Unitary rigid (tensor and) 2-categories and their actions on operator algebras

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Goal: understand categorical structures behind

- classification of group(-like thing) actions $\Gamma \curvearrowright M$
  - cocycle actions
  - cocycle conjugacy
- classification of subfactors $N \subset M$
  - standard invariants

What is a unitary 2-category?

Def <u>unitary category</u> (C^*-cat.) $\mathcal{C}$ is given by

- collection of objects $X, Y, \dots$
- morphism sets $\mathcal{C}(X,Y)$ s.t.
  - $\mathcal{C}(X) = \mathcal{C}(X,X)$ is a C^*-algebra
  - $\mathcal{C}(X,Y)$ is a right Hilbert $\mathcal{C}(X)$-module
- direct sum $X \oplus Y$

Def unitary (nonstrict) __2-category__ is given by

- collection of 0-cells : $i, j, k, \ldots$
- unitary categories $\mathcal{C}_{ij}$
  - objects $X, Y, \ldots$ of $\mathcal{C}_{ij}$: 1-cells $\quad i \xrightarrow{X} j$
  - morphisms $T \in \mathcal{C}_{ij}(X, Y)$: 2-cells $\quad \begin{tikzcd} \bullet \arrow[r, bend left, "X"] \arrow[r, bend right, "Y"'] \arrow[r, phantom, "\Downarrow T"] & \bullet \end{tikzcd}$
- monoidal structure $\mathcal{C}_{ij} \times \mathcal{C}_{jk} \to \mathcal{C}_{ik}$
  - $X \otimes Y \in \mathcal{C}_{ik}$ for $X \in \mathcal{C}_{ij}$, $Y \in \mathcal{C}_{jk}$
  - "units" $\mathbb{1}_i \in \mathcal{C}_{ii}$

- associator: natural unitary morphisms

$$\Phi_{X,Y,Z} : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$$

satisfying standard consistency conditions

e.g. pentagon equation

$$\begin{tikzcd}
 & (X \otimes (Y \otimes Z)) \otimes W \arrow[dr, "\Phi_{X, Y\otimes Z, W}"] & \\
((X \otimes Y) \otimes Z) \otimes W \arrow[ur, "\Phi_{X,Y,Z} \otimes id_W"] \arrow[d, "\Phi_{X\otimes Y, Z, W}"'] & & X \otimes ((Y \otimes Z) \otimes W) \arrow[d, "id_X \otimes \Phi_{Y,Z,W}"] \\
(X \otimes Y) \otimes (Z \otimes W) \arrow[rr, "\Phi_{X,Y,Z\otimes W}"'] & & X \otimes (Y \otimes (Z \otimes W))
\end{tikzcd}$$

Rem: one 0-cell $\rightsquigarrow$ unitary <u>tensor</u> <u>category</u>

Where do 2-categories come from?

Bimodule category

- 0-cells : von Neumann algs $M, N, \dots$
  (often: with faithful traces, ...)
- $\mathcal{C}_{M,N} = \mathrm{Bim}_{M,N}$
  - 1-cells : Hilbert spaces ${}_M H_N, {}_M K_N, \dots$
    with (normal) M-N-bimod. structure
    (often: finite vN dim over M & N, ...)
  - 2-cells : M-N-bimod. homomorphisms
- monoidal structure : Connes fusion

Homomorphism category

- 0-cells: properly infinite vN algs M, N,
- $\mathcal{C}_{M,N} = \mathrm{Hom}(N, M)$
  - 1-cells: (nonunital) $*$-homs $\rho : N \to M$
  - 2-cells: <u>intertwiners</u> $x \in M$ s.t.

$$x\rho(y) = \sigma(y)x \quad \forall\, y \in N$$

- monoidal structure:

  composition as homs $\rho\sigma$

  for intertwiners $\underset{\rho' \leftarrow \rho}{x} \otimes y = x\overbrace{\rho(y)}^{\text{"}\mathrm{id}_\rho \otimes y\text{"}} = \rho'(y)x$

