Crossed products by partial actions

Eusebio Gardella,
Universität Münster

(joint with Fernando Abadie and Shirly Geffen)

January 28, 2021

Historical motivation

For an automorphism $\alpha \in \mathrm{Aut}(A)$, the *Pimsner-Voiclescu* 6-term exact sequence

$$\begin{array}{ccccc}
K_0(A) & \xrightarrow{\mathrm{id}-K_0(\alpha)} & K_0(A) & \longrightarrow & K_0(A \rtimes_\alpha \mathbb{Z}) \\
\uparrow & & & & \downarrow \\
K_1(A \rtimes_\alpha \mathbb{Z}) & \longleftarrow & K_1(A) & \xleftarrow[1-K_1(\alpha)]{} & K_1(A)
\end{array}$$

is a fundamental tool in the computation of the crossed product $A \rtimes_\alpha \mathbb{Z}$.

Historical motivation

For an automorphism $\alpha \in \mathrm{Aut}(A)$, the *Pimsner-Voiclescu* 6-term exact sequence

$$\begin{array}{ccccc} K_0(A) & \xrightarrow{\mathrm{id}-K_0(\alpha)} & K_0(A) & \longrightarrow & K_0(A \rtimes_\alpha \mathbb{Z}) \\ \uparrow & & & & \downarrow \\ K_1(A \rtimes_\alpha \mathbb{Z}) & \longleftarrow & K_1(A) & \xleftarrow[1-K_1(\alpha)]{} & K_1(A) \end{array}$$

is a fundamental tool in the computation of the crossed product $A \rtimes_\alpha \mathbb{Z}$. For example, $K_*(A_\theta)$ is most easily computed using it.

Historical motivation

For an automorphism $\alpha \in \mathrm{Aut}(A)$, the *Pimsner-Voiclescu* 6-term exact sequence

$$\begin{array}{ccccc} K_0(A) & \xrightarrow{\mathrm{id}-K_0(\alpha)} & K_0(A) & \longrightarrow & K_0(A \rtimes_\alpha \mathbb{Z}) \\ \uparrow & & & & \downarrow \\ K_1(A \rtimes_\alpha \mathbb{Z}) & \longleftarrow & K_1(A) & \xleftarrow[1-K_1(\alpha)]{} & K_1(A) \end{array}$$

is a fundamental tool in the computation of the crossed product $A \rtimes_\alpha \mathbb{Z}$. For example, $K_*(A_\theta)$ is most easily computed using it. To apply it, one wants to know which algebras are $\mathbb{Z}$-crossed products.

Historical motivation

For an automorphism $\alpha \in \mathrm{Aut}(A)$, the *Pimsner-Voiclescu* 6-term exact sequence

$$\begin{array}{ccccc} K_0(A) & \xrightarrow{\mathrm{id}-K_0(\alpha)} & K_0(A) & \longrightarrow & K_0(A \rtimes_\alpha \mathbb{Z}) \\ \uparrow & & & & \downarrow \\ K_1(A \rtimes_\alpha \mathbb{Z}) & \longleftarrow & K_1(A) & \xleftarrow[1-K_1(\alpha)]{} & K_1(A) \end{array}$$

is a fundamental tool in the computation of the crossed product $A \rtimes_\alpha \mathbb{Z}$. For example, $K_*(A_\theta)$ is most easily computed using it. To apply it, one wants to know which algebras are $\mathbb{Z}$-crossed products. Sometimes this is given,

Historical motivation

For an automorphism $\alpha \in \mathrm{Aut}(A)$, the *Pimsner-Voiclescu* 6-term exact sequence

$$\begin{array}{ccccc} K_0(A) & \xrightarrow{\mathrm{id}-K_0(\alpha)} & K_0(A) & \longrightarrow & K_0(A \rtimes_\alpha \mathbb{Z}) \\ \uparrow & & & & \downarrow \\ K_1(A \rtimes_\alpha \mathbb{Z}) & \longleftarrow & K_1(A) & \xleftarrow[1-K_1(\alpha)]{} & K_1(A) \end{array}$$

is a fundamental tool in the computation of the crossed product $A \rtimes_\alpha \mathbb{Z}$. For example, $K_*(A_\theta)$ is most easily computed using it. To apply it, one wants to know which algebras are $\mathbb{Z}$-crossed products. Sometimes this is given, but if not there is a result of Landstad which tells you when a C^*-algebra is a $\mathbb{Z}$-crossed product.

Historical motivation

For an automorphism $\alpha \in \mathrm{Aut}(A)$, the *Pimsner-Voiclescu* 6-term exact sequence

$$\begin{array}{ccccc} K_0(A) & \xrightarrow{\mathrm{id}-K_0(\alpha)} & K_0(A) & \longrightarrow & K_0(A \rtimes_\alpha \mathbb{Z}) \\ \uparrow & & & & \downarrow \\ K_1(A \rtimes_\alpha \mathbb{Z}) & \longleftarrow & K_1(A) & \xleftarrow[1-K_1(\alpha)]{} & K_1(A) \end{array}$$

is a fundamental tool in the computation of the crossed product $A \rtimes_\alpha \mathbb{Z}$. For example, $K_*(A_\theta)$ is most easily computed using it. To apply it, one wants to know which algebras are $\mathbb{Z}$-crossed products. Sometimes this is given, but if not there is a result of Landstad which tells you when a C^*-algebra is a $\mathbb{Z}$-crossed product.

Theorem (Landstad)

A unital C^*-algebra A is isomorphic to a $\mathbb{Z}$-crossed product if and only if

Historical motivation

For an automorphism $\alpha \in \mathrm{Aut}(A)$, the *Pimsner-Voiclescu* 6-term exact sequence

$$\begin{array}{ccccc}
K_0(A) & \xrightarrow{\mathrm{id}-K_0(\alpha)} & K_0(A) & \longrightarrow & K_0(A \rtimes_\alpha \mathbb{Z}) \\
\uparrow & & & & \downarrow \\
K_1(A \rtimes_\alpha \mathbb{Z}) & \longleftarrow & K_1(A) & \xleftarrow[1-K_1(\alpha)]{} & K_1(A)
\end{array}$$

is a fundamental tool in the computation of the crossed product $A \rtimes_\alpha \mathbb{Z}$. For example, $K_*(A_\theta)$ is most easily computed using it. To apply it, one wants to know which algebras are $\mathbb{Z}$-crossed products. Sometimes this is given, but if not there is a result of Landstad which tells you when a C^*-algebra is a $\mathbb{Z}$-crossed product.

Theorem (Landstad)

A unital C^*-algebra A is isomorphic to a $\mathbb{Z}$-crossed product if and only if there exist a circle action $\gamma \colon \mathbb{T} \to \mathrm{Aut}(A)$

Historical motivation

For an automorphism $\alpha \in \mathrm{Aut}(A)$, the *Pimsner-Voiclescu* 6-term exact sequence

$$\begin{array}{ccccc}
K_0(A) & \xrightarrow{\mathrm{id}-K_0(\alpha)} & K_0(A) & \longrightarrow & K_0(A \rtimes_\alpha \mathbb{Z}) \\
\uparrow & & & & \downarrow \\
K_1(A \rtimes_\alpha \mathbb{Z}) & \longleftarrow & K_1(A) & \xleftarrow[1-K_1(\alpha)]{} & K_1(A)
\end{array}$$

is a fundamental tool in the computation of the crossed product $A \rtimes_\alpha \mathbb{Z}$. For example, $K_*(A_\theta)$ is most easily computed using it. To apply it, one wants to know which algebras are $\mathbb{Z}$-crossed products. Sometimes this is given, but if not there is a result of Landstad which tells you when a C^*-algebra is a $\mathbb{Z}$-crossed product.

Theorem (Landstad)

A unital C^*-algebra A is isomorphic to a $\mathbb{Z}$-crossed product if and only if there exist a circle action $\gamma\colon \mathbb{T} \to \mathrm{Aut}(A)$ and a unitary $u \in \mathcal{U}(A)$ satisfying $\gamma_z(u) = zu$ for all $z \in \mathbb{T}$.

