Distortion for II_1 multifactor inclusions and bimodules

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Actions of Tensor Categories on C^*-algebras
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j.w.: M. Bischoff, I. Charlesworth,
S. Evington, D. Penneys.

arxiv: 2010.01067

Aim of "distortion":

classify inclusions $A \subset B$
up to $*$isomorphisms

- finite Jones index
- finite depth
- A, B tracial, hyperfinite von Neumann algebras, finite-dimensional centers $\neq \mathbb{C}\mathbb{1}$

classify "representations" $\mathcal{C} \hookrightarrow \mathrm{Bim}\, A$

unitary multi-fusion category

"multi": tensor unit I, $I = \bigoplus_{i=1}^{n} I_i$, I_i simples
is not simple

Q, how to complete Popa's standard invariant for multifactors?
Q, why not complete ??

Inclusions: $1\!\!1 \in A \subset B$ as in [GHJ89, Ch. 3]

- "II_1 multifactors" $A \cong \bigoplus_{i=1}^{a} A_i$, $B \cong \bigoplus_{j=1}^{b} B_j$

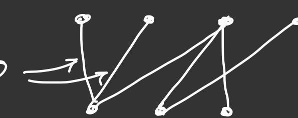
  $A_i := p_i A$, $p_i \in Z(A)$ minimal central projections, A_i, B_j II_1 factors
  $B_j := q_j B$, $q_j \in Z(B)$

- connected $Z(A) \cap Z(B) = \mathbb{C}1\!\!1$
  (w.l.o.g.)

  i.e. the inclusion graph of $A \subset B$ is connected

$j = 1, \dots, b$

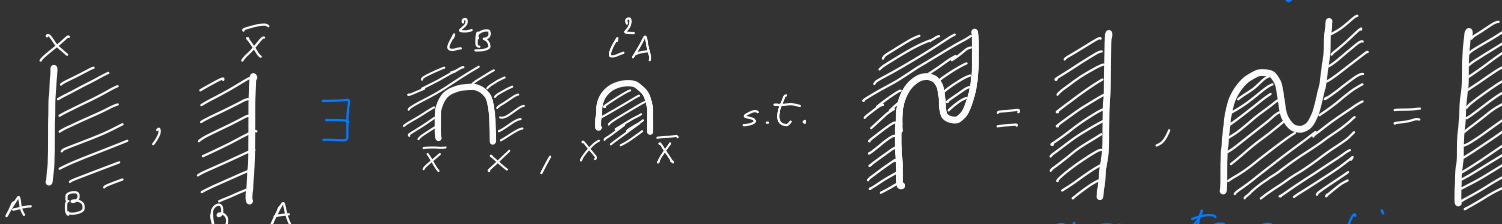
$p_i q_j \neq 0$



B
$\cup$
A

$i = 1, \dots, a$

- finite index i.e. the standard bimodule ${}_A L^2 B_B =: X$ is dualizable
  has "finite dimension"



conjugate equations

def <u>standard invariant</u> of $A \subset B$:

$\mathcal{C}_{A\subset B}$:= $\langle X \rangle$ rigid multitensor C^*-category generated by X

$\boxtimes, \bar{\cdot}, \oplus, <$

choice of <u>generator</u> X

<u>solution of</u> the conj. eqn.s <u>s.t.</u>

$= d_X \cdot \mathbb{1}$ in $Z(B)$

$= d_X \cdot \mathbb{1}$ in $Z(A)$ (*)

loops need not be numbers

<u>the same</u> <u>scalar</u> $d_X \geq 1$

why (*) ?
we want Jones projections :

how many  are there fulfilling (*) ?

- [BCEGP 20] for fusion only one
- [G, Longo 19] "standard" solutions

Thm [BCEGP 20] :

if A, B hyperfinite and $A \subset B$ finite depth (i.e. $\langle X \rangle$ is multifusion)

then

$$A \subset B \xleftrightarrow[\text{sa}]{1:1} \left(\mathcal{C}_{A \subset B} \,,\, \delta_{A \subset B} \right)$$

up to $*$-isomorphism

the standard invariant of $A \subset B$ (purely categorical)

$\boxed{\text{new}}$ the distortion of $A \subset B$ (spatial / tracial)

Examples :

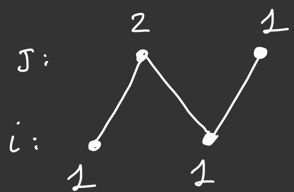
$$A \subset B := \left(\bigoplus_i M_{m_i}(\mathbb{C}) \hookrightarrow \bigoplus_J M_{m_J}(\mathbb{C}) \right) \otimes R$$

hyperfinite II_1 factor

why the standard invariant is not enough ?

