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Outer actions of amenable groups on von Neumann algebras

Actions of Tensor Categories on C^*-algebras

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Stefaan Vaes – KU Leuven

Connes' classification of amenable factors

Theorem (Connes, 1976)

The **hyperfinite II_1 factor** R is the unique amenable II_1 factor.

Realization: $R = \bigotimes_{\mathbb{N}} (M_2(\mathbb{C}), \tau)$.

A factor $M \subset B(H)$ is **amenable** if there exists a conditional expectation $P : B(H) \to M$.

Theorem (Connes 1976 and Haagerup 1985)

Amenable factors (with separable predual) are completely classified by:

- their type: I_n with $n \in \{1, 2, \ldots, \infty\}$, II_1, II_∞, III_λ with $\lambda \in [0, 1]$,
- and the flow of weights in the III_0 case, i.e. an ergodic action of $\mathbb{R}$.

Side remark: ergodic flows are in a sense unclassifiable.

Symmetries of II_1 factors

- Having classified amenable factors: can we classify their symmetries ?
- Classification "up to what" ?
- Up to **conjugacy** : $\alpha \sim \theta \circ \alpha \circ \theta^{-1}$ for $\theta \in \mathrm{Aut}(R)$. $\rightsquigarrow$ **Completely unwieldy.**
- Up to **outer conjugacy** : $\alpha \sim (\mathrm{Ad}\, u) \circ \theta \circ \alpha \circ \theta^{-1}$ for $u \in \mathcal{U}(R)$ and $\theta \in \mathrm{Aut}(R)$.
- Not restricted to single automorphisms, but in general **group actions** $\Gamma \curvearrowright^{\alpha} R$.
- And **far beyond** : subfactors, quantum symmetries, "actions" of tensor categories, "actions" of groupoids on fields of factors.

Group actions and crossed products

Let $\Gamma \curvearrowright^{\alpha} R$ be a **free** action: if $g \neq e$, then α_g is **outer**, i.e. $\alpha_g \neq \mathrm{Ad}\, u$ for $u \in \mathcal{U}(R)$.

Crossed product $M = R \rtimes_{\alpha} \Gamma$ is a II_1 factor.

- $R \subset M$ is **irreducible** : $R' \cap M = \mathbb{C}1$.
- $R \subset M$ is **regular** : the normalizer $\mathcal{N}_M(R) = \{u \in \mathcal{U}(M) \mid uRu^* = R\}$ generates M.

$\rightsquigarrow$ Is every irreducible, regular inclusion of this form ? NO.

When $R \subset M = R \rtimes \Gamma$ is a crossed product inclusion,

- we recover $\Gamma = \mathcal{N}_M(R)/\mathcal{U}(R)$,
- we have a canonical lifting homomorphism $\Gamma \to \mathcal{N}_M(R)$.

Cocycle actions

Let $R \subset M$ be an **arbitrary** irreducible, regular subfactor.

- Define the group $\Gamma = \mathcal{N}_M(R)/\mathcal{U}(R)$.
- Choose a lift $\Gamma \to \mathcal{N}_M(R) : g \mapsto u_g$.
- We have $u_g u_h = v(g,h)\, u_{gh}$ for all $g, h \in \Gamma$.

⤳ The formula $\alpha_g(a) = u_g a u_g^*$ for $g \in \Gamma$ and $a \in R$ defines:

Definition

A **cocycle action** $\Gamma \curvearrowright^{\alpha, v} R$ consists of

- $\alpha_g \in \mathrm{Aut}(R)$ for every $g \in \Gamma$,
- $\alpha_g \circ \alpha_h = (\mathrm{Ad}\, v(g,h)) \circ \alpha_{gh}$,
- the natural 2-cocycle relation on $v(g,h) \in \mathcal{U}(R)$.

⤳ $R \subset M$ is the **cocycle crossed product** inclusion.

The cohomology vanishing problem

Let $\Gamma \curvearrowright^{\alpha,v} R$ be a cocycle action.

Usual assumption: for every $g \neq e$, the automorphism α_g is outer. We call α a **free cocycle action**.

