x \otimes I + I \otimes x$; $t \in \mathfrak{g} \otimes \mathfrak{g}$ tensor coming from the Killing form, with "trivial" R-matrix $R_{KZ} = e^{ht/2}$.
- Drinfeld constructed

$$\Phi_{KZ} \quad \text{(Drinfeld associator)}$$

from Knizhnik-Zamolodgikov differential equations.
- $U(\mathfrak{g})$ is a quasi-Hopf algebra, and $\mathrm{Rep}(U(\mathfrak{g}))$ is a braided tensor category.
- There is an algebra isomorphism $U_h(\mathfrak{g}) \simeq U(\mathfrak{g})[[h]]$.
- Drinfeld-Kohno theorem relates the Hopf algebra $U_h(\mathfrak{g})$, h formal variable with the quasi-Hopf algebra $U(\mathfrak{g})$ via a twist F making the non-trivial R_h-matrix of $U_h(\mathfrak{g})$ into a "trivial" R_{KZ}-matrix of $U(\mathfrak{g})$ and a trivial associator 1 of $U_h(\mathfrak{g})$ into the non-trivial associator Φ_{KZ} of $U(\mathfrak{g})$ which turns out 3-coboundary,

$$(R_h)_F = R_{KZ}, \quad 1 \otimes \Delta(F^{-1}) I \otimes F^{-1} F \otimes I \Delta \otimes 1(F) = \Phi_{KZ}$$

Drinfeld coboundary matrix

- In $U_h(\mathfrak{g})$ we have $R_{21}R = e^{ht}$.
- Drinfeld considered the coboundary matrix $\overline{R} = Re^{-ht/2}$.
- On a tensor product representation

$$\overline{c} := \Sigma\overline{R} : \rho \otimes \sigma \to \sigma \otimes \rho$$

  is a morphism satisfying $\overline{c}^2 = 1$.
- One of the arguments entering in Drinfeld first proof of Drinfeld-Kohno to show existence is the twist $F = \overline{R}^{1/2}$.
- It follows that $R_F = e^{ht/2} = R_{KZ}$,
- It also follows that $\Phi_F = \Phi_{KZ}$ up to a 2-coboundary deformation by a uniqueness theorem.

Back to $\mathcal{C}(\mathfrak{g}, q)$

- We consider Drinfeld-Jimbo quantum groups $U_q(\mathfrak{g})$ at roots of unity following a construction of Lusztig. Specifically, we specialize at $q \in \mathbb{C}$ s.t. q^2 is a primitive root of unity of order $\ell d > \check{h}$, with $d = 1$ ADE, $d = 2$ BCF, $d = 3$ G_2.
- $U_q(\mathfrak{g})$ is a ribbon Hopf algebra in a topological sense. In particular, R and $\overline{R}$ act on tensor product of representation spaces.
- $U_q(\mathfrak{g})$ has non-semisimple representations. Andersen defined tilting modules, and there is a notion of negligible tilting modules and morphisms.
- Constructions of Andersen, Reshetikin-Turaev, Wenzl, Gelfand-Kazhdan, associate a semisimple ribbon fusion category $\mathcal{C}(\mathfrak{g}, q)$ with irreducibles parameterized by a certain finite set Λ_ℓ by factoring through negligible modules and morphisms.
- Consider the case $q = e^{i\pi/\ell d}$. Then $\mathcal{C}(\mathfrak{g}, e^{i\pi/\ell d}) = \mathcal{C}(\mathfrak{g}, e^{i\pi/\ell d})$ is a modular fusion category (Kirillov; Turaev, Wenzl).

A connection between Drinfeld coboundary matrix and Wenzl unitarity

- By works of Kirillov, Wenzl, Xu, $\mathcal{C}(\mathfrak{g}, q)$ admits a unitary braided tensor structure.
- Following Wenzl, the following are our understanding of some of the ideas concerning the unitary structure.
- The most difficult part is the construction of a well defined fusion space of two irreducible objects $V \boxtimes W$, and of a positive inner product on the fusion space.
- The inner product of $V \boxtimes W$ is twisted with respect to the tensor product inner product $V \otimes W$, by Drinfeld coboundary matrix $\overline{R}$.