Historical motivation

For an automorphism $\alpha \in \mathrm{Aut}(A)$, the *Pimsner-Voiclescu* 6-term exact sequence

$$\begin{array}{ccccc} K_0(A) & \xrightarrow{\mathrm{id}-K_0(\alpha)} & K_0(A) & \longrightarrow & K_0(A \rtimes_\alpha \mathbb{Z}) \\ \uparrow & & & & \downarrow \\ K_1(A \rtimes_\alpha \mathbb{Z}) & \longleftarrow & K_1(A) & \xleftarrow[1-K_1(\alpha)]{} & K_1(A) \end{array}$$

is a fundamental tool in the computation of the crossed product $A \rtimes_\alpha \mathbb{Z}$. For example, $K_*(A_\theta)$ is most easily computed using it. To apply it, one wants to know which algebras are $\mathbb{Z}$-crossed products. Sometimes this is given, but if not there is a result of Landstad which tells you when a C^*-algebra is a $\mathbb{Z}$-crossed product.

Theorem (Landstad)

A unital C^*-algebra A is isomorphic to a $\mathbb{Z}$-crossed product if and only if there exist a circle action $\gamma\colon \mathbb{T} \to \mathrm{Aut}(A)$ and a unitary $u \in \mathcal{U}(A)$ satisfying $\gamma_z(u) = zu$ for all $z \in \mathbb{T}$. In this case $A \cong A^\gamma \rtimes_{\mathrm{Ad}(u)} \mathbb{Z}$.

Historical motivation

As it turns out, there are many interesting C^*-algebras with canonical circle actions that don't arise as $\mathbb{Z}$-crossed products,

Historical motivation

As it turns out, there are many interesting C^*-algebras with canonical circle actions that don't arise as $\mathbb{Z}$-crossed products, for example AF-algebras, $\mathcal{O}_n$, $\mathcal{O}_E$, Toeplitz algebra.

Historical motivation

As it turns out, there are many interesting C^*-algebras with canonical circle actions that don't arise as $\mathbb{Z}$-crossed products, for example AF-algebras, $\mathcal{O}_n$, $\mathcal{O}_E$, Toeplitz algebra.

In his PhD thesis, Exel showed that under mild assumptions on the circle action

Historical motivation

As it turns out, there are many interesting C^*-algebras with canonical circle actions that don't arise as $\mathbb{Z}$-crossed products, for example AF-algebras, $\mathcal{O}_n$, $\mathcal{O}_E$, Toeplitz algebra.

In his PhD thesis, Exel showed that under mild assumptions on the circle action (much weaker than having an equivariant unitary),

Historical motivation

As it turns out, there are many interesting C^*-algebras with canonical circle actions that don't arise as $\mathbb{Z}$-crossed products, for example AF-algebras, $\mathcal{O}_n$, $\mathcal{O}_E$, Toeplitz algebra.

In his PhD thesis, Exel showed that under mild assumptions on the circle action (much weaker than having an equivariant unitary), the given algebra A is isomorphic to the crossed product of what he called a *partial automorphism*:

Historical motivation

As it turns out, there are many interesting C^*-algebras with canonical circle actions that don't arise as $\mathbb{Z}$-crossed products, for example AF-algebras, $\mathcal{O}_n$, $\mathcal{O}_E$, Toeplitz algebra.

In his PhD thesis, Exel showed that under mild assumptions on the circle action (much weaker than having an equivariant unitary), the given algebra A is isomorphic to the crossed product of what he called a *partial automorphism*: an isomorphism $\alpha\colon I \to J$ between ideals I and J of A.

Historical motivation

As it turns out, there are many interesting C^*-algebras with canonical circle actions that don't arise as $\mathbb{Z}$-crossed products, for example AF-algebras, $\mathcal{O}_n$, $\mathcal{O}_E$, Toeplitz algebra.

In his PhD thesis, Exel showed that under mild assumptions on the circle action (much weaker than having an equivariant unitary), the given algebra A is isomorphic to the crossed product of what he called a *partial automorphism*: an isomorphism $\alpha\colon I \to J$ between ideals I and J of A. This applies to the algebras mentioned above.

Historical motivation

As it turns out, there are many interesting C^*-algebras with canonical circle actions that don't arise as $\mathbb{Z}$-crossed products, for example AF-algebras, $\mathcal{O}_n$, $\mathcal{O}_E$, Toeplitz algebra.

In his PhD thesis, Exel showed that under mild assumptions on the circle action (much weaker than having an equivariant unitary), the given algebra A is isomorphic to the crossed product of what he called a *partial automorphism*: an isomorphism $\alpha\colon I \to J$ between ideals I and J of A. This applies to the algebras mentioned above.

To complement this, he proved a version of the Pimsner-Voiculescu sequence:

Historical motivation

As it turns out, there are many interesting C^*-algebras with canonical circle actions that don't arise as $\mathbb{Z}$-crossed products, for example AF-algebras, $\mathcal{O}_n$, $\mathcal{O}_E$, Toeplitz algebra.

In his PhD thesis, Exel showed that under mild assumptions on the circle action (much weaker than having an equivariant unitary), the given algebra A is isomorphic to the crossed product of what he called a *partial automorphism*: an isomorphism $\alpha\colon I \to J$ between ideals I and J of A. This applies to the algebras mentioned above.

To complement this, he proved a version of the Pimsner-Voiculescu sequence:

$$\begin{array}{ccccc} K_0(I) & \xrightarrow{K_0(\iota)-K_0(\alpha)} & K_0(A) & \longrightarrow & K_0(A \rtimes_\alpha \mathbb{Z}) \\ \uparrow & & & & \downarrow \\ K_1(A \rtimes_\alpha \mathbb{Z}) & \longleftarrow & K_1(A) & \xleftarrow[K_1(\iota)-K_1(\alpha)]{} & K_1(I) \end{array}$$

Historical motivation

As it turns out, there are many interesting C^*-algebras with canonical circle actions that don't arise as $\mathbb{Z}$-crossed products, for example AF-algebras, $\mathcal{O}_n$, $\mathcal{O}_E$, Toeplitz algebra.

In his PhD thesis, Exel showed that under mild assumptions on the circle action (much weaker than having an equivariant unitary), the given algebra A is isomorphic to the crossed product of what he called a *partial automorphism*: an isomorphism $\alpha\colon I \to J$ between ideals I and J of A. This applies to the algebras mentioned above.

To complement this, he proved a version of the Pimsner-Voiculescu sequence:

$$\begin{array}{ccccc} K_0(I) & \xrightarrow{K_0(\iota)-K_0(\alpha)} & K_0(A) & \longrightarrow & K_0(A \rtimes_\alpha \mathbb{Z}) \\ \uparrow & & & & \downarrow \\ K_1(A \rtimes_\alpha \mathbb{Z}) & \longleftarrow & K_1(A) & \xleftarrow[K_1(\iota)-K_1(\alpha)]{} & K_1(I) \end{array}$$

Combining these things, one can (re-)compute the K-theory of many relevant C^*-algebras.

A brief introduction to partial actions

These successes promted the study of partial actions of more general groups.

A brief introduction to partial actions

These successes promted the study of partial actions of more general groups.

Definition (Exel, McClanahan)

A *partial action* of a discrete group G on a C^*-algebra A

A brief introduction to partial actions

These successes promted the study of partial actions of more general groups.

Definition (Exel, McClanahan)

A *partial action* of a discrete group G on a C^*-algebra A is a collection $(A_g)_{g\in G}$ of ideals of A, together with isomorphisms $\alpha_g : A_{g^{-1}} \to A_g$, for $g \in G$,

A brief introduction to partial actions

These successes promted the study of partial actions of more general groups.

Definition (Exel, McClanahan)

A *partial action* of a discrete group G on a C^*-algebra A is a collection $(A_g)_{g \in G}$ of ideals of A, together with isomorphisms $\alpha_g \colon A_{g^{-1}} \to A_g$, for $g \in G$, such that $\alpha_1 = \mathrm{id}_A$ and α_{gh} extends $\alpha_g \circ \alpha_h$ wherever the composition is well-defined.

A brief introduction to partial actions

These successes promted the study of partial actions of more general groups.

Definition (Exel, McClanahan)

A *partial action* of a discrete group G on a C^*-algebra A is a collection $(A_g)_{g \in G}$ of ideals of A, together with isomorphisms $\alpha_g \colon A_{g^{-1}} \to A_g$, for $g \in G$, such that $\alpha_1 = \mathrm{id}_A$ and α_{gh} extends $\alpha_g \circ \alpha_h$ wherever the composition is well-defined.

There are two upshots:

A brief introduction to partial actions

These successes promted the study of partial actions of more general groups.