- **Question :** when can the 2-cocycle v be untwisted ?
- This means: does there exist $w_g \in \mathcal{U}(R)$ such that $v(g,h) = w_g \, \alpha_g(w_h) \, w_{gh}^*$?
- If yes, then the inner perturbation $\beta_g = (\mathrm{Ad}\, w_g) \circ \alpha_g$ defines an ordinary free action $\Gamma \curvearrowright^{\beta} R$.
- **Equivalent problem :** when is it possible to write an irreducible regular inclusion $R \subset M$ as an ordinary crossed product $R \subset R \rtimes \Gamma$?

The cocycle conjugacy problem

A **natural** notion: the subfactors $R_1 \subset M_1$ and $R_2 \subset M_2$ are called isomorphic if there exists a $*$-isomorphism $\theta : M_1 \to M_2$ such that $\theta(R_1) = R_2$.

$\rightsquigarrow$ **This leads to:**

Definition

Two actions $\Gamma \curvearrowright^{\alpha} R$ and $\Gamma \curvearrowright^{\beta} R$ are called **cocycle conjugate** if

- there exists a 1-cocycle w_g for α, i.e. $w_g \in \mathcal{U}(R)$ and $w_g\, \alpha_g(w_h) = w_{gh}$,
- such that the action $(\mathrm{Ad}\, w_g) \circ \alpha_g$ is conjugate to β, i.e. there exists $\theta \in \mathrm{Aut}(R)$ such that $\beta_g = \theta \circ (\mathrm{Ad}\, w_g) \circ \alpha_g \circ \theta^{-1}$.

$\rightsquigarrow$ In that case, there is a natural $*$-isomorphism $\psi : R \rtimes_{\alpha} \Gamma \to R \rtimes_{\beta} \Gamma$ with $\psi|_R = \theta$.

Question: when are two actions cocycle conjugate ?

Connes – Jones – Ocneanu

Theorem (Ocneanu, 1985)

Let Γ be an amenable group.

- If $\Gamma \curvearrowright^{\alpha,v} R$ is a free cocycle action on the hyperfinite II_1 factor, then v is a coboundary.

  Thus: cohomology vanishing holds for all free actions of amenable groups on R.

- Up to cocycle conjugacy, there is a **unique** free action of Γ on R.

- For cyclic groups: Connes, 1976.
- For finite groups (and then, uniqueness up to conjugacy): Jones, 1979.

Examples – constructions

Let Γ be an infinite group.

- **Bernoulli action.** Take (A_0, τ) with $A_0 = M_k(\mathbb{C})$ or $A_0 = R$.

  Then consider $\Gamma \curvearrowright R_1 = \bigotimes_\Gamma (A_0, \tau)$ by shifting.

- **Stabilized version.** Consider $\Gamma \curvearrowright^\beta R_1 \mathbin{\overline{\otimes}} R_2 : \beta_g = \alpha_g \otimes \mathrm{id}$.

- **Connes-Størmer Bernoulli action.** Let φ be a state on $M_k(\mathbb{C})$.

  Then consider $\Gamma \curvearrowright (P, \varphi) = \bigotimes_\Gamma (M_k(\mathbb{C}), \varphi)$ by shifting.

  We have $R \cong P^\varphi$ and $\Gamma \curvearrowright R$.

$\rightsquigarrow$ These are all free actions on R. None of them are "obviously" cocycle conjugate.

A digression to subfactors

- **Jones index** of $N \subset M$ is defined as $[M : N] = \dim_N L^2(M)$.
- Let $\Gamma \curvearrowright^{\alpha} N$ be a free action on a II_1 factor N. Let $g_1, \dots, g_n \in \Gamma$. Define the **diagonal subfactor** $N \hookrightarrow M_n(\mathbb{C}) \otimes N : a \mapsto \begin{pmatrix} \alpha_{g_1}(a) & & \\ & \ddots & \\ & & \alpha_{g_n}(a) \end{pmatrix}$
- We can "encode" the action $\Gamma \curvearrowright^{\alpha} N$ as a finite index subfactor.
- We can view a finite index subfactor $N \subset M$ as "quantum symmetries" of N.
- The group Γ is replaced by the **standard invariant** of $N \subset M$.
- **Popa (1992) :** for hyperfinite subfactors with amenable standard invariant, this is a complete invariant.
- In particular : a subfactor proof of Ocneanu's theorem.

