The abstract approach to classifying C^*-algebras
Part I: Classifying algebras

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CRC 1442 Geometry: Deformations and Rigidity

Goal/Caveat

In these two expository talks, we aim to give participants working primarily in quantum symmetries, subfactors, tensor categories, etc. a working overview of the classification program in C^*-algebras:

- Capstone Results
- Key Ingredients

Our purpose is not to give a full account of the program or its history but to prime the participants to ask questions and discuss ideas in next week's sessions.

Theorem (2015, Many hands)

Simple, separable, unital, nuclear, $\mathcal{Z}$-stable C^*-algebras in the UCT class are classified by K-theory and traces.

A tale of two algebras

C^*- and W^*-algebras

In the commutative case,

C^*-algebras
are $C_0(X)$,
locally compact Hausdorff X.

W^*-algebras
are $L^\infty(X)$,
regular Borel measure space X.

Even in the noncommutative setting,

C^*-algebras
are more topological.

W^*-algebras
are more measure-theoretic.

Group Algebras

Given a discrete group Γ and its left regular representation $\lambda : \Gamma \to \mathcal{U}(\ell^2(\Gamma))$, where $\lambda_g(\delta_h) = \delta_{gh}$ for $g, h \in \Gamma$,

$\rightsquigarrow$ $\mathrm{C}^*_\lambda(\Gamma) := \overline{\mathbb{C}\lambda(\Gamma)}^{\|\cdot\|}$ is the reduced group C^*-algebra.

$\rightsquigarrow$ $\mathrm{L}(\Gamma) := \overline{\mathbb{C}\lambda(\Gamma)}^{SOT}$ is the group von Neumann algebra.

Example

If Γ is abelian, then $\mathrm{C}^*_\lambda(\Gamma) = C(\hat{\Gamma})$ and $\mathrm{L}(\Gamma) = L^\infty(\hat{\Gamma})$.

- $\mathrm{C}^*_\lambda(\mathbb{Z}) = C(\mathbb{T}) \not\simeq C(\mathbb{T}^2) = \mathrm{C}^*_\lambda(\mathbb{Z}^2)$
- $\mathrm{L}(\mathbb{Z}) = L^\infty(\mathbb{T}) \simeq L^\infty(\mathbb{T}^2) = \mathrm{L}(\mathbb{Z}^2)$

Crossed Products and Dynamics

Let X compact Hausdorff and Γ discrete with $\alpha : \Gamma \curvearrowright X$ by homeomorphisms.

$\rightsquigarrow \Gamma \curvearrowright C(X)$ by $g \cdot f = f \circ \alpha_g^{-1}$, for $g \in \Gamma$, $f \in C(X)$

$\rightsquigarrow$ Crossed product C^*-algebra $C(X) \rtimes_{\lambda,\alpha} \Gamma$

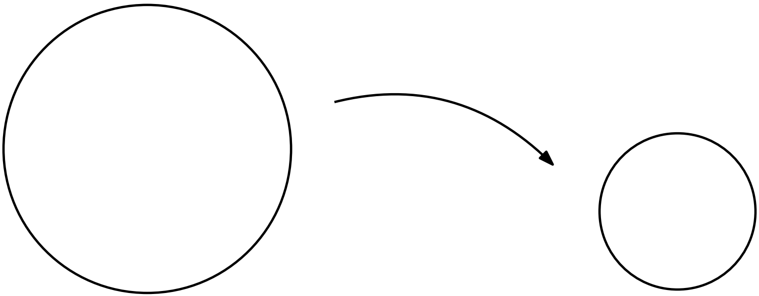
- generated by copies of $\lambda(\Gamma)$ and $C(X)$ (in $B(L^2(X) \otimes \ell^2(\Gamma))$)
- where α is implemented via conjugation: $\lambda_g f \lambda_g^* = \alpha_g(f)$ for $g \in \Gamma$, $f \in C(X)$.

A similar construction with a regular Borel probability space (Y, μ) with probability measure preserving (pmp) $\beta : \Gamma \curvearrowright Y$ yields $L^\infty(Y) \rtimes_\beta \Gamma$.