Definition (Exel, McClanahan)

A *partial action* of a discrete group G on a C^*-algebra A is a collection $(A_g)_{g \in G}$ of ideals of A, together with isomorphisms $\alpha_g \colon A_{g^{-1}} \to A_g$, for $g \in G$, such that $\alpha_1 = \mathrm{id}_A$ and α_{gh} extends $\alpha_g \circ \alpha_h$ wherever the composition is well-defined.

There are two upshots: one can construct new C^*-algebras,

A brief introduction to partial actions

These successes promted the study of partial actions of more general groups.

Definition (Exel, McClanahan)

A *partial action* of a discrete group G on a C^*-algebra A is a collection $(A_g)_{g\in G}$ of ideals of A, together with isomorphisms $\alpha_g \colon A_{g^{-1}} \to A_g$, for $g \in G$, such that $\alpha_1 = \mathrm{id}_A$ and α_{gh} extends $\alpha_g \circ \alpha_h$ wherever the composition is well-defined.

There are two upshots: one can construct new C^*-algebras, and one can also realize known C^*-algebras as partial crossed products, providing more tools to study them.

These successes promted the study of partial actions of more general groups.

Definition (Exel, McClanahan)

A *partial action* of a discrete group G on a C^*-algebra A is a collection $(A_g)_{g \in G}$ of ideals of A, together with isomorphisms $\alpha_g \colon A_{g^{-1}} \to A_g$, for $g \in G$, such that $\alpha_1 = \mathrm{id}_A$ and α_{gh} extends $\alpha_g \circ \alpha_h$ wherever the composition is well-defined.

There are two upshots: one can construct new C^*-algebras, and one can also realize known C^*-algebras as partial crossed products, providing more tools to study them.

Examples

A brief introduction to partial actions

These successes promted the study of partial actions of more general groups.

Definition (Exel, McClanahan)

A *partial action* of a discrete group G on a C^*-algebra A is a collection $(A_g)_{g\in G}$ of ideals of A, together with isomorphisms $\alpha_g : A_{g^{-1}} \to A_g$, for $g \in G$, such that $\alpha_1 = \mathrm{id}_A$ and α_{gh} extends $\alpha_g \circ \alpha_h$ wherever the composition is well-defined.

There are two upshots: one can construct new C^*-algebras, and one can also realize known C^*-algebras as partial crossed products, providing more tools to study them.

Examples

1. Global actions: take $A_g = A$ for all $g \in G$.

A brief introduction to partial actions

These successes promted the study of partial actions of more general groups.

Definition (Exel, McClanahan)

A *partial action* of a discrete group G on a C^*-algebra A is a collection $(A_g)_{g \in G}$ of ideals of A, together with isomorphisms $\alpha_g : A_{g^{-1}} \to A_g$, for $g \in G$, such that $\alpha_1 = \mathrm{id}_A$ and α_{gh} extends $\alpha_g \circ \alpha_h$ wherever the composition is well-defined.

There are two upshots: one can construct new C^*-algebras, and one can also realize known C^*-algebras as partial crossed products, providing more tools to study them.

Examples

1. Global actions: take $A_g = A$ for all $g \in G$.
2. The trivial partial action: take $A_g = \{0\}$ for all $g \in G \setminus \{1\}$.

A brief introduction to partial actions

These successes promted the study of partial actions of more general groups.

Definition (Exel, McClanahan)

A *partial action* of a discrete group G on a C^*-algebra A is a collection $(A_g)_{g\in G}$ of ideals of A, together with isomorphisms $\alpha_g\colon A_{g^{-1}} \to A_g$, for $g \in G$, such that $\alpha_1 = \mathrm{id}_A$ and α_{gh} extends $\alpha_g \circ \alpha_h$ wherever the composition is well-defined.

There are two upshots: one can construct new C^*-algebras, and one can also realize known C^*-algebras as partial crossed products, providing more tools to study them.

Examples

1. Global actions: take $A_g = A$ for all $g \in G$.
2. The trivial partial action: take $A_g = \{0\}$ for all $g \in G \setminus \{1\}$.
3. Solutions to differential equations give partial actions of $\mathbb{R}$.

A brief introduction to partial actions

Prototypical example: restriction to an ideal of a global action

A brief introduction to partial actions

Prototypical example: restriction to an ideal of a global action

Start with a global action $\beta\colon G \to \mathrm{Aut}(B)$, and let A be an ideal in B.

A brief introduction to partial actions

Prototypical example: restriction to an ideal of a global action

Start with a global action $\beta\colon G \to \mathrm{Aut}(B)$, and let A be an ideal in B. Then β induces (by restriction) a partial action α on A:

A brief introduction to partial actions

Prototypical example: restriction to an ideal of a global action

Start with a global action $\beta\colon G \to \mathrm{Aut}(B)$, and let A be an ideal in B. Then β induces (by restriction) a partial action α on A:

$$A_g = A \cap \beta_g(A) \quad \text{and} \quad \alpha_g = \beta_g|_{A_{g^{-1}}} : A_{g^{-1}} \to A_g.$$

A brief introduction to partial actions

Prototypical example: restriction to an ideal of a global action

Start with a global action $\beta\colon G \to \mathrm{Aut}(B)$, and let A be an ideal in B. Then β induces (by restriction) a partial action α on A:

$$A_g = A \cap \beta_g(A) \quad \text{and} \quad \alpha_g = \beta_g|_{A_{g^{-1}}} \colon A_{g^{-1}} \to A_g.$$

Partial actions of this form are called *globalizable*.

A brief introduction to partial actions

Prototypical example: restriction to an ideal of a global action

Start with a global action $\beta\colon G \to \mathrm{Aut}(B)$, and let A be an ideal in B. Then β induces (by restriction) a partial action α on A:

$$A_g = A \cap \beta_g(A) \quad \text{and} \quad \alpha_g = \beta_g|_{A_{g^{-1}}} \colon A_{g^{-1}} \to A_g.$$

Partial actions of this form are called *globalizable*.

There are many tools to study partial actions:

A brief introduction to partial actions

Prototypical example: restriction to an ideal of a global action

Start with a global action $\beta\colon G \to \mathrm{Aut}(B)$, and let A be an ideal in B. Then β induces (by restriction) a partial action α on A:

$$A_g = A \cap \beta_g(A) \quad \text{and} \quad \alpha_g = \beta_g|_{A_{g^{-1}}} : A_{g^{-1}} \to A_g.$$

Partial actions of this form are called *globalizable*.

There are many tools to study partial actions: Takai duality;

A brief introduction to partial actions

Prototypical example: restriction to an ideal of a global action

Start with a global action $\beta\colon G \to \mathrm{Aut}(B)$, and let A be an ideal in B. Then β induces (by restriction) a partial action α on A:

$$A_g = A \cap \beta_g(A) \quad \text{and} \quad \alpha_g = \beta_g|_{A_{g^{-1}}} : A_{g^{-1}} \to A_g.$$

Partial actions of this form are called *globalizable*.

There are many tools to study partial actions: Takai duality; numerous connections to groupoids and Fell bundles;

A brief introduction to partial actions

Prototypical example: restriction to an ideal of a global action

Start with a global action $\beta \colon G \to \mathrm{Aut}(B)$, and let A be an ideal in B. Then β induces (by restriction) a partial action α on A:

$$A_g = A \cap \beta_g(A) \quad \text{and} \quad \alpha_g = \beta_g|_{A_{g^{-1}}} \colon A_{g^{-1}} \to A_g .$$

Partial actions of this form are called *globalizable*.

There are many tools to study partial actions: Takai duality; numerous connections to groupoids and Fell bundles; notions of amenability;

A brief introduction to partial actions

Prototypical example: restriction to an ideal of a global action

Start with a global action $\beta\colon G \to \mathrm{Aut}(B)$, and let A be an ideal in B. Then β induces (by restriction) a partial action α on A:

$$A_g = A \cap \beta_g(A) \quad \text{and} \quad \alpha_g = \beta_g|_{A_{g^{-1}}} : A_{g^{-1}} \to A_g.$$

Partial actions of this form are called *globalizable*.

There are many tools to study partial actions: Takai duality; numerous connections to groupoids and Fell bundles; notions of amenability; Morita globalizations; etc.