Example 1

A_4 :



$$\mathbb{C} \oplus \mathbb{C} \hookrightarrow M_2(\mathbb{C}) \oplus \mathbb{C}$$

$$\Lambda = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad \|\Lambda\|_2^2 = \phi^2 \simeq 2{,}618\ldots$$

Bratteli matrix

ϕ = golden ratio

- $A \subset B := A_4 \otimes R$ does not admit an infinite tunnel [Popa 95]
  i.e. it is not "homogeneous"

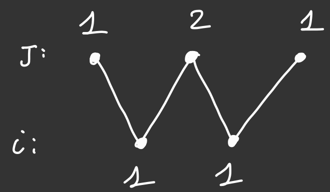
- $A \subset B$ does not admit a single downward basic construction and also it is not uniquely determined by $\mathscr{C}_{A \subset B}$

$$\underbrace{A \subset B}\subset \underbrace{B_1 \subset B_2}$$

- same $\mathscr{C}$, but different $\mathcal{S}$
- $B_1 \subset B_2$ admits two downward basic constructions

Example 2

A_5 :



$\mathbb{C} \oplus \mathbb{C} \hookrightarrow \mathbb{C} \oplus M_2(\mathbb{C}) \oplus \mathbb{C}$

$\Lambda = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$, $\quad \|\Lambda\|_2^2 = 3$

- $A \subset B := A_5 \otimes R$ is "homogeneous" and "distortion free"

it does admit an infinite tunnel

it has "standard" distortion

standard = purely categorical

<u>def</u> <u>distortion</u> of ${}_A X_B$ <u>dualizable</u> <u>A-B bimodule</u>

(e.g. $X := {}_A L^2 B_B$)

p_i $i=1,\dots,a$ minimal central projections, let $X_{ij} := p_i X q_j$ A_i-B_j bimodule

q_j $j=1,\dots,b$

if $X_{ij} \neq \langle 0 \rangle$

i.e.

$p_i q_j^{op} \neq 0$ on X

$$(\delta_X)_{i,j} := \sqrt{\frac{\dim_{A_i} X_{ij}}{\dim X_{ij\, B_j}}}$$

<u>distortion of X</u>

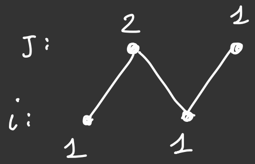
(partially defined $a \times b$ matrix)

<u>Note:</u> if A, B II_1 factors

for $X := {}_A L^2 B_B$

$$\sqrt{\dim_A X \cdot \dim X_B} = [B:A]^{1/2}$$

compute the distortion :

A_4 : 

has $\delta_{A_4} = \begin{pmatrix} 2 & * \\ 2 & 1 \end{pmatrix}$

$$\Lambda = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

$$\mathbb{C} \oplus \mathbb{C} \hookrightarrow M_2(\mathbb{C}) \oplus \mathbb{C}$$

$$\boxed{\lambda}\,\boxed{\mu} \longmapsto \begin{bmatrix} \lambda & 0 \\ 0 & \cancel{\mu} \end{bmatrix} \boxed{\mu} \subset \begin{bmatrix} a & b \\ c & d \end{bmatrix} \boxed{e}$$

$$p_1 = \begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix} \qquad q_1 = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 0 \end{pmatrix}$$

$$p_2 = \begin{pmatrix} 0 & & \\ & 1 & \\ & & 1 \end{pmatrix} \qquad q_2 = \begin{pmatrix} 0 & & \\ & 0 & \\ & & 1 \end{pmatrix}$$

e.g. for the $(1,1)$ entry :

$$X = L^2 B = B$$

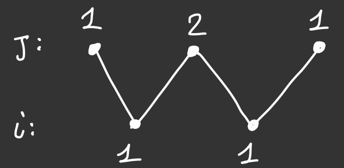
$$X_{11} = p_1 \, M_2(\mathbb{C}) \oplus \mathbb{C} \, q_1 \cong \mathbb{C}^2$$

$$A_1 = p_1(\mathbb{C} \oplus \mathbb{C}) \cong \mathbb{C}$$

$$B_1 = q_1(M_2(\mathbb{C}) \oplus \mathbb{C}) \cong M_2(\mathbb{C})$$

$$\delta_{11} = \sqrt{\frac{\dim_{A_1} X_{11}}{\dim X_{11 \, B_1}}} = \sqrt{\frac{2}{1/2}} = 2$$