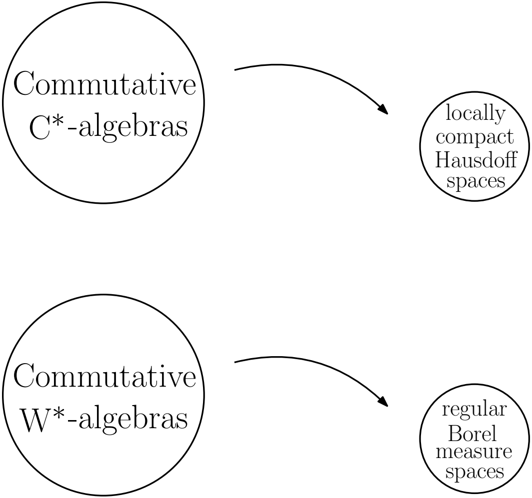
Example

$\mathbb{Z} \curvearrowright \mathbb{T}$ by irrational rotation, $\theta : \mathrm{e}^{2\pi \mathrm{i} t} \mapsto \mathrm{e}^{2\pi \mathrm{i}(\theta + t)}$.
$\rightsquigarrow$ *Irrational rotation algebras* $A_\theta := C(\mathbb{T}) \rtimes_{\lambda,\theta} \mathbb{Z}$ and $L^\infty(\mathbb{T}) \rtimes_\theta \mathbb{Z}$.

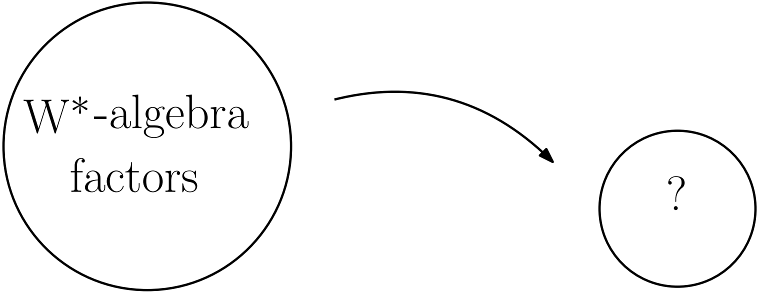
Classification



Classification: Commutative Setting



W^*-Classification: Factors



Let's focus on the story for II_1-factors.

Examples of II_1-factors

- $\mathrm{L}(\Gamma)$ for an infinite conjugacy class (ICC) group Γ

$$\tau(x) := \langle x\delta_e, \delta_e \rangle$$

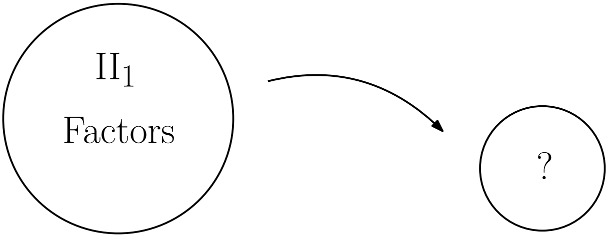
- $L^\infty(X) \rtimes \Lambda$ for a free, ergodic, pmp action $\Lambda \curvearrowright X$ of a group Λ on a probability space (X, μ)

$$\tau\left(\sum_{g\in\Lambda} f_g \lambda_g\right) := \int_X f_e \; d\mu$$

- $$\begin{array}{ccccccc} \mathbb{M}_2 & \hookrightarrow & \mathbb{M}_2 \otimes \mathbb{M}_2 & \hookrightarrow & \dots & \hookrightarrow & \overline{\bigotimes_{k=1}^{\infty} \mathbb{M}_2}^{SOT} \\ \updownarrow & & \updownarrow & & & & \updownarrow \\ \mathbb{M}_2 & \xhookrightarrow{a \mapsto a \oplus a} & \mathbb{M}_4 & \hookrightarrow & \dots & \hookrightarrow & \overline{\bigcup_{k=1}^{\infty} \mathbb{M}_{2^k}}^{SOT} \end{array}$$

  The closure is with respect to the GNS representation for $\tau := \otimes_{k=1}^{\infty} \tau_2$, with τ_2 the normalized trace on $\mathbb{M}_2$.

W^*-Classification: II_1-factors



In general, this is still a lot to ask.
For instance, we still don't know if $L(\mathbb{F}_2) \simeq L(\mathbb{F}_3)$.

We need an additional "smallness criteria".

Smallness Criteria 1: Approximately Finite

A separably acting von Neumann algebra is *Approximately Finite Dimensional* (AFD) if it is the SOT closure of an increasing union of finite dimensional subalgebras.

A tracial AFD von Neumann algebras is called *hyperfinite*.