A brief introduction to partial actions

Prototypical example: restriction to an ideal of a global action

Start with a global action $\beta\colon G \to \mathrm{Aut}(B)$, and let A be an ideal in B. Then β induces (by restriction) a partial action α on A:

$$A_g = A \cap \beta_g(A) \quad \text{and} \quad \alpha_g = \beta_g|_{A_{g^{-1}}} : A_{g^{-1}} \to A_g.$$

Partial actions of this form are called *globalizable*.

There are many tools to study partial actions: Takai duality; numerous connections to groupoids and Fell bundles; notions of amenability; Morita globalizations; etc.

However, things don't work the same way as for global actions,

A brief introduction to partial actions

Prototypical example: restriction to an ideal of a global action

Start with a global action $\beta\colon G \to \mathrm{Aut}(B)$, and let A be an ideal in B. Then β induces (by restriction) a partial action α on A:

$$A_g = A \cap \beta_g(A) \quad \text{and} \quad \alpha_g = \beta_g|_{A_{g^{-1}}} : A_{g^{-1}} \to A_g.$$

Partial actions of this form are called *globalizable*.

There are many tools to study partial actions: Takai duality; numerous connections to groupoids and Fell bundles; notions of amenability; Morita globalizations; etc.

However, things don't work the same way as for global actions, and many of the most used results in the global setting simply <u>fail</u> for partial actions.

Partial actions of finite groups: what goes wrong

We now focus on finite groups.

Partial actions of finite groups: what goes wrong

We now focus on finite groups. In the global setting, averaging arguments are used all the time.

Partial actions of finite groups: what goes wrong

We now focus on finite groups. In the global setting, averaging arguments are used all the time. For example, there is a faithful conditional expectation $E\colon A \to A^\alpha$, given by $E(a) = \frac{1}{|G|}\sum_{g\in G}\alpha_g(a)$.

Partial actions of finite groups: what goes wrong

We now focus on finite groups. In the global setting, averaging arguments are used all the time. For example, there is a faithful conditional expectation $E\colon A \to A^\alpha$, given by $E(a) = \frac{1}{|G|}\sum_{g\in G}\alpha_g(a)$. This does not work in the partial setting!

Partial actions of finite groups: what goes wrong

We now focus on finite groups. In the global setting, averaging arguments are used all the time. For example, there is a faithful conditional expectation $E\colon A \to A^\alpha$, given by $E(a) = \frac{1}{|G|}\sum_{g\in G}\alpha_g(a)$. This does not work in the partial setting!

Definition (fixed point algebra)

For a partial action $\alpha = \big((A_g)_{g\in G}, (\alpha_g)_{g\in G}\big)$, we set

Partial actions of finite groups: what goes wrong

We now focus on finite groups. In the global setting, averaging arguments are used all the time. For example, there is a faithful conditional expectation $E\colon A \to A^\alpha$, given by $E(a) = \frac{1}{|G|}\sum_{g\in G}\alpha_g(a)$. This does not work in the partial setting!

Definition (fixed point algebra)

For a partial action $\alpha = \big((A_g)_{g\in G}, (\alpha_g)_{g\in G}\big)$, we set

$$A^\alpha = \{a \in A\colon \alpha_g(ax_{g^{-1}}) = a\alpha_g(x_{g^{-1}}) \text{ for all } x_{g^{-1}} \in A_{g^{-1}}\}.$$

Partial actions of finite groups: what goes wrong

We now focus on finite groups. In the global setting, averaging arguments are used all the time. For example, there is a faithful conditional expectation $E\colon A \to A^{\alpha}$, given by $E(a) = \frac{1}{|G|}\sum_{g\in G}\alpha_g(a)$. This does not work in the partial setting!

Definition (fixed point algebra)

For a partial action $\alpha = \big((A_g)_{g\in G}, (\alpha_g)_{g\in G}\big)$, we set

$$A^{\alpha} = \{a \in A\colon \alpha_g(ax_{g^{-1}}) = a\alpha_g(x_{g^{-1}}) \text{ for all } x_{g^{-1}} \in A_{g^{-1}}\}.$$

Fixed point algebras are often too small:

Partial actions of finite groups: what goes wrong

We now focus on finite groups. In the global setting, averaging arguments are used all the time. For example, there is a faithful conditional expectation $E\colon A \to A^\alpha$, given by $E(a) = \frac{1}{|G|}\sum_{g\in G}\alpha_g(a)$. This does not work in the partial setting!

Definition (fixed point algebra)

For a partial action $\alpha = \big((A_g)_{g\in G}, (\alpha_g)_{g\in G}\big)$, we set

$$A^\alpha = \{a \in A\colon \alpha_g(ax_{g^{-1}}) = a\alpha_g(x_{g^{-1}}) \text{ for all } x_{g^{-1}} \in A_{g^{-1}}\}.$$

Fixed point algebras are often too small: there does not always exist a faithful conditional expectation $E\colon A \to A^\alpha$.

Partial actions of finite groups: what goes wrong

We now focus on finite groups. In the global setting, averaging arguments are used all the time. For example, there is a faithful conditional expectation $E\colon A \to A^\alpha$, given by $E(a) = \frac{1}{|G|}\sum_{g\in G} \alpha_g(a)$. This does not work in the partial setting!

Definition (fixed point algebra)

For a partial action $\alpha = \big((A_g)_{g\in G}, (\alpha_g)_{g\in G}\big)$, we set

$$A^\alpha = \{a \in A\colon \alpha_g(ax_{g^{-1}}) = a\alpha_g(x_{g^{-1}}) \text{ for all } x_{g^{-1}} \in A_{g^{-1}}\}.$$

Fixed point algebras are often too small: there does not always exist a faithful conditional expectation $E\colon A \to A^\alpha$.

Another important feature of finite group actions is the fact that there is a corner embedding

$$c\colon A^\alpha \to A \rtimes_\alpha G$$

Partial actions of finite groups: what goes wrong

We now focus on finite groups. In the global setting, averaging arguments are used all the time. For example, there is a faithful conditional expectation $E\colon A \to A^\alpha$, given by $E(a) = \frac{1}{|G|}\sum_{g\in G} \alpha_g(a)$. This does not work in the partial setting!

Definition (fixed point algebra)

For a partial action $\alpha = \big((A_g)_{g\in G}, (\alpha_g)_{g\in G}\big)$, we set

$$A^\alpha = \{a \in A \colon \alpha_g(ax_{g^{-1}}) = a\alpha_g(x_{g^{-1}}) \text{ for all } x_{g^{-1}} \in A_{g^{-1}}\}.$$

Fixed point algebras are often too small: there does not always exist a faithful conditional expectation $E\colon A \to A^\alpha$.

Another important feature of finite group actions is the fact that there is a corner embedding

$$c\colon A^\alpha \to A \rtimes_\alpha G$$

given by $c(a) = \frac{1}{|G|}\sum_{g\in G} u_g a$.

Partial actions of finite groups: what goes wrong

We now focus on finite groups. In the global setting, averaging arguments are used all the time. For example, there is a faithful conditional expectation $E\colon A \to A^{\alpha}$, given by $E(a) = \frac{1}{|G|}\sum_{g\in G}\alpha_g(a)$. This does not work in the partial setting!

Definition (fixed point algebra)

For a partial action $\alpha = \big((A_g)_{g\in G}, (\alpha_g)_{g\in G}\big)$, we set

$$A^{\alpha} = \{a \in A\colon \alpha_g(a x_{g^{-1}}) = a\alpha_g(x_{g^{-1}}) \ \text{ for all } x_{g^{-1}} \in A_{g^{-1}}\}.$$

Fixed point algebras are often too small: there does not always exist a faithful conditional expectation $E\colon A \to A^{\alpha}$.

Another important feature of finite group actions is the fact that there is a corner embedding

$$c\colon A^{\alpha} \to A \rtimes_{\alpha} G$$

given by $c(a) = \frac{1}{|G|}\sum_{g\in G} u_g a$. This also fails in the partial setting.

Our projects

The initial goal of the project was to study the structure of crossed products and fixed point algebras by partial actions of finite groups.

Our projects

The initial goal of the project was to study the structure of crossed products and fixed point algebras by partial actions of finite groups. In the global setting, a lot can be said if one assumes, for example, the Rokhlin property,

Our projects

The initial goal of the project was to study the structure of crossed products and fixed point algebras by partial actions of finite groups. In the global setting, a lot can be said if one assumes, for example, the Rokhlin property, so we wanted to extend this notion to the partial setting and prove preservation results.