Duality and dimension for unitary 2-categories

Def $\mathcal{C}$: unitary 2-category

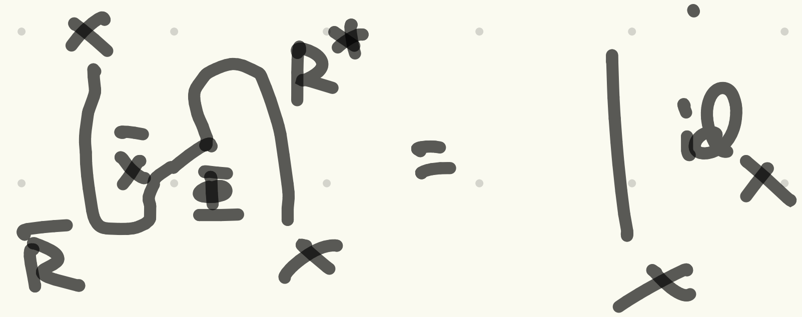
$X \in \mathcal{C}_{ij}$: 1-cell

A **dual** of X is given by

- 1-cell $\bar{X} \in \mathcal{C}_{ji}$
- 2-cells $R : \mathbb{1}_j \to \bar{X} \otimes X$, $\bar{R} : \mathbb{1}_i \to X \otimes \bar{X}$

s.t. $(\mathrm{id}_X \otimes R^*)\, \Phi_{X,\bar{X},X}\, (\bar{R} \otimes \mathrm{id}_X) = \mathrm{id}_X$, etc.

graphically:



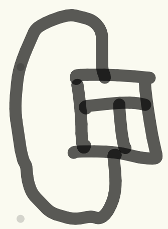
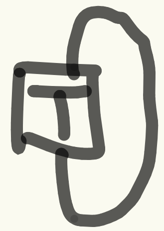
Def assuming $\mathcal{C}_k(\mathbb{1}_k) = \mathbb{C}\,\mathrm{id}_{\mathbb{1}_k}$ (simplicity)

- the **categorical dimension** of $X \in \mathcal{C}_{ij}$ is

$$d(X) = \min_{(R,\bar{R})} \|R\| \cdot \|\bar{R}\|$$

- **categorical trace** on $\mathcal{C}_{ij}(X)$

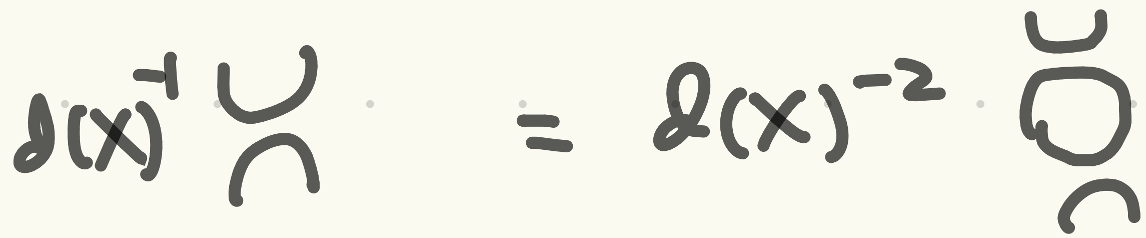
$$\mathrm{Tr}_X(T) = R^*(\mathrm{id}_{\bar{X}} \otimes T)R = \bar{R}^*(T \otimes \mathrm{id}_{\bar{X}})\bar{R}$$

  (sphericity)

for minimizing solutions with $\|R\| = \|\bar{R}\|$

Rem $d(X \oplus Y) = d(X) + d(Y)$, $d(X \otimes Y) = d(X)\,d(Y)$

Rem $X \in \mathcal{C}_i$, $T \geq 0$ in $\mathcal{C}_i(X)$ $\Rightarrow$ $\mathrm{Tr}_X(T) \geq d(X)^{-1} \|T\|$



proj.

$\rightsquigarrow$ 

$\mathrm{id}_X \otimes \mathrm{Tr}_X(T)$

in particular $\dim \mathcal{C}_i(X) < \infty$

$\dim \mathcal{C}_{ij}(X, Y) < \infty$

$\Rightarrow$ $\mathcal{C}_{ij}$ is <u>semisimple</u> ($\exists$ irreducible decomp.