- The Hopf algebra $U_q(\mathfrak{g})$ has a *-involution satisfying an anticommutativity relation
  $\Delta(a)^* = \Delta^{\mathrm{op}}(a^*) = R\Delta(a^*)R^{-1} = \overline{R}\Delta(a^*)\overline{R}^{-1}$.
- Every irreducible $V_q(\lambda)$ $\lambda \in \Lambda_\ell$ becomes a unitary representation of $U_q(\mathfrak{g})$.
- This gives a faithful functor $W : \mathcal{C}(\mathfrak{g}, \ell) \to \mathrm{Hilb}$. It follows that $A_W = \mathrm{Nat}(W)$ is a semisimple C^*-algebra.
- A_W is a subquotient $U_q(\mathfrak{g}) \to A_W$ as a *-algebra, but the ideal is not a coideal.
- To construct an analogue of $U_h(\mathfrak{g})$ we need a coproduct for A_W, a tensor structure on $W : \mathcal{C}(\mathfrak{g}, q) \to \mathrm{Vec}$ in a weak sense (Tannakian duality).

- For λ, $\mu \in \Lambda_\ell$, there is a fusion tensor product $V_q(\lambda) \boxtimes V_q(\mu)$ which is a subrepresentation of the full tensor product $V(\lambda) \otimes V_q(\mu)$ defined by an idempotent $P_{\lambda,\mu} : V_q(\lambda) \otimes V_q(\mu) \to V_q(\lambda) \boxtimes V_q(\mu)$.
- $V_q(\lambda) \boxtimes V_q(\mu)$ becomes unitary with a scalar product twisted by Drinfeld coboundary matrix $\overline{R}$ of $U_q(\mathfrak{g})$.
- Wenzl work suggests that to construct a tensor equivalence $\mathcal{C}(\mathfrak{g}, q) \simeq \mathrm{Rep}(V_{\mathfrak{g}_k})$ we may try to untwist the tensor product inner product. That is try to study a square root construction of $\overline{R}$ in $\mathcal{C}(\mathfrak{g}, q)$.

wqh algebras

Definition Let $\mathcal{C}$ be a unitary fusion category and $\mathcal{F} : \mathcal{C} \to \mathrm{Hilb}_{\mathrm{f.d.}}$ a linear *-functor. A weak quasi-tensor structure on $\mathcal{F}$ is given by a natural transformation

$$F_2 : \mathcal{F}(V) \otimes \mathcal{F}(W) \to \mathcal{F}(V \boxtimes W)$$

such that $F_2 F_2^{-1} = 1$.

- It follows that $A = \mathrm{Nat}(\mathcal{F})$ is a weak quasi-Hopf algebra in the extension of Drinfeld definition considered by Mack-Schomerus. Weak refers to the fact that the coproduct is not necessarily unital.
- A admits a twisted unitary structure ($\Delta(a^*)^* = \Omega\Delta(a)\Omega^{-1}$..., $\Omega = F_2^* F_2$).

A unitary wqt structure is defined by $F_2^* = F_2^{-1}$ (i.e. F_2^*, F_2^{-1} are isometries).

wh algebras

Definition A weak tensor structure is defined by two commutative diagrams describing compatibility of associativity morphisms of $\mathcal{C}$, Vec. F_2 and F_2^{-1} in a weak sense.

$$\begin{array}{ccccc}
F((\rho\sigma)\tau) & \xrightarrow{F_2^{-1}} & F(\rho\sigma)\otimes F(\tau) & \xrightarrow{F_2^{-1}\otimes 1} & (F(\rho)F(\sigma))F(\tau) \\
\Big\downarrow{\scriptstyle F(\alpha)} & & & & \Big\downarrow \\
F(\rho(\sigma\tau)) & \xleftarrow{F_2} & F(\rho)\otimes F(\sigma\tau) & \xleftarrow{1\otimes F_2} & F(\rho)(F(\sigma)F(\tau))
\end{array}$$

In this case, $\mathrm{Nat}(\mathcal{F})$ is a Hopf algebra in a weak sense (w-Hopf algebra).