Our projects

The initial goal of the project was to study the structure of crossed products and fixed point algebras by partial actions of finite groups. In the global setting, a lot can be said if one assumes, for example, the Rokhlin property, so we wanted to extend this notion to the partial setting and prove preservation results. The first problem was to find the right definition;

Our projects

The initial goal of the project was to study the structure of crossed products and fixed point algebras by partial actions of finite groups. In the global setting, a lot can be said if one assumes, for example, the Rokhlin property, so we wanted to extend this notion to the partial setting and prove preservation results. The first problem was to find the right definition; this took some years (!) to get settled.

Our projects

The initial goal of the project was to study the structure of crossed products and fixed point algebras by partial actions of finite groups. In the global setting, a lot can be said if one assumes, for example, the Rokhlin property, so we wanted to extend this notion to the partial setting and prove preservation results. The first problem was to find the right definition; this took some years (!) to get settled. But the biggest problem was that the usual arguments from the global setting don't work here.

Our projects

The initial goal of the project was to study the structure of crossed products and fixed point algebras by partial actions of finite groups. In the global setting, a lot can be said if one assumes, for example, the Rokhlin property, so we wanted to extend this notion to the partial setting and prove preservation results. The first problem was to find the right definition; this took some years (!) to get settled. But the biggest problem was that the usual arguments from the global setting don't work here. The Rokhlin property is an averaging property by construction,

Our projects

The initial goal of the project was to study the structure of crossed products and fixed point algebras by partial actions of finite groups. In the global setting, a lot can be said if one assumes, for example, the Rokhlin property, so we wanted to extend this notion to the partial setting and prove preservation results. The first problem was to find the right definition; this took some years (!) to get settled. But the biggest problem was that the usual arguments from the global setting don't work here. The Rokhlin property is an averaging property by construction, but for partial actions all these arguments break down.

Our projects

The initial goal of the project was to study the structure of crossed products and fixed point algebras by partial actions of finite groups. In the global setting, a lot can be said if one assumes, for example, the Rokhlin property, so we wanted to extend this notion to the partial setting and prove preservation results. The first problem was to find the right definition; this took some years (!) to get settled. But the biggest problem was that the usual arguments from the global setting don't work here. The Rokhlin property is an averaging property by construction, but for partial actions all these arguments break down.

So we had to take a step back and first try to understand partial actions of finite groups better (without any Rokhlin-type assumptions).

Our projects

The initial goal of the project was to study the structure of crossed products and fixed point algebras by partial actions of finite groups. In the global setting, a lot can be said if one assumes, for example, the Rokhlin property, so we wanted to extend this notion to the partial setting and prove preservation results. The first problem was to find the right definition; this took some years (!) to get settled. But the biggest problem was that the usual arguments from the global setting don't work here. The Rokhlin property is an averaging property by construction, but for partial actions all these arguments break down.

So we had to take a step back and first try to understand partial actions of finite groups better (without any Rokhlin-type assumptions). This brings us to the decomposition property.

The decomposition property

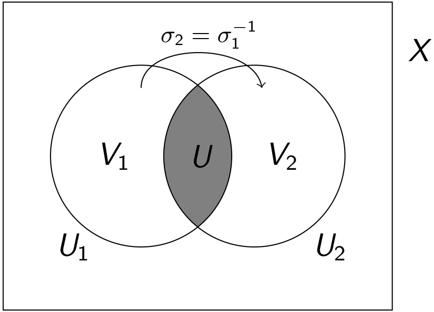
We begin with a motivating example.

The decomposition property

We begin with a motivating example. An action of $\mathbb{Z}_3 = \{0, 1, 2\}$ on a compact Hausdorff space X is given by the choice of two open subsets $U_1, U_2 \subseteq X$, and a homeomorphism $\sigma_1 \colon U_2 \to U_1$.

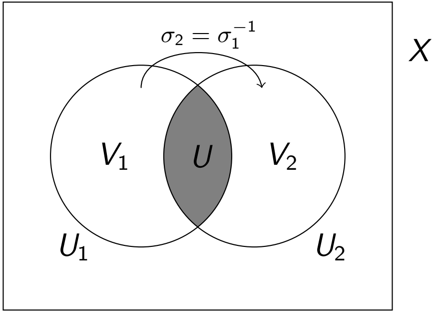
The decomposition property

We begin with a motivating example. An action of $\mathbb{Z}_3 = \{0, 1, 2\}$ on a compact Hausdorff space X is given by the choice of two open subsets $U_1, U_2 \subseteq X$, and a homeomorphism $\sigma_1 \colon U_2 \to U_1$.



The decomposition property

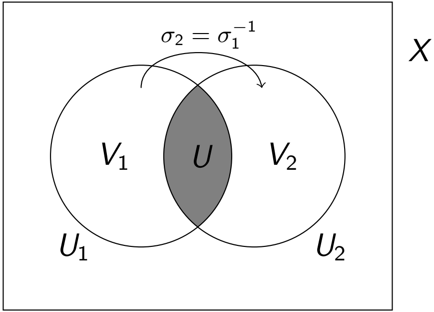
We begin with a motivating example. An action of $\mathbb{Z}_3 = \{0, 1, 2\}$ on a compact Hausdorff space X is given by the choice of two open subsets $U_1, U_2 \subseteq X$, and a homeomorphism $\sigma_1 \colon U_2 \to U_1$.



Set $U = U_1 \cap U_2$, $V_1 = U_1 \setminus U$, $V_2 = U_2 \setminus U$, and $Y = X \setminus (U_1 \cup U_2)$.

The decomposition property

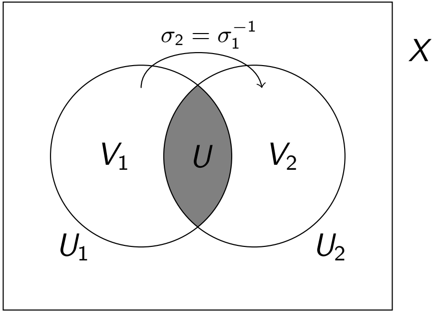
We begin with a motivating example. An action of $\mathbb{Z}_3 = \{0, 1, 2\}$ on a compact Hausdorff space X is given by the choice of two open subsets $U_1, U_2 \subseteq X$, and a homeomorphism $\sigma_1 \colon U_2 \to U_1$.



Set $U = U_1 \cap U_2$, $V_1 = U_1 \setminus U$, $V_2 = U_2 \setminus U$, and $Y = X \setminus (U_1 \cup U_2)$. The restriction of σ to U is global;

The decomposition property

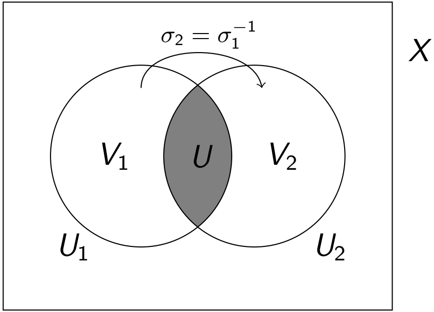
We begin with a motivating example. An action of $\mathbb{Z}_3 = \{0, 1, 2\}$ on a compact Hausdorff space X is given by the choice of two open subsets $U_1, U_2 \subseteq X$, and a homeomorphism $\sigma_1 \colon U_2 \to U_1$.



Set $U = U_1 \cap U_2$, $V_1 = U_1 \setminus U$, $V_2 = U_2 \setminus U$, and $Y = X \setminus (U_1 \cup U_2)$. The restriction of σ to U is global; the restriction of σ to Y is the trivial partial action;

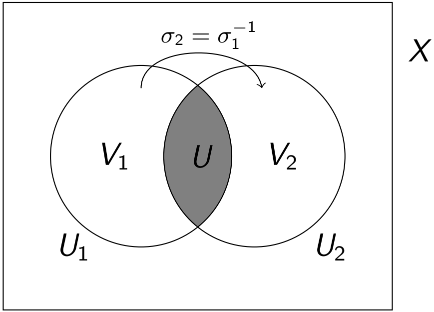
The decomposition property

We begin with a motivating example. An action of $\mathbb{Z}_3 = \{0, 1, 2\}$ on a compact Hausdorff space X is given by the choice of two open subsets $U_1, U_2 \subseteq X$, and a homeomorphism $\sigma_1 \colon U_2 \to U_1$.