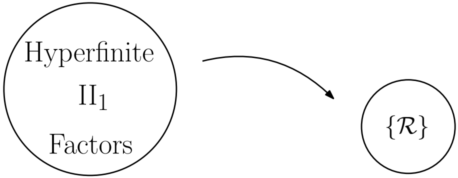
Example

$$\mathbb{M}_2 \xhookrightarrow{a \mapsto a \oplus a} \mathbb{M}_4 \hookrightarrow \dots \hookrightarrow \overline{\bigcup_{k=1}^{\infty} \mathbb{M}_{2^k}}^{SOT}$$

Theorem (Murray, von Neumann)

There is a unique (separably acting) hyperfinite II_1*-factor.*

We denote it by $\mathcal{R}$.



Smallness Criteria 2: Amenability

A group Γ is *amenable* if it admits a finitely additive left invariant probability measure (a mean) on its subsets.

- Includes finite groups and abelian groups
- Closed under subgroups, quotients, extensions, direct limits.
- Does not include non-abelian free groups.

There exists an analogous property for von Neumann algebras, which is more than an analogy for group von Neumann algebras:

$$\mathrm{L}(\Gamma) \text{ is amenable } \iff \Gamma \text{ is amenable.}$$

$\rightsquigarrow \mathrm{L}(\mathbb{F}_n)$ are non-amenable.

Smallness Criteria 2: Amenability

It's "easy" to show that

$$\text{Hyperfinite (AFD)} \implies \text{Amenable.}$$

But, amenability is often easier to verify.

Example

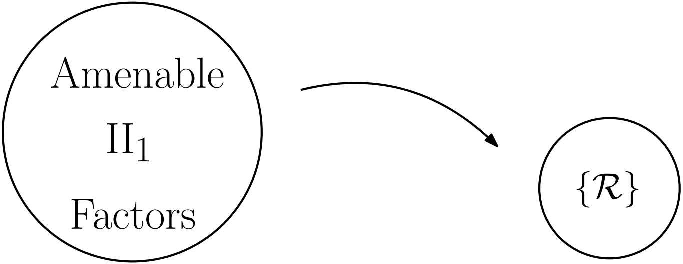
- Finite dimensional and commutative von Neumann algebras
- $\mathrm{L}(\Gamma)$ for an amenable group Γ
- $L^\infty(X) \rtimes \Gamma$ for Γ amenable
- $\mathcal{R}$

Classification of Amenable II_1-factors

Theorem (Connes)

A von Neumann algebra is hyperfinite (AFD) iff it is amenable.

With Murray and von Neumann's classification of hyperfinite II_1-factors:



Ingredients, Scope, and Further Results

Some consequences:

- $\mathrm{L}(\Gamma) \simeq \mathrm{L}(\Lambda)$ for all amenable ICC groups Γ, Λ.
- $L^\infty(\mathbb{T}) \rtimes_\theta \mathbb{Z}$ is independent of $\theta \notin \mathbb{Q}$.
- (Ornstein-Weiss) All free, ergodic, pmp actions of infinite amenable groups are orbit equivalent.

Classification of amenable factors of other types was completed by Connes and Haagerup.

An important factor in Connes work is the fact that any (separably acting) amenable II_1-factor $\mathcal{M}$ is *McDuff*, i.e.,

$$\mathcal{M} \simeq \mathcal{M} \bar{\otimes} \mathcal{R}.$$

From W^*- to C^*-Classification

With a striking and incredibly useful classification result for infinite dimensional, separably acting (tracial) von Neumann factors satisfying certain "smallness criteria," we turn to ask the same for comparable C^*-algebras.

We consider simple, separable, infinite dimensional C^*-algebras (with tracial states), but how do the "smallness criteria" translate to this setting?

Smallness Criteria 1: Approximately Finite

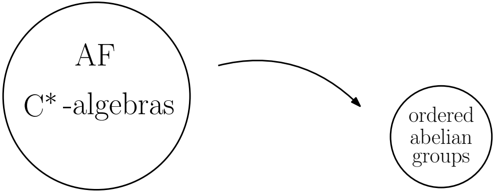
A C^*-algebra is called AF if it is the *norm* closure of an increasing union of finite dimensional subalgebras, e.g., the CAR algebra

$$\mathbb{M}_{2^\infty} := \overline{\bigcup_{k=1}^{\infty} \mathbb{M}_{2^k}} = \overline{\bigotimes_{k=1}^{\infty} \mathbb{M}_2}.$$

Unlike with hyperfinite II_1-factors, simple AF algebras are not all the same:

$$\mathbb{M}_{2^\infty} \not\simeq \mathbb{M}_{3^\infty} := \overline{\bigotimes_{k=1}^{\infty} \mathbb{M}_3}.$$

Still, AF algebras can be classified by their (ordered) K_0-groups.