A_5 :

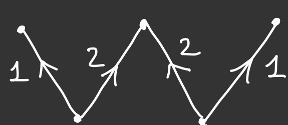


similarly has $\delta_{A_5} = \begin{pmatrix} 1 & 2 & * \\ * & 2 & 1 \end{pmatrix}$

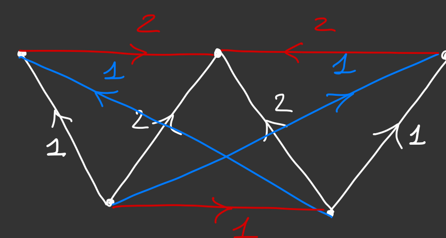
$$\Lambda = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

Q. how to "complete" the distortion matrix ?

$\delta_{A_5} =$

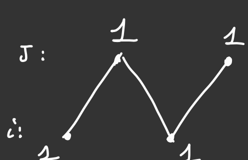


then $\tilde{\delta}_{A_5} =$

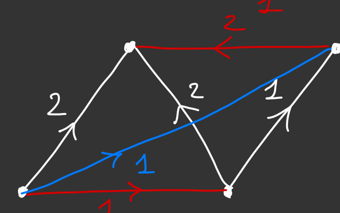


$= \begin{pmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \end{pmatrix}$

A_4 :



then $\tilde{\delta}_{A_4} =$



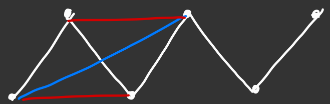
$= \begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix}$

"Extremality":

$A \subset B$ finite index, connected, finite multifactors (not necessarily multimatrices)

TFAE:

- $E^{tr} = E^0$ (Markov trace) (=: $A \subset B$ is "extremal")
  in particular $[B:A] = [B:A]_0$



- $D = \Delta$ and δ extends to the complete bipartite graph and to a groupoid homomorphism on the vertices

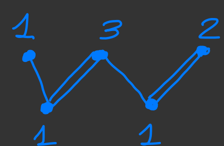
  D: minimal index matrix ("categorical dimension")

  Δ: Jones index matrix

$\iff$

$$\delta_{iJ} = \frac{\xi_J}{\eta_i} \quad \text{for some vectors } \vec{\xi}, \vec{\eta} \text{ in } \mathbb{R}^a_{>0}, \mathbb{R}^b_{>0}$$

For multimatrices: (always extremal) $\Lambda = D = \Delta$,



$\vec{\xi} = (1,3,2)$

$\vec{\eta} = (1,1)$

Distortion in the tower :

if $A \subset B$ is in addition extremal :

$$A \overset{D}{\subset} B \overset{D^T}{\subset} B_1 \overset{D}{\subset} B_2 \overset{D^T}{\subset} B_3 \subset \cdots$$

$$\delta^{(0)}_{iJ} = \frac{\xi_J}{\eta_i} \qquad \delta^{(1)}_{Ji} = \frac{(\vec{\xi} D^T)_i}{\xi_J} \qquad \delta^{(2)}_{iJ} = \frac{(\vec{\xi} D^T D)_J}{(\vec{\xi} D^T)_i} \quad , \quad \cdots$$

Frobenius Perron data of D

$\exists$ limit point for the sequence $\lim\limits_{K \to \infty} \delta^{(2K)}_{iJ} = \dfrac{d\,\beta_J}{\alpha_i}$

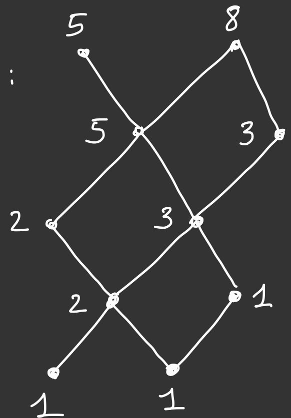
$$\begin{cases} d \cdot \vec{\beta} = \vec{\alpha} D \\ d \cdot \vec{\alpha} = \vec{\beta} D^T \end{cases}$$