Popa's cohomology vanishing theorem

Theorem (Popa, 2018)

Let Γ be an amenable group and $\Gamma \curvearrowright^{\alpha,v} N$ a free cocycle action on **any** II_1 factor with separable predual.

Then, v is a coboundary : $v(g,h) = w_g\, \alpha_g(w_h)\, w_{gh}^*$.

Notation : $\Gamma \in \mathcal{VC}$.

Open problem : characterize this class $\mathcal{VC}$.

- If $\Gamma_1, \Gamma_2 \in \mathcal{VC}$, then $\Gamma_1 * \Gamma_2 \in \mathcal{VC}$. Also $\Gamma_1 *_K \Gamma_2 \in \mathcal{VC}$ if $K < \Gamma_i$ is a finite subgroup.
- $\Gamma \notin \mathcal{VC}$ if Γ admits an infinite subgroup $\Lambda < \Gamma$ with the relative property (T).
- $\Gamma \notin \mathcal{VC}$ if Γ admits an infinite subgroup $\Lambda < \Gamma$ with nonamenable centralizer $C_\Gamma(\Lambda)$.
- **Speculation :** $\mathcal{VC}$ is closely related to the class of treeable groups.

Connes-Jones cocycles

Let Γ be a countable group and $\pi : \mathbb{F}_\infty \to \Gamma$ a surjective group homomorphism.

- Let $\Lambda = \operatorname{Ker} \pi$.
- We canonically have $L(\mathbb{F}_\infty) = L(\Lambda) \rtimes_{\alpha,v} \Gamma$ for some cocycle action.
- If cohomology vanishing holds, we find $L(\Gamma) \hookrightarrow L(\mathbb{F}_\infty)$.
- Thus: if $\Gamma \in \mathcal{VC}$, then $L(\Gamma)$ is embeddable in a free group factor.
- Many group von Neumann algebras are known to be non-embeddable into $L(\mathbb{F}_\infty)$.
- It is **wide open** to characterize the groups Γ such that $L(\Gamma)$ is embeddable into $L(\mathbb{F}_\infty)$.

  Also here: possible relation to treeable groups.

Proof of Popa's cohomology vanishing theorem

The proof of the theorem is really a **subfactor proof**.

Theorem (Popa, 2018)

Let Γ be an amenable group and $\Gamma \curvearrowright^{\alpha,v} N$ a free cocycle action on **any** II_1 factor with separable predual.

There exists a copy of $R \subset N$ and an inner perturbation $\beta_g = (\mathrm{Ad}\, w_g) \circ \alpha_g$ with corresponding $v'(g,h)$ such that

- $\beta_g(R) = R$ and the restriction of β to R is a free action,
- $v'(g,h) \in R$ for all $g, h \in \Gamma$.

$\rightsquigarrow$ Next, apply Ocneanu's theorem to $\Gamma \curvearrowright^{\beta,v'} R$.

$\rightsquigarrow$ The above property characterizes amenability.

Approximate vanishing of 1-cohomology

- For the (unique, up to cocycle conjugacy) free action $\Gamma \curvearrowright^{\alpha} R$ of an infinite amenable group, the space of 1-cocycles (w_g) is **huge**.

  Recall : $w_g\, \alpha_g(w_h) = w_{gh}$. We say $w \sim w'$ if $w'_g = a\, w_g\, \alpha_g(a^*)$ for some $a \in \mathcal{U}(R)$.
- That is "hidden" behind Γ having free actions with very different ergodic theoretic properties: compact, mixing, invariant elements or not, etc.

Theorem (Popa – Shlyakhtenko – V, 2018)

Let Γ be amenable and $\Gamma \curvearrowright^{\alpha} N$ a free action on **any** II_1 factor with separable predual.