Smallness Criteria 2: Amenability

There is also a notion of amenability for C^*-algebras, which is also more than an analogy for group C^*-algebras:

$$\mathrm{C}^*_\lambda(\Gamma) \text{ amenable } \iff \Gamma \text{ amenable.}$$

We prefer one of the following two characterizations:

Definition

A is *nuclear* if given any C^*-algebra B, there is a unique way to complete $A \otimes_{\mathrm{alg}} B$ to a C^*-algebra $A \otimes B$.

Definition

A satisfies the *completely positive approximation property* (CPAP) if there exist completely positive contractive (cpc) maps $A \xrightarrow{\psi_i} \mathbb{M}_{n_i} \xrightarrow{\varphi_i} A$ so that

$$\|\varphi_i \circ \psi_i(a) - a\| \to 0 \;\forall\; a \in A.$$

A map $\phi : A \to B$ is *completely positive* if $\phi^{(n)}(\mathbb{M}_n(A)_+) \subset \mathbb{M}_n(B)_+ \;\forall\; n$.

Smallness Criteria 2: Amenability

Example

- Finite dimensional and commutative C^*-algebras
- $\mathrm{C}^*_\lambda(\Gamma)$ for Γ amenable
- $C(X) \rtimes_{\lambda,\alpha} \Gamma$ for Γ amenable
- C^*-algebras that can be built from these via ideals, tensor products, quotients, extensions, and direct limits:
  - AF C^*-algebras
  - A_θ for θ irrational
- Cuntz algebras $\mathcal{O}_n$

Like with von Neumann algebras, AF implies amenability.
Unlike with von Neumann algebras, this class goes way beyond AF.

Classifying Simple Nuclear C^*-algebras?

- If we want to classify a larger class than simple AF C^*-algebras, we need a larger invariant than the ordered K_0-group.
- A larger class of C^*-algebras, which includes A_θ, can be distinguished by ordered K_0-groups together with K_1-groups.
- The final ingredient in the invariant is traces, i.e. the simplex of tracial states $T(A)$ of a C^*-algebra.

Together, these are referred to as *"K-theory and traces"*.

Classification by K-Theory and Traces?

What is the class of *classifiable* C^*-algebras, i.e., those that can be classified by K-theory and traces?

Analogous to the von Neumann setting, we want

- Simple
- Separable
- Infinite dimensional
- Unital

$\rightsquigarrow$ We have to disregard commutative and reduced group C^*-algebras.

We also want to stay in the realm of

- Nuclear
  There are infinitely many simple, separable, unital, exact, non-nuclear C^*-algebras that are indistinguishable from $\mathcal{O}_2$ using just K-theory and traces.

But we keep simple AF, irrational rotation algebras, Cuntz algebras, and crossed products with a free minimal action.

Is that it?

Can we classify simple, separable, infinite dimensional, unital, nuclear C^*-algebras by K-theory and traces?

Still no.
Using higher dimensional topological phenomena one can construct simple, separable, unital, nuclear C^*-algebras that cannot be distinguished by K-theory and traces. (Villadsen, Rørdam, Toms)

In particular, Rørdam gives and example of a simple, separable, unital, nuclear C^*-algebra that is finite but has no traces.

We need additional structural criteria.

Finite nuclear dimension:
Or how I learned to stop worrying and love $\mathcal{Z}$

Classification: Finite nuclear dimension

Theorem (2015, Many hands)

Simple, separable, unital, infinite dimensional, nuclear C^**-algebras with finite nuclear dimension in the UCT class are classified by K-theory and traces.*

Finite nuclear dimension is a refinement of the CPAP for C^*-algebras that incorporates a generalized notion of Lebesgue covering dimension, e.g., $\dim_{\mathrm{nuc}}(C(X)) = \dim(X)$.