Definition A compatible coboundary wqh algebra is a complex semisimple wqh A endowed with

- a C^*-algebra involution s.t. $\tilde{A} = A^{\mathrm{op}}$ as quasitriangular algebras,
- a ribbon structure (R, v) with unitary ribbon element v
- a unitary square root w of v s.t. $S(w) = w$, with S the antipode.

It follows that $\Omega = Rw^{-1} \otimes w^{-1}\Delta(w)$ is automatically selfadjoint. When it is positive, we call A a unitary coboundary wqh.

Under certain structural circumstances, $\mathrm{Rep}^+(A)$ is automatically unitary ribbon.

This definition is stronger than that we considered an year ago, in that as for $U_q(\mathfrak{g})$ we are now requiring an extra compatibility axiom $\Delta(a)^* = \Delta^{\mathrm{op}}(a)$, and accordingly we have added the adjective "compatible".

An example is the discrete Hopf algebra associated to a compact group G.

Theorem Let $\mathfrak{g}$ be a complex simple Lie algebra and $q = e^{i\pi/\ell d}$, $\ell > \check{h}$. Then Wenzl fusion tensor product induces a weak tensor structure on $W : \mathcal{C}(\mathfrak{g}, q) \to \mathrm{Hilb}$, thus A_W becomes a compatible unitary coboundary w-Hopf algebra such that $\mathrm{Rep}^+(A_W) \simeq \mathcal{C}(\mathfrak{g}, q)$ as unitary ribbon categories.

- A year ago we did not know whether the structure of A_W was weak tensor (except for $\mathfrak{sl}_N$ that two of us studied earlier with different methods.) To show this we use Gelfand-Kazhdan description of negligible modules of $U_q(\mathfrak{g})$.
- This wh property allows vanishing of a cohomological obstruction and Wenzl work on unitarity implies the extra compatibility property $\Delta^{\mathrm{op}}(a^*) = \Delta(a)^*$ for all Lie types.
- The compatibility property is useful for us to formulate an abstract Drinfeld-Kohno theorem in the setting of tensor categories.
- This theorem deals with a square root twist construction that makes the R-matrix "trivial", untwists the unitary structure into a trivial one, but the associator changes into a 3-coboundary deformation, similarly to Drinfeld case.

Vertex Operator Algebras

- A VOA is a $\mathbb{Z}_{\geq 0}$-graded vector space V together with a linear map into the set of operator valued formal distributions acting on V, i.e. $a \in V \to Y(a,z) = \sum_{n\in\mathbb{Z}} a(n) z^{-n-1}$, $a(n) \in \mathrm{End}(V)$, + axioms.
- A V-module is a graded vector space M together with a linear map $a \in V \to Y^M(a,z) = \sum_{n\in\mathbb{Z}} a^M_{(n)} z^{-n-1}$, $a^M_{(n)} \in \mathrm{End}(M)$ which is compatible with the vertex algebra structure of V.
- For an irreducible V-module, $M = \oplus_{n=0}^{\infty} M_n$, $M_n = \mathrm{Ker}(L_0^M - (h+n))$, L_0^M the conformal Hamiltonian on M.
- Let V be a completely rational VOA, and unitary in the sense of CKLW. Then Carpi, Weiner, and Xu define the complex linear category $\mathrm{Rep}^+(V)$ of unitary V-modules, which is a C^*-category.

The Zhu algebra

- Zhu associates to V an associative algebra $A(V)$ which is a quotient of V as a vector space. It has the property that $Z : M \to M_0$ is a bijective correspondence up to isom. between V-modules and $A(V)$-modules. Furthermore, if V satisfies certain rationality conditions, then $A(V)$ is semisimple.
- $A(V)$ admits a canonical *-involution induced by the PCT operator giving the unitary structure on V and an involutive automorphism considered by Zhu.
- When every V-module is unitarizable, $A(V)$ is a C^*-algebra. Moreover $M \to M_0$ gives an equivalence of C^*-categories, $Z : \mathrm{Rep}^+(V) \simeq \mathrm{Rep}^+(A(V))$.