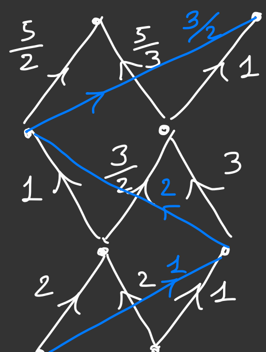
"standard distorsion"

$d = \| D \|_2$

$\| \vec{\alpha} \|_2 = \| \vec{\beta} \|_2 = 1$

A_4:





$$D = \Lambda = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

$$S^{(2\infty)} = \begin{pmatrix} \phi^2 & \phi \\ \phi & 1 \end{pmatrix} = \left(\frac{d\beta_J}{\alpha_i} \right)_{i,J}$$

$$\frac{F_{n+m}}{F_n} \xrightarrow[n\to\infty]{} \phi^m, \quad m \in \mathbb{N}$$

Fibonacci numbers : 1, 1, 2, 3, 5, 8, ...

$(2K)$, $K \to \infty$

$$S^{(2)} = \begin{pmatrix} 5/2 & 3/2 \\ 5/3 & 1 \end{pmatrix}$$

(2)

$$S^{(1)} = \begin{pmatrix} 1 & 3/2 \\ 2 & 3 \end{pmatrix}$$

$$S^{(0)} = \begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix}$$

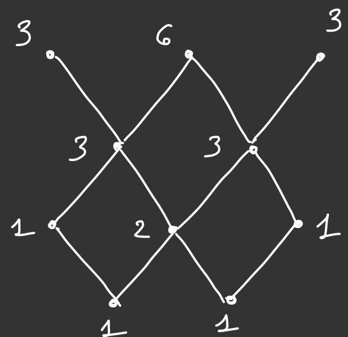
Perron-Frobenius data of D:

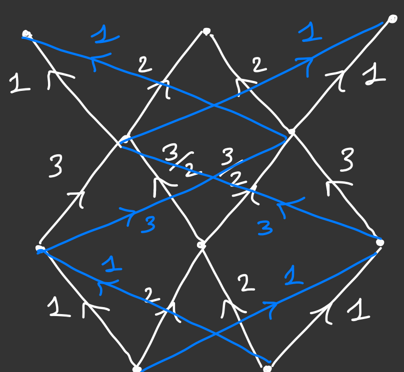
$$\underbrace{(\phi \;\; 1)}_{\vec{\beta}} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \phi \cdot (1 \;\; \phi)$$

$$\underbrace{(1 \;\; \phi)}_{\vec{\alpha}} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \underbrace{\phi}_{= d} \cdot (\phi \;\; 1)$$

$$\|D\|_2 = \phi = d$$

A_5 :





$$D = \Lambda = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

$$S^{(2K)} = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \end{pmatrix}$$

$$\vdots$$

$$S^{(2)} = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \end{pmatrix}$$

$$S^{(1)} = \begin{pmatrix} 3 & 3 \\ 3/2 & 3/2 \\ 3 & 3 \end{pmatrix}$$

$$S^{(0)} = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \end{pmatrix} = \left(\frac{d\beta_J}{\alpha_i} \right)_{i,J}$$

$$S^{(0)} \xrightarrow{(2)} S^{(2)}, \qquad S^{(0)} \xrightarrow{(2K)} S^{(2K)}, \qquad S^{(0)} = S^{(2)}$$

$$\| D \|_2 = \sqrt{3} = d$$

$$\frac{(1\ 1)}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} = \sqrt{3} \cdot \frac{(1\ 2\ 1)}{\sqrt{6}}$$

$$\frac{(1\ 2\ 1)}{\sqrt{6}} \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} = \sqrt{3} \cdot \frac{(1\ 1)}{\sqrt{2}}$$

Popa's "homogeneity":

$A \subset B$ extremal, finite index, connected II_1 multifactors

TFAE:

- $A \subset B$ admits an infinite tunnel (=: $A \subset B$ is "homogeneous")
- $Tr^{Z}(e) = \frac{1}{d^2} \cdot \mathbb{1}$, e Jones projection of E^{tr}, $e \in B_1$
  E^{tr} Markov trace-preserving expectation