$X \simeq X_1 \oplus \cdots \oplus X_n$, $\mathcal{C}_{ij}(X_k) \simeq \mathbb{C}\,\mathrm{id}$

Amenability for unitary 2-categories

Def $\mathcal{C}$: unitary rigid 2-category

$X \in \mathcal{C}_{ij}$: 1-cell

$\rightsquigarrow$ the <u>fusion</u> <u>operator</u> Γ_X is

$$\ell^2(\mathrm{Irr}\,\mathcal{C}_{jk}) \to \ell^2(\mathrm{Irr}\,\mathcal{C}_{ik}),$$

$$\delta_Y \mapsto \sum_{k=1}^{n} \delta_{Z_k} \quad \text{if} \quad X \otimes Y \simeq Z_1 \oplus \cdots \oplus Z_n$$

Rem $\|\Gamma_X\| \leq d(X)$ from $d(X \otimes Y) = d(X)d(Y)$, etc.

Def. $\mathcal{C}$ is <u>amenable</u> if $\|\Gamma_X\| = d(X)$ holds for all 1-cells

Rem. - $\mathcal{C}$ <u>amenable</u> $\Leftrightarrow$ $\exists/\forall$ i : $\mathcal{C}_i$ is amenable as rigid unitary tensor cat.

- pointed monoidal category ($\mathrm{Irr}\,\mathcal{C} = \Gamma$ for some discrete group, with "same" prod.) is amenable $\Leftrightarrow$ Γ is amenable

$$\|\lambda_{g_1} + \cdots + \lambda_{g_n}\| = n$$

Duality with Frobenius algebra objects

2-category $\mathcal{C}$ $\rightsquigarrow$

- tensor categories $\mathcal{C}_i := \mathcal{C}_{ii}$
- bimodule categories $\mathcal{C}_i \curvearrowright \mathcal{C}_{ij} \curvearrowleft \mathcal{C}_j$
- $\mathcal{C}_j \simeq^{\otimes op} \mathrm{End}_{\mathcal{C}_i}(\mathcal{C}_{ij})$, $\mathcal{C}_i \simeq^{\otimes} \mathrm{End}_{\mathcal{C}_j}(\mathcal{C}_{ij})$

  ($\mathcal{C}_{ij} \simeq \mathrm{End}_{\mathcal{C}_k}(\mathcal{C}_{jk}, \mathcal{C}_{ik})$)

$\rightsquigarrow$ can we internalize such structure (in some corner $\mathcal{C}_i$) ?

Ostrik, --) $\mathcal{C}$ tensor category

$\mathcal{M}$: right $\mathcal{C}$-module category ($\mathcal{M} \curvearrowleft \mathcal{C}$)

with generating obj $X \in \mathcal{M}$

then (morally) $\exists$ <u>algebra</u> object $A \in \mathcal{C}$ s.t.

$\mathcal{M} \simeq A\text{-mod}_{\mathcal{C}} = \{(M \in \mathcal{C}, A \otimes M \to M)\}$

as $\mathcal{C}$-module categories

Idea : <u>internal</u> <u>endomorphism</u> ring

$$\underline{\mathrm{End}}(X) = \bigoplus_{U_k \in \mathrm{Irr}\,\mathcal{C}} \mathcal{M}(X \otimes U_k, X)\, U_k \quad \text{as } A$$

Rem properness $|\{k : \mathcal{M}(X \otimes U_k, X) \neq 0\}| < \infty$

Müger, ...) rigidity for bimod. category

$A\text{-bimod}_{\mathcal{C}} \curvearrowright A\text{-mod}_{\mathcal{C}} \curvearrowleft \mathcal{C}$

corresponds to the <u>Frobenius algebra</u> condition on A:

$m^*: A \to A \otimes A$ is A-bimod. hom for the product morphism $m: A \otimes A \to A$

Morally: "C^*-algebra with faithful state"

Action on von Neumann algebras

Def (··, Tomatsu) $\mathcal{C}$: unitary 2-category

<u>action</u> of $\mathcal{C}$ on von Neumann algebras

$(M_i)_{i:\text{0-cell}}$ is given by a unitary

<u>2</u>-<u>cat</u>eg<u>ory</u> <u>functor</u>

$F: \mathcal{C} \to$ (von Neumann algs, bimodules)

s.t. $i \to M_i$ for 0-cells

<u>Free</u> action : fully faithful functor

Unpacking ingredients

- $X \in \mathcal{C}_{ij} \rightsquigarrow$ bimodule ${}_{M_i}X_{M_j}$, hom $M_j \xrightarrow{\rho_X} M_i$