- We consider the special case where V is an affine VOA $V_{\mathfrak{g}_k}$ associated to a complex simple Lie algebra g and a positive integer level k.
- $V_{\mathfrak{g}_k}$ is associated to an affine Kac-Moody algebra $\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}\hat{k}$. Irreducible modules of $V_{\mathfrak{g}_k}$ are parameterized by a certain finite set $\Lambda^{(k)}$ (Frenkel, Zhu).
- $A(V_{\mathfrak{g}_k})$ is canonically isomorphic to a subquotient of $U(\mathfrak{g})$
- $A(V_{\mathfrak{g}_k})$ is isomorphic to $A_W(\mathfrak{g}, q)$.

Theorem The Zhu C^*-algebra $A(V_{\mathfrak{g}_k})$ admits a canonical structure of compatible coboundary weak quasi-Hopf algebra with unitary wqt structure on the forgetful functor obtained by transferring the untwisted structure of $A_W(\mathfrak{g}, q, \ell)$ via Drinfeld-Kohno theorem and Wenzl continuous path argument. The linear category $\mathrm{Rep}(V_{\mathfrak{g}_k})$ becomes a unitary modular tensor category.

The linear *-structure identifies with that of CWX, braiding and unitary structure seems from conformal net theory (Guido-Longo, Wassermann, Toledano-Laredo).

Amenability

- The Grothendieck ring $\mathrm{Gr}(\mathcal{C})$ of a rigid semisimple tensor category $\mathcal{C}$ is called amenable if it admits a dimension function satisfying a certain analytic property.
- By the work of Hiai-Izumi, Popa, Longo-Roberts, Yamagami such a function, d_a called amenable, is unique and satisfies $d_a(x) \leq d'(x)$ for any other dimension function d'.
- if we consider a weak dimension function D (i.e. positive valued and submultiplicative) then we still have $d_a(x) \leq D(x)$.
- For a fusion category, the Frobenius-Perron dimension is the unique positive dimension function and is amenable.
- A wqh algebra has always associated a positive integral valued wdf given by $D([\rho]) = \dim(\mathcal{F}(\rho))$.
- Thus we have for a fusion category when $\mathcal{F} : \mathcal{C} \to \mathrm{Vec}_{f.d.}$ has a weak quasi-tensor structure then

$$\dim(\mathcal{F}(\rho)) \geq \mathrm{FPdim}(\rho).$$

Amenability

- Moreover, if $\mathcal{C}$ and $\mathcal{C}'$ are rigid tensor C^*-categories such that $\mathrm{Gr}(\mathcal{C}')$ has an amenable dimension function and $\mathcal{F} : \mathcal{C} \to \mathcal{C}'$ is a weak tensor functor then extending results by Longo and Roberts we have

$$d_a(\rho) \leq d_i(\mathcal{F}(\rho)) \leq \|F\|\|F^{-1}\|d_i(\rho),$$

  with d_i the intrinsic dimension functions.
- In particular if $\mathcal{F} : \mathcal{C} \to \mathrm{Hilb}_{f.d.}$ has F_2^* and F_2^{-1} isometries (unitarity) then $d_i(\rho) = \dim(\mathcal{F}(\rho))$ and $\mathcal{F}$ is tensor, that is $\mathrm{Nat}(\mathcal{F})$ is a Hopf algebra.

- In other words when a specific *-functor

$$\mathcal{F} : \mathcal{C} \to \mathrm{Hilb}$$

on a fusion C^*-category is given such that the intrinsic dimension differs from the associated vector space dimension then $\mathcal{F}$ admits no unitary weak tensor structure $F_2^* = F_2^{-1}$.
- On the other hand, we know that non-unitary weak tensor structures exist on $W : \mathcal{C}(\mathfrak{g}, q) \to \mathrm{Hilb}$ at level $k > 1$ and their unitarization is a non-trivial 3-coboundary.

Thank You!