Theorem (Castillejos, Evington, Tikuisis, White, Winter^2)

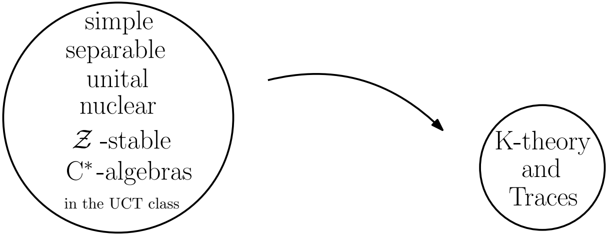
A simple, separable, unital, infinite dimensional, nuclear C^**-algebra* A *has finite nuclear dimension iff it is stable with respect to tensoring with the Jiang-Su algebra* $\mathcal{Z}$*, i.e.*

$$A \otimes \mathcal{Z} \simeq A.$$

Classification: $\mathcal{Z}$-stability

Theorem (2015, Many hands)

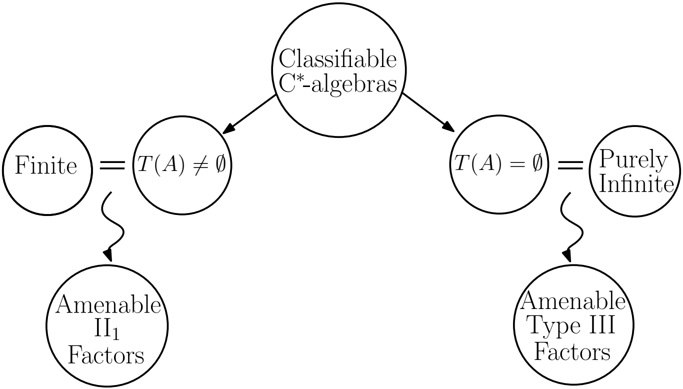
Simple, separable, unital, nuclear, $\mathcal{Z}$-stable C^*-algebras in the UCT class are classified by K-theory and traces.



Tracial Dichotomy

Theorem (Kirchberg)

If A is simple, unital, and nuclear with $T(A) = \emptyset$, then $\mathcal{Z}$-stability implies pure infiniteness.



What is $\mathcal{Z}$?

The Jiang-Su algebra $\mathcal{Z}$ is a simple, separable, unital, nuclear C^*-algebra, which acts like an "infinite-dimensional version of $\mathbb{C}$," in particular

- $\mathcal{Z} \otimes \mathcal{Z} \simeq \mathcal{Z}$
- $\mathcal{Z}$ has the same K-theory and traces as $\mathbb{C}$.

Any unital C^*-algebra A has the same K-theory and traces as $A \otimes \mathcal{Z}$.

$\rightsquigarrow$ We can only classify *up to* $\mathcal{Z}$-stability, i.e., $\mathcal{Z}$-stability is necessary for classification.

But what **is** $\mathcal{Z}$??

$\mathcal{Z}$ can be constructed as the inductive limit of certain so-called *dimension drop* algebras:

$$Z_{p,q} = \{f \in C([0,1], \mathbb{M}_p \otimes \mathbb{M}_q) : f(0) \in \mathbb{M}_p, f(1) \in \mathbb{M}_q\},$$

with p, q co-prime.

But we are more interested in $\mathcal{Z}$ with regards to its role in delineating classifiable C^*-algebras.

Actually, we are mostly interested in $\mathcal{Z}$-stability.

What **is** $\mathcal{Z}$-Stability?

In the tracial setting, $\mathcal{Z}$-stability is the C^*-analogue to the McDuff property ($\mathcal{R}$-stability). In particular,

- McDuff characterized $\mathcal{R}$-stability as having approximately central matrix subalgebras.
- Whereas $\mathcal{Z}$-stability is characterized by having “suitably large” approximately central matrix cones (Rørdam-Winter).

Separable $\mathcal{Z}$-stable C^*-algebras with *-homomorphisms up to approximate unitary equivalence forms monoidal category whose unit is $\mathcal{Z}$.

$\rightsquigarrow$ The class of simple, separable, unital, nuclear C^*-algebras localized at $\mathcal{Z}$ yields the class of classifiable C^*-algebras (modulo UCT).

About the UCT

We also require that a classifiable C^*-algebra satisfies the Universal Coefficient Theorem (UCT).

- Essentially, this says K-theory of a C^*-algebra is enough to describe its KK-theory.
- More formally, A satisfies the UCT iff $KK(A, C) = 0$ whenever $K_*(C) = 0$ (equivalently A is KK-equivalent to a commutative C^*-algebra).