- $T \in \mathcal{C}_{ij}(X,Y) \rightsquigarrow$ bimod. hom ${}_{M_i}X_{M_j} \to {}_{M_i}Y_{M_j}$

  intertwiner $T \in (\rho_X, \rho_Y) \subset M_i$

- unitary morphisms ${}_{M_i}X \underset{M_j}{\otimes} Y_{M_k} \xrightarrow{F_2} {}_{M_i}(X \otimes Y)_{M_k}$

with compatibility with associators, . .
(2-cocycle condition)

Example

free action of the pointed category $\underline{\Gamma}$ on properly infinite M

$\equiv$ $\Gamma \curvearrowright M$ outer cocycle action

Rem. $\mathcal{C}$ rigid $\curvearrowright (M_i)_i$

$\rightsquigarrow$ ${}_{M_i}X_{M_j}$ "finite index" bimodules

Singly generated rigid 2-category

0-cells : 0, 1

generating 1-cell : $X \in \mathcal{C}_{10}$

i.e., any object $Y \in \mathcal{C}_{ij}$ embeds in

$$(X \text{ or } \bar{X}) \otimes \cdots \otimes \bar{X} \otimes X \otimes \bar{X} \otimes \cdots \otimes (X \text{ or } \bar{X})$$

$\rightsquigarrow$ look for "generators and relations"

presentation of $\mathcal{C}$

Standard invariant of $(\mathcal{C}, X \in \mathcal{C}_{10})$

Ocneanu) paragroup & flat connection

- fusion graphs $\Gamma_X^{(\ell)}$, $\Gamma_X^{(r)}$
- matrix coefficients ("6j-symbol") of

$$\mathcal{C}_{10}(Y, (X\otimes Z)\otimes X) \xrightarrow[\Phi_*]{} \mathcal{C}_{10}(Y, X\otimes(Z\otimes X))$$

Popa) λ-lattice $\qquad d^{-1}\substack{\cup \\ \cap}$

- $A_{m,n} = \mathrm{End}(\underbrace{\cdots\otimes\bar{X}}_{m}\otimes\underbrace{X\otimes\cdots}_{n}) \ni$ Jones proj
- trace $\mathrm{Tr}_{\cdots\otimes\bar{X}\otimes X\otimes\cdots}$ on $A_{m,n}$

Jones) (shaded) planar algebra

- $P_n^{(0)} = e_0(1_0, \underbrace{\bar{X} \otimes X \otimes \cdots \otimes X}_{2n})$

  $P_n^{(1)} = e_1(1_1, \underbrace{X \otimes \bar{X} \otimes \cdots \otimes \bar{X}}_{2n})$

- 2-cat. operations from $(R, \bar{R})$

Classification

Popa, Tomatsu

$\mathcal{C}$: amenable rigid unitary 2-cat.

$(M_i)_{i:\text{0-cells}}$: properly infinite injective factors

then free actions $\mathcal{C} \overset{F,G}{\curvearrowright} (M_i)_i$ are unique up to <u>isomorphism</u> of unitary 2-functors $F \cong G$

Unpacking isom. of 2-functors $F \to G$

$u_X : {}_{M_i}F(X)_{M_j} \to {}_{M_i}G(X)_{M_j}$ unitary bimod. hom

$$\begin{array}{ccc} {}_{M_i}F(X) \underset{M_j}{\otimes} F(Y)_{M_k} & \xrightarrow{u_X \otimes u_Y} & {}_{M_i}G(X) \underset{M_j}{\otimes} G(Y)_{M_k} \\ F_2 \downarrow & & \\ {}_{M_i}F(X \otimes Y)_{M_k} & \xrightarrow[u_{X \otimes Y}]{} & {}_{M_i}G(X \otimes Y)_{M_k} \end{array}$$

compatible with associators

("$\underline{1}$-cocycle")

Ocneanu) $\underset{\text{amen.}}{\Gamma} \curvearrowright \underset{\text{inj.}}{M}$ outer

Popa) $N \subset M$ injective subfactor
with amenable standard invariant
generated by $X = {}_M L^2(M)_N$

Masuda-Tomatsu) $\hat{G} \curvearrowright M_{\text{injective}}$
discr. amen. quantum group

Arano) $\underset{\text{fusion cat}}{\mathcal{C}} \curvearrowright \underset{\text{Kirchbergs als}}{A}$ "Rokhlin action"