Any 1-cocycle (w_g) is an **approximate coboundary** : there exists a sequence $a_n \in \mathcal{U}(N)$ such that $\lim_n \|w_g - a_n\, \alpha_g(a_n^*)\|_2 = 0$ for all $g \in \Gamma$.

$\rightsquigarrow$ Also this is a characterization of amenability.

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Beyond amenability: no-go theorems

Let Γ be a **nonamenable** group.

- (Jones, 1982) The group Γ admits at least two free actions on R that are not cocycle conjugate (and not even outer conjugate).

  Invariant: existence of a nontrivial central sequence $a_n \in \mathcal{U}(R)$ with $\lim_n \|\alpha_g(a_n) - a_n\|_2 = 0$ for all $g \in \Gamma$.

  The Bernoulli action $\Gamma \curvearrowright^{\alpha} R_1$ **does not admit** this.

  The stabilized action $\alpha_g \otimes \mathrm{id}$ on $R_1 \mathbin{\overline{\otimes}} R_2$ **admits** this.

- (Popa, 2001) If Γ admits an infinite normal subgroup with the relative property (T), then Γ admits uncountably many free actions on R that are not outer conjugate.

  Invariant: the **fundamental group** of a free action $\Gamma \curvearrowright^{\alpha} R$.

Popa's fundamental group of a free action

Let $\Gamma \curvearrowright^{\alpha} R$ be a free action. Let $0 < t < 1$.

- Pick a projection $p \in R$ with $\tau(p) = t$.
- For every $g \in \Gamma$, $\alpha_g(p) \sim p$. Choose $w_g \in R$ with $w_g^* w_g = \alpha_g(p)$ and $w_g w_g^* = p$.
- Define the cocycle action $\alpha^t : \Gamma \curvearrowright pRp : \alpha_g^t(a) = w_g \alpha_g(a) w_g^*$.
- Similarly, α^t for all $t > 0$. Well defined up to cocycle conjugacy.
- Define $\mathcal{F}(\alpha) = \{t > 0 \mid \alpha^t \text{ and } \alpha \text{ are outer conjugate}\}$.
- Then, $\mathcal{F}(\alpha) \subset \mathbb{R}_+^*$ is a subgroup. It is an outer conjugacy invariant.

Theorem (Popa, 2001)

When Γ admits an infinite normal subgroup with the relative property (T) and $\Gamma \curvearrowright^{\alpha} R$ is a Connes-Størmer Bernoulli action with eigenvalues t_i, then $\mathcal{F}(\alpha) = \langle t_i / t_j \rangle$.

Beyond amenability: no-go theorems

Theorem (Brothier - V, 2013)

A nonamenable group Γ admits uncountably many non outer conjugate free actions on R.

There even is a concrete list of them.

- Let $\mathcal{G}$ be the class of amenable groups Λ with the following property: if $g \in \Lambda \setminus \{e\}$ has a finite conjugacy class, then g has infinite order.
- Whenever Γ is nonamenable and $\Lambda \in \mathcal{C}$, realize

$$R = \left(M_2(\mathbb{C})^{\Gamma \times \Lambda} \mathbin{\overline{\otimes}} M_2(\mathbb{C})^{\Lambda}\right) \rtimes \Lambda.$$

- We naturally have $\alpha_\Lambda : \Gamma \curvearrowright R$.
- **Theorem:** α_Λ and $\alpha_{\Lambda'}$ are outer conjugate if and only if $\Lambda \cong \Lambda'$.

And what about subfactors

Many, many open problems !

- Which standard invariants $\mathcal{G}$ come from hyperfinite subfactors ?
- Recall: any countable group acts freely on R. So, is there any restriction on $\mathcal{G}$ at all ?
- Even open for: Temperley-Lieb-Jones standard invariant !
- And when a nonamenable standard invariant $\mathcal{G}$ "acts freely" on R, are there infinitely/uncountably many non "outer conjugate" actions ?
- Very few results: essentially Bisch-Haagerup type subfactors $R^H \subset R \rtimes K$, based on non outer conjugacy of actions $\langle H, K \rangle \curvearrowright R$.

  Their standard invariant "is" the countable group $\langle H, K \rangle$.