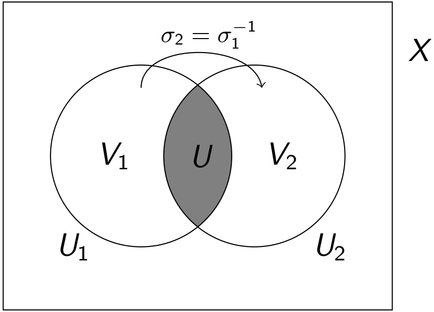


Set $U = U_1 \cap U_2$, $V_1 = U_1 \setminus U$, $V_2 = U_2 \setminus U$, and $Y = X \setminus (U_1 \cup U_2)$. The restriction of σ to U is global; the restriction of σ to Y is the trivial partial action; and the restriction of σ to $V_1 \sqcup V_2$ exchanges V_1 and V_2.

The decomposition property

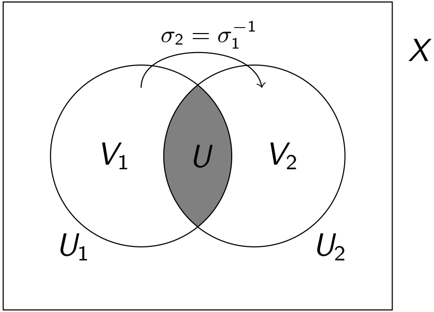


The decomposition property



The restriction of σ to U is global; the restriction of σ to Y is the trivial partial action; and the restriction of σ to $V_1 \sqcup V_2$ exchanges V_1 and V_2.

The decomposition property

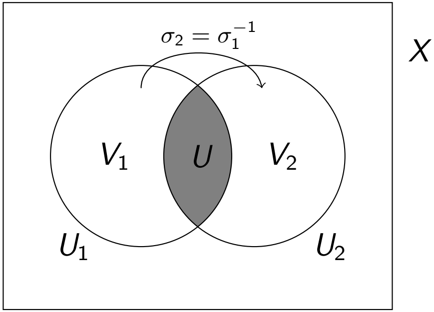


The restriction of σ to U is global; the restriction of σ to Y is the trivial partial action; and the restriction of σ to $V_1 \sqcup V_2$ exchanges V_1 and V_2. We get equivariant extensions

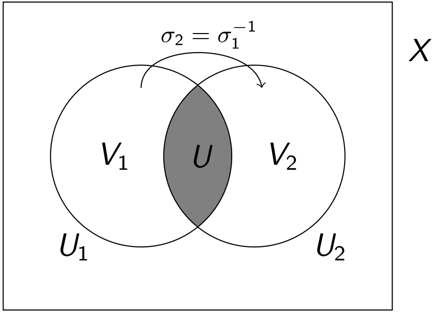
$$0 \longrightarrow C_0(U) \longrightarrow C(X) \longrightarrow C(X \setminus U) \longrightarrow 0, \quad \text{and}$$

$$0 \longrightarrow C_0(V_1 \sqcup V_2) \longrightarrow C(X \setminus U) \longrightarrow C(Y) \longrightarrow 0.$$

The decomposition property

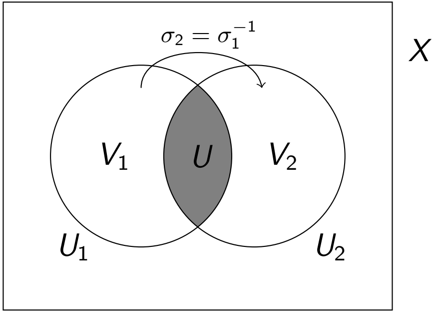


The decomposition property



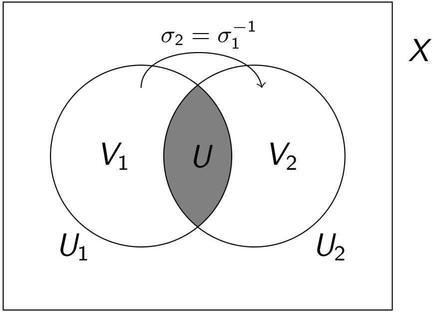
We understand the partial action on U (it's global);

The decomposition property



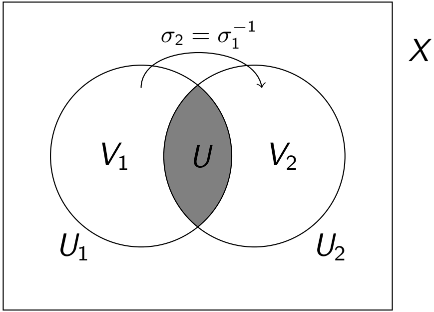
We understand the partial action on U (it's global); the partial action on $X \setminus (U_1 \cup U_2)$ (it's trivial);

The decomposition property



We understand the partial action on U (it's global); the partial action on $X \setminus (U_1 \cup U_2)$ (it's trivial); and the action on $V_1 \sqcup V_2$ looks like a translation plus an internal symmetry (and some restriction).

The decomposition property



We understand the partial action on U (it's global); the partial action on $X \setminus (U_1 \cup U_2)$ (it's trivial); and the action on $V_1 \sqcup V_2$ looks like a translation plus an internal symmetry (and some restriction). The entire complexity of the system is then encoded in the system $V_1 \sqcup V_2$, together with the different gluings.

The decomposition property

This can be done in greater generality, but the combinatorics get extremely complicated.

The decomposition property

This can be done in greater generality, but the combinatorics get extremely complicated. One gets $|G| - 1$ extensions, where in each of them the ideal is "like" the action on $V_1 \sqcup V_2$ we had before.

The decomposition property

This can be done in greater generality, but the combinatorics get extremely complicated. One gets $|G| - 1$ extensions, where in each of them the ideal is "like" the action on $V_1 \sqcup V_2$ we had before. We isolated the main conceptual property satisfied with these systems and studied it independently.

The decomposition property

This can be done in greater generality, but the combinatorics get extremely complicated. One gets $|G| - 1$ extensions, where in each of them the ideal is “like” the action on $V_1 \sqcup V_2$ we had before. We isolated the main conceptual property satisfied with these systems and studied it independently. We call it the *decomposition property*.

The decomposition property

This can be done in greater generality, but the combinatorics get extremely complicated. One gets $|G| - 1$ extensions, where in each of them the ideal is “like” the action on $V_1 \sqcup V_2$ we had before. We isolated the main conceptual property satisfied with these systems and studied it independently. We call it the *decomposition property*.

Given $n \in \mathbb{N}$, set $\mathcal{T}_n(G) = \{\tau \subseteq G : 1 \in \tau, |\tau| = n\}$.

The decomposition property

This can be done in greater generality, but the combinatorics get extremely complicated. One gets $|G|-1$ extensions, where in each of them the ideal is "like" the action on $V_1 \sqcup V_2$ we had before. We isolated the main conceptual property satisfied with these systems and studied it independently. We call it the *decomposition property*.

Given $n \in \mathbb{N}$, set $\mathcal{T}_n(G) = \{\tau \subseteq G : 1 \in \tau, |\tau| = n\}$. For a partial action α of G on A, we set $A_\tau = \bigcap_{g\in\tau} A_g$ for $\tau \in \mathcal{T}_n(G)$.

The decomposition property

This can be done in greater generality, but the combinatorics get extremely complicated. One gets $|G| - 1$ extensions, where in each of them the ideal is "like" the action on $V_1 \sqcup V_2$ we had before. We isolated the main conceptual property satisfied with these systems and studied it independently. We call it the *decomposition property*.

Given $n \in \mathbb{N}$, set $\mathcal{T}_n(G) = \{\tau \subseteq G : 1 \in \tau, |\tau| = n\}$. For a partial action α of G on A, we set $A_\tau = \bigcap_{g \in \tau} A_g$ for $\tau \in \mathcal{T}_n(G)$. Then A_τ is an ideal in A.

The decomposition property

This can be done in greater generality, but the combinatorics get extremely complicated. One gets $|G|-1$ extensions, where in each of them the ideal is "like" the action on $V_1 \sqcup V_2$ we had before. We isolated the main conceptual property satisfied with these systems and studied it independently. We call it the *decomposition property*.

Given $n \in \mathbb{N}$, set $\mathcal{T}_n(G) = \{\tau \subseteq G : 1 \in \tau, |\tau| = n\}$. For a partial action α of G on A, we set $A_\tau = \bigcap_{g\in\tau} A_g$ for $\tau \in \mathcal{T}_n(G)$. Then A_τ is an ideal in A.