  Popa → (for λ-Markov inclusions)
- $S_{ij} = d \dfrac{\beta_j}{\alpha_i}$ (=: $A \subset B$ has the "standard distortion")
- $tr(p_i) = \alpha_i^2$ (=: $A \subset B$ is "super-extremal" [GL19])

$\Longleftrightarrow$

Markov trace = "spherical state"

(immediate to check for multimatrix inclusions)

Classification :

$A \subset B$ finite index, connected, II_1 multifactors, hyperfinite and finite depth (i.e. $\mathcal{C}_{A\subset B}$ unitary multifusion)

then

$$\big(\underbrace{\mathcal{C}_{A\subset B}}_{\text{standard invariant}} , \underbrace{\delta_{A\subset B}}_{\text{distortion}} \big) \text{ is a } \boxed{\text{complete invariant}}$$

$$A \subset B \cong \tilde{A} \subset \tilde{B} \iff \begin{cases} \mathcal{C}_{A\subset B} \simeq \mathcal{C}_{\tilde{A}\subset\tilde{B}} & \text{unitary tensor equivalence} \\ (\delta_{A\subset B})_{p_i, q_j} = (\delta_{\tilde{A}\subset\tilde{B}})_{p_{\tilde{\imath}}, q_{\tilde{\jmath}}} \end{cases}$$

main ingredients :

- Popa's Uniqueness thm for homogeneous $A \subset B$ $(*)$ — $\delta = \delta^{std}$
- Morita equivalence $A \subset B \simeq A' \subset B'$ $(*)$ — $\delta \rightsquigarrow \delta'$

A1 "Morita equivalence":

$A \subset B \subset B_1 \subset B_2$ 2 steps in the Jones tower

$\underbrace{A \subset B}$ and $\underbrace{B_1 \subset B_2}$ are Morita equivalent: $H_A := \overline{L^2 B_A}$

then $A' := (A^{op})' \cap B(H) = (J_B A J_B)' = B_1$

$K_B := H_A \underset{A}{\boxtimes} L^2 B \cong L^2 B_1$ [Sauvageot 83]

then $B' := (B^{op})' \cap B(K) \cong (J_{B_1} B J_{B_1})' = B_2$

$\Rightarrow$ $A \subset B \simeq B_1 \subset B_2$ Morita equivalent via H_A

$\underline{{}_A L^2 B_B} \longmapsto H \underset{A}{\boxtimes} L^2 B \underset{B}{\boxtimes} \overline{K} \cong \underline{{}_{B_1} L^2 B_2 {}_{B_2}}$

$\mathcal{C}_{A \subset B} \simeq \mathcal{C}_{B_1 \subset B_2}$

A2 | **Distortion and Morita equivalence** :

$A \subset B$, $\quad H_A$ faithful right A-module

$\rightsquigarrow$ $A' \subset B'$ Morita equivalent inclusion via H_A

then

$$\underline{\delta'_{i,j}} = \delta_{i,j} \; \rho_i^{-1} \sum_{k=1}^{a} \rho_k \Delta_{kj} \delta_{kj}^{-1}$$

where $\rho_i := \dim H_{i\,A_i}$

with the same proof : $\underline{\Delta'} = \Delta$

A3 Classification of representations :

$\mathcal{C}$ unitary multifusion category , $I = \bigoplus_{i=1}^{m} I_i$, M, N hyperfinite II_1 multifactors
m-dim centers

morphism between $\mathcal{C} \overset{\alpha}{\hookrightarrow} \mathrm{Bim}\, N$ and $\mathcal{C} \overset{\beta}{\hookrightarrow} \mathrm{Bim}\, M$ is

${}_M X_N$ s.t. $X \underset{N}{\boxtimes} \alpha(c) \cong \beta(c) \underset{M}{\boxtimes} X$ for every $c \in \mathcal{C}^{obj}$

compatible with the tensorators of α, β

X isomorphism between α and β if it is invertible as an M-N bimodule

$$\delta^\alpha : \mathrm{Irr}(\mathcal{C}) \longrightarrow \mathbb{R}_{>0}$$
$$c \longmapsto \delta(\alpha(c))$$

$\delta^\alpha = \delta^\beta \iff \exists\, N \cong M$, $*$-isomorphism $\varphi : N \longrightarrow M$
s.t. ${}_M X_N := L^2 M_\varphi$ is an isomorphism $\alpha \simeq \beta$