The UCT Problem:

- **Open Question:** Does every separable, nuclear C^*-algebra satisfy the UCT? What about every classifiable C^*-algebra?
- The answer is yes for virtually any example one can write down.
- (Barlak-Li, Tu) A separable, nuclear C^*-algebra satisfies the UCT if it has a Cartan subalgebra.

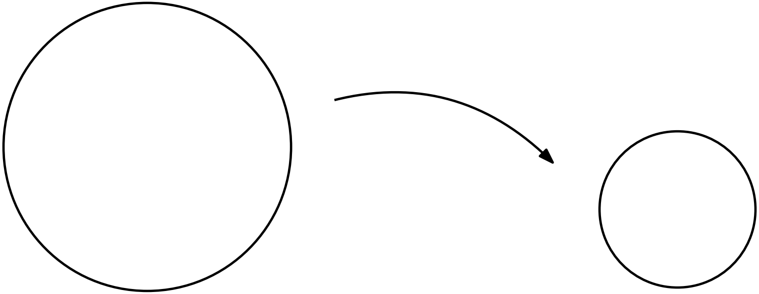
Classifiable C^*-algebras

Classifiable C^*-algebras include

- Simple, unital, infinite dimensional AF algebras,
- Irrational rotation algebras A_θ,
- Cuntz algebras $\mathcal{O}_n$,
- $\mathcal{Z}$-stabilization of Rørdam's and Tom's counterexamples.
- The C^*-algebra $\mathrm{C}^*_\pi(\Gamma)$ generated by an irreducible unitary representation π of a finitely generated nilpotent group, (Eckhardt-Gillaspy)
- $C(X) \rtimes_{\lambda,\alpha} \Gamma$ arising from free minimal actions of groups with local subexponential growth on finite-dimensional spaces (Many Hands $\neq$ the hands in the classification)

"I am prepared to stick my neck out and say that this should hold for all amenable groups - though that's still a long way off."– S. White.

Thanks!



Appendix

Operator K-theory: K_0

Operator algebraic K-theory is the noncommutative extension of topological K-theory of Atiyah and Hirzebruch.

Suppose A is a unital C^*-algebra.

$K_0(A)$ is an ordered abelian group which captures the structure of projections in A and its matrix amplifications $\mathbb{M}_n(A)$.

More precisely, it is the Grothendieck group of its MvN semigroup of projections:

$$\{p \in \bigcup_n \mathbb{M}_n(A) : p \text{ a projection } \} / \sim_{\mathrm{MvN}} .$$

Operator K-theory: K_1

Suppose A is a unital C^*-algebra.

$K_1(A)$ is an abelian group which captures the structure of unitaries in C^*-algebra A and its matrix amplifications $\mathbb{M}_n(A)$.

More precisely, writing $\mathcal{U}_\infty(A) := \bigcup_n \mathcal{U}_n(A)$,

$$K_1(A) := \mathcal{U}_\infty(A)/\sim_h .$$

For $n \geq 0$, we have $K_n(A) \simeq K_{n+2}(SA)$ where $SA := C_0(0,1) \otimes A$.

Bott Periodicity: $K_1(A) \simeq K_0(SA)$ and $K_0(A) \simeq K_1(SA)$.
$\rightsquigarrow$ higher K-groups are redundant.

KK-Theory

Kasparov's $KK(\cdot,\cdot)$ is a bivariant functor on separable C^*-algebras, generalizing both K-homology and K-theory.

Think of KK-equivalence as a loose notion of homotopy equivalence.

(Cuntz) $KK(A,B)$ is an abelian group consisting of homotopy classes of pairs of $*$-homomorphisms $A \to M(B \otimes \mathcal{K})$ who agree modulo $B \otimes \mathcal{K}$.

$\rightsquigarrow$ We can consider a category whose objects are separable C^*-algebras and whose morphisms are KK-elements, i.e. Cuntz pairs. Here $KK(A,B)$ are Hom sets, and isomorphisms are *KK-equivalences*.

Rørdam's Examples

Rørdam constructs

- A simple, separable, unital, nuclear, UCT-class that contains a (nonzero) finite projection and an infinite projection.
- B simple, separable, unital, nuclear, UCT-class such that B is finite but $\mathbb{M}_2(B)$ is (properly) infinite (in particular B is finite but not stably finite).
- Moreover, B is finite but has no traces.