Definition

We say that α has the *n-decomposition property* if

The decomposition property

This can be done in greater generality, but the combinatorics get extremely complicated. One gets $|G| - 1$ extensions, where in each of them the ideal is "like" the action on $V_1 \sqcup V_2$ we had before. We isolated the main conceptual property satisfied with these systems and studied it independently. We call it the *decomposition property*.

Given $n \in \mathbb{N}$, set $\mathcal{T}_n(G) = \{\tau \subseteq G : 1 \in \tau, |\tau| = n\}$. For a partial action α of G on A, we set $A_\tau = \bigcap_{g \in \tau} A_g$ for $\tau \in \mathcal{T}_n(G)$. Then A_τ is an ideal in A.

Definition

We say that α has the *n-decomposition property* if $A \cong \bigoplus_{\tau \in \mathcal{T}_n(G)} A_\tau$.

The decomposition property

This can be done in greater generality, but the combinatorics get extremely complicated. One gets $|G|-1$ extensions, where in each of them the ideal is "like" the action on $V_1 \sqcup V_2$ we had before. We isolated the main conceptual property satisfied with these systems and studied it independently. We call it the *decomposition property*.

Given $n \in \mathbb{N}$, set $\mathcal{T}_n(G) = \{\tau \subseteq G \colon 1 \in \tau, |\tau| = n\}$. For a partial action α of G on A, we set $A_\tau = \bigcap_{g\in\tau} A_g$ for $\tau \in \mathcal{T}_n(G)$. Then A_τ is an ideal in A.

Definition

We say that α has the *n-decomposition property* if $A \cong \bigoplus_{\tau \in \mathcal{T}_n(G)} A_\tau$. We say that α is *decomposable* if it has the n-decomposition property for some $n \in \mathbb{N}$.

The decomposition property

This can be done in greater generality, but the combinatorics get extremely complicated. One gets $|G| - 1$ extensions, where in each of them the ideal is "like" the action on $V_1 \sqcup V_2$ we had before. We isolated the main conceptual property satisfied with these systems and studied it independently. We call it the *decomposition property*.

Given $n \in \mathbb{N}$, set $\mathcal{T}_n(G) = \{\tau \subseteq G \colon 1 \in \tau, |\tau| = n\}$. For a partial action α of G on A, we set $A_\tau = \bigcap_{g \in \tau} A_g$ for $\tau \in \mathcal{T}_n(G)$. Then A_τ is an ideal in A.

Definition

We say that α has the *n-decomposition property* if $A \cong \bigoplus_{\tau \in \mathcal{T}_n(G)} A_\tau$. We say that α is *decomposable* if it has the n-decomposition property for some $n \in \mathbb{N}$.

The decomposition property

Again: α is n-decomposable if $A \cong \bigoplus_{\tau \in \mathcal{T}_n(G)} A_\tau$.

The decomposition property

Again: α is n-decomposable if $A \cong \bigoplus_{\tau \in \mathcal{T}_n(G)} A_\tau$. Any partial action of a finite group can be written as an iterated extension of decomposable ones:

The decomposition property

Again: α is n-decomposable if $A \cong \bigoplus_{\tau \in \mathcal{T}_n(G)} A_\tau$. Any partial action of a finite group can be written as an iterated extension of decomposable ones:

Definition

If G is finite, then there are canonical equivariant extensions

The decomposition property

Again: α is n-decomposable if $A \cong \bigoplus_{\tau \in \mathcal{T}_n(G)} A_\tau$. Any partial action of a finite group can be written as an iterated extension of decomposable ones:

Definition

If G is finite, then there are canonical equivariant extensions

$$0 \longrightarrow (D^{(k)}, \delta^{(k)}) \longrightarrow (A^{(k)}, \alpha^{(k)}) \longrightarrow (A^{(k-1)}, \alpha^{(k-1)}) \longrightarrow 0,$$

for $2 \leq k \leq |G|$

The decomposition property

Again: α is n-decomposable if $A \cong \bigoplus_{\tau \in \mathcal{T}_n(G)} A_\tau$. Any partial action of a finite group can be written as an iterated extension of decomposable ones:

Definition

If G is finite, then there are canonical equivariant extensions

$$0 \longrightarrow (D^{(k)}, \delta^{(k)}) \longrightarrow (A^{(k)}, \alpha^{(k)}) \longrightarrow (A^{(k-1)}, \alpha^{(k-1)}) \longrightarrow 0,$$

for $2 \leq k \leq |G|$ with $(A^{(|G|)}, \alpha^{(|G|)}) = (A, \alpha)$,

The decomposition property

Again: α is n-decomposable if $A \cong \bigoplus_{\tau \in \mathcal{T}_n(G)} A_\tau$. Any partial action of a finite group can be written as an iterated extension of decomposable ones:

Definition

If G is finite, then there are canonical equivariant extensions

$$0 \longrightarrow (D^{(k)}, \delta^{(k)}) \longrightarrow (A^{(k)}, \alpha^{(k)}) \longrightarrow (A^{(k-1)}, \alpha^{(k-1)}) \longrightarrow 0,$$

for $2 \leq k \leq |G|$ with $(A^{(|G|)}, \alpha^{(|G|)}) = (A, \alpha)$, and such that

1. $\delta^{(k)}$ has the k-decomposition property for all k;

The decomposition property

Again: α is n-decomposable if $A \cong \bigoplus_{\tau \in \mathcal{T}_n(G)} A_\tau$. Any partial action of a finite group can be written as an iterated extension of decomposable ones:

Definition

If G is finite, then there are canonical equivariant extensions

$$0 \longrightarrow (D^{(k)}, \delta^{(k)}) \longrightarrow (A^{(k)}, \alpha^{(k)}) \longrightarrow (A^{(k-1)}, \alpha^{(k-1)}) \longrightarrow 0,$$

for $2 \leq k \leq |G|$ with $(A^{(|G|)}, \alpha^{(|G|)}) = (A, \alpha)$, and such that

1. $\delta^{(k)}$ has the k-decomposition property for all k;
2. $\alpha^{(1)}$ has the 1-decomposition property.

The decomposition property

Again: α is n-decomposable if $A \cong \bigoplus_{\tau \in \mathcal{T}_n(G)} A_\tau$. Any partial action of a finite group can be written as an iterated extension of decomposable ones:

Definition

If G is finite, then there are canonical equivariant extensions

$$0 \longrightarrow (D^{(k)}, \delta^{(k)}) \longrightarrow (A^{(k)}, \alpha^{(k)}) \longrightarrow (A^{(k-1)}, \alpha^{(k-1)}) \longrightarrow 0,$$

for $2 \leq k \leq |G|$ with $(A^{(|G|)}, \alpha^{(|G|)}) = (A, \alpha)$, and such that

1. $\delta^{(k)}$ has the k-decomposition property for all k;
2. $\alpha^{(1)}$ has the 1-decomposition property.

This means that in order to understand α it suffices to understand all the $\delta^{(k)}$, and the extension problems.

The decomposition property

Again: α is n-decomposable if $A \cong \bigoplus_{\tau \in \mathcal{T}_n(G)} A_\tau$. Any partial action of a finite group can be written as an iterated extension of decomposable ones:

Definition

If G is finite, then there are canonical equivariant extensions

$$0 \longrightarrow (D^{(k)}, \delta^{(k)}) \longrightarrow (A^{(k)}, \alpha^{(k)}) \longrightarrow (A^{(k-1)}, \alpha^{(k-1)}) \longrightarrow 0,$$

for $2 \leq k \leq |G|$ with $(A^{(|G|)}, \alpha^{(|G|)}) = (A, \alpha)$, and such that

1. $\delta^{(k)}$ has the k-decomposition property for all k;
2. $\alpha^{(1)}$ has the 1-decomposition property.

This means that in order to understand α it suffices to understand all the $\delta^{(k)}$, and the extension problems. We have described the internal structure of decomposable partial actions,

The decomposition property

Again: α is n-decomposable if $A \cong \bigoplus_{\tau \in \mathcal{T}_n(G)} A_\tau$. Any partial action of a finite group can be written as an iterated extension of decomposable ones:

Definition

If G is finite, then there are canonical equivariant extensions

$$0 \longrightarrow (D^{(k)}, \delta^{(k)}) \longrightarrow (A^{(k)}, \alpha^{(k)}) \longrightarrow (A^{(k-1)}, \alpha^{(k-1)}) \longrightarrow 0,$$

for $2 \leq k \leq |G|$ with $(A^{(|G|)}, \alpha^{(|G|)}) = (A, \alpha)$, and such that

1. $\delta^{(k)}$ has the k-decomposition property for all k;
2. $\alpha^{(1)}$ has the 1-decomposition property.

This means that in order to understand α it suffices to understand all the $\delta^{(k)}$, and the extension problems. We have described the internal structure of decomposable partial actions, to the extent that we understand them "as good" as we understand global actions.

The decomposition property

Again: any partial action of a finite group is an iterated extension of decomposable partial actions.

The decomposition property

Again: any partial action of a finite group is an iterated extension of decomposable partial actions.

For $\tau \in \mathcal{T}_n(G)$, recall $A_\tau = \bigcap_{g\in\tau} A_g$.

The decomposition property

Again: any partial action of a finite group is an iterated extension of decomposable partial actions.

For $\tau \in \mathcal{T}_n(G)$, recall $A_\tau = \bigcap_{g \in \tau} A_g$. We write H_τ for the largest subgroup of G that acts globally on A_τ (potentially the trivial subgroup).

The decomposition property

Again: any partial action of a finite group is an iterated extension of decomposable partial actions.

For $\tau \in \mathcal{T}_n(G)$, recall $A_\tau = \bigcap_{g \in \tau} A_g$. We write H_τ for the largest subgroup of G that acts globally on A_τ (potentially the trivial subgroup).

Theorem (Decomposable actions behave like global actions)

The decomposition property

Again: any partial action of a finite group is an iterated extension of decomposable partial actions.

For $\tau \in \mathcal{T}_n(G)$, recall $A_\tau = \bigcap_{g \in \tau} A_g$. We write H_τ for the largest subgroup of G that acts globally on A_τ (potentially the trivial subgroup).

Theorem (Decomposable actions behave like global actions)

Suppose that α has the decomposition property. Then

1. α is globalizable.

The decomposition property

Again: any partial action of a finite group is an iterated extension of decomposable partial actions.

For $\tau \in \mathcal{T}_n(G)$, recall $A_\tau = \bigcap_{g \in \tau} A_g$. We write H_τ for the largest subgroup of G that acts globally on A_τ (potentially the trivial subgroup).

Theorem (Decomposable actions behave like global actions)

Suppose that α has the decomposition property. Then

1. α is globalizable.
2. There are a canonical faithful conditional expectation $E\colon A \to A^\alpha$,

The decomposition property

Again: any partial action of a finite group is an iterated extension of decomposable partial actions.

For $\tau \in \mathcal{T}_n(G)$, recall $A_\tau = \bigcap_{g\in\tau} A_g$. We write H_τ for the largest subgroup of G that acts globally on A_τ (potentially the trivial subgroup).

Theorem (Decomposable actions behave like global actions)

Suppose that α has the decomposition property. Then

1. α is globalizable.
2. There are a canonical faithful conditional expectation $E\colon A \to A^\alpha$, and a canonical corner-embedding $c\colon A^\alpha \to A \rtimes_\alpha G$.

The decomposition property

Again: any partial action of a finite group is an iterated extension of decomposable partial actions.

For $\tau \in \mathcal{T}_n(G)$, recall $A_\tau = \bigcap_{g\in\tau} A_g$. We write H_τ for the largest subgroup of G that acts globally on A_τ (potentially the trivial subgroup).

Theorem (Decomposable actions behave like global actions)

Suppose that α has the decomposition property. Then

1. α is globalizable.
2. There are a canonical faithful conditional expectation $E\colon A \to A^\alpha$, and a canonical corner-embedding $c\colon A^\alpha \to A \rtimes_\alpha G$.
3. There are canonical identifications $A \rtimes_\alpha G \cong \bigoplus_\tau A_\tau \rtimes H_\tau$

The decomposition property

Again: any partial action of a finite group is an iterated extension of decomposable partial actions.

For $\tau \in \mathcal{T}_n(G)$, recall $A_\tau = \bigcap_{g\in\tau} A_g$. We write H_τ for the largest subgroup of G that acts globally on A_τ (potentially the trivial subgroup).

Theorem (Decomposable actions behave like global actions)

Suppose that α has the decomposition property. Then

1. α is globalizable.
2. There are a canonical faithful conditional expectation $E\colon A \to A^\alpha$, and a canonical corner-embedding $c\colon A^\alpha \to A \rtimes_\alpha G$.
3. There are canonical identifications $A \rtimes_\alpha G \cong \bigoplus_\tau A_\tau \rtimes H_\tau$ and $A^G \cong \bigoplus_\tau A_\tau^{H_\tau}$.

The decomposition property

Again: any partial action of a finite group is an iterated extension of decomposable partial actions.

For $\tau \in \mathcal{T}_n(G)$, recall $A_\tau = \bigcap_{g\in\tau} A_g$. We write H_τ for the largest subgroup of G that acts globally on A_τ (potentially the trivial subgroup).

Theorem (Decomposable actions behave like global actions)

Suppose that α has the decomposition property. Then

1. α is globalizable.
2. There are a canonical faithful conditional expectation $E\colon A \to A^\alpha$, and a canonical corner-embedding $c\colon A^\alpha \to A \rtimes_\alpha G$.
3. There are canonical identifications $A \rtimes_\alpha G \cong \bigoplus_\tau A_\tau \rtimes H_\tau$ and $A^G \cong \bigoplus_\tau A_\tau^{H_\tau}$.

Note: in (3), $H_\tau \curvearrowright A_\tau$ is a <u>global</u> action, and $A_\tau \rtimes H_\tau$ is a global crossed product.

One simple application

Again: any partial action is a recursive extension of *decomposable* partial actions, and each decomposable partial action "behaves like a global action".

One simple application

Again: any partial action is a recursive extension of *decomposable* partial actions, and each decomposable partial action "behaves like a global action". Since we understand decomposable actions very well, the complexity of a general α lies in how the different decomposable actions are glued together.

One simple application

Again: any partial action is a recursive extension of *decomposable* partial actions, and each decomposable partial action "behaves like a global action". Since we understand decomposable actions very well, the complexity of a general α lies in how the different decomposable actions are glued together.

Corollary

The following properties pass from A to $A \rtimes_{\alpha} G$, for an arbitrary partial action of finite G:

One simple application

Again: any partial action is a recursive extension of *decomposable* partial actions, and each decomposable partial action "behaves like a global action". Since we understand decomposable actions very well, the complexity of a general α lies in how the different decomposable actions are glued together.

Corollary

The following properties pass from A to $A \rtimes_\alpha G$, for an arbitrary partial action of finite G:

1. Nuclearity/exactness;
2. Finiteness of the stable/real rank;
3. Being type I.

One simple application

Again: any partial action is a recursive extension of *decomposable* partial actions, and each decomposable partial action "behaves like a global action". Since we understand decomposable actions very well, the complexity of a general α lies in how the different decomposable actions are glued together.

Corollary

The following properties pass from A to $A \rtimes_\alpha G$, for an arbitrary partial action of finite G:

1. Nuclearity/exactness;
2. Finiteness of the stable/real rank;
3. Being type I.

One shows this first for decomposable partial actions:

One simple application

Again: any partial action is a recursive extension of *decomposable* partial actions, and each decomposable partial action "behaves like a global action". Since we understand decomposable actions very well, the complexity of a general α lies in how the different decomposable actions are glued together.

Corollary

The following properties pass from A to $A \rtimes_\alpha G$, for an arbitrary partial action of finite G:

1. Nuclearity/exactness;
2. Finiteness of the stable/real rank;
3. Being type I.

One shows this first for decomposable partial actions: use previous theorem and the fact that global actions preserve the above properties.

One simple application

Again: any partial action is a recursive extension of *decomposable* partial actions, and each decomposable partial action "behaves like a global action". Since we understand decomposable actions very well, the complexity of a general α lies in how the different decomposable actions are glued together.

Corollary

The following properties pass from A to $A \rtimes_\alpha G$, for an arbitrary partial action of finite G:

1. Nuclearity/exactness;
2. Finiteness of the stable/real rank;
3. Being type I.

One shows this first for decomposable partial actions: use previous theorem and the fact that global actions preserve the above properties. In general, one decomposes α recursively with iterated extensions of decomposable actions, and uses that the above properties are preserved by extensions.

Thank you.