Outer actions of amenable groups on von Neumann algebras

Actions of Tensor Categories on C*-algebras

IPAM workshop – 21-28 January 2021

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Connes’ classification of amenable factors

Theorem (Connes, 1976)

The hyperfinite II$_1$ factor $R$ is the unique amenable II$_1$ factor.

Realization: $R = \bigotimes_{N}(M_2(\mathbb{C}), \tau)$. A factor $M \subset B(H)$ is amenable if there exists a conditional expectation $P : B(H) \to M$.

Theorem (Connes 1976 and Haagerup 1985)

Amenable factors (with separable predual) are completely classified by:

- their type: $I_n$ with $n \in \{1, 2, \ldots, \infty\}$, II$_1$, II$_\infty$, III$_\lambda$ with $\lambda \in [0, 1]$, and
- the flow of weights in the III$_0$ case, i.e. an ergodic action of $\mathbb{R}$.

Side remark: ergodic flows are in a sense unclassifiable.
Symmetries of $\text{II}_1$ factors

- Having classified amenable factors: can we classify their symmetries?

- Classification “up to what”?

- Up to **conjugacy** : $\alpha \sim \theta \circ \alpha \circ \theta^{-1}$ for $\theta \in \text{Aut}(R)$.  
  
  Completely unwieldy.

- Up to **outer conjugacy** : $\alpha \sim (\text{Ad } u) \circ \theta \circ \alpha \circ \theta^{-1}$ for $u \in \mathcal{U}(R)$ and $\theta \in \text{Aut}(R)$.

- Not restricted to single automorphisms, but in general **group actions** $\Gamma \curvearrowright \alpha \ R$.

- And **far beyond** : subfactors, quantum symmetries, “actions” of tensor categories, “actions” of groupoids on fields of factors.
Group actions and crossed products

Let $\Gamma \curvearrowright^{\alpha} R$ be a free action: if $g \neq e$, then $\alpha_g$ is outer, i.e. $\alpha_g \neq \text{Ad} u$ for $u \in \mathcal{U}(R)$.

Crossed product $M = R \rtimes_{\alpha} \Gamma$ is a $\text{II}_1$ factor.

- $R \subset M$ is irreducible: $R' \cap M = \mathbb{C}1$.
- $R \subset M$ is regular: the normalizer $\mathcal{N}_M(R) = \{ u \in \mathcal{U}(M) \mid uRu^* = R \}$ generates $M$.

Is every irreducible, regular inclusion of this form? NO.

When $R \subset M = R \rtimes \Gamma$ is a crossed product inclusion,

- we recover $\Gamma = \mathcal{N}_M(R)/\mathcal{U}(R)$,
- we have a canonical lifting homomorphism $\Gamma \to \mathcal{N}_M(R)$. 

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Cocycle actions
Let $R \subset M$ be an arbitrary irreducible, regular subfactor.

- Define the group $\Gamma = \mathcal{N}_M(R)/\mathcal{U}(R)$.
- Choose a lift $\Gamma \to \mathcal{N}_M(R) : g \mapsto u_g$.
- We have $u_g u_h = \nu(g, h) u_{gh}$ for all $g, h \in \Gamma$.

The formula $\alpha_g(a) = u_g a u_g^*$ for $g \in \Gamma$ and $a \in R$ defines:

**Definition**

A **cocycle action** $\Gamma \actson^{\alpha, \nu} R$ consists of

- $\alpha_g \in \text{Aut}(R)$ for every $g \in \Gamma$,
- $\alpha_g \circ \alpha_h = (\text{Ad} \nu(g, h)) \circ \alpha_{gh}$,
- the natural 2-cocycle relation on $\nu(g, h) \in \mathcal{U}(R)$.

$R \subset M$ is the cocycle crossed product inclusion.
The cohomology vanishing problem

Let $\Gamma \curvearrowright^{\alpha, v} R$ be a cocycle action.

Usual assumption: for every $g \neq e$, the automorphism $\alpha_g$ is outer. We call $\alpha$ a free cocycle action.

- **Question**: when can the 2-cocycle $v$ be untwisted?
  - This means: does there exist $w_g \in U(R)$ such that $v(g, h) = w_g \alpha_g(w_h) w_{gh}^*$?
  - If yes, then the inner perturbation $\beta_g = (\text{Ad } w_g) \circ \alpha_g$ defines an ordinary free action $\Gamma \curvearrowright^{\beta} R$.
  - **Equivalent problem**: when is it possible to write an irreducible regular inclusion $R \subset M$ as an ordinary crossed product $R \subset R \rtimes \Gamma$?
The cocycle conjugacy problem

A natural notion: the subfactors $R_1 \subset M_1$ and $R_2 \subset M_2$ are called isomorphic if there exists a $\ast$-isomorphism $\theta : M_1 \rightarrow M_2$ such that $\theta(R_1) = R_2$.

This leads to:

Definition

Two actions $\Gamma \curvearrowright^\alpha R$ and $\Gamma \curvearrowright^\beta R$ are called cocycle conjugate if

- there exists a 1-cocycle $w_g$ for $\alpha$, i.e. $w_g \in \mathcal{U}(R)$ and $w_g \alpha_g(w_h) = w_{gh}$,

- such that the action $(\text{Ad } w_g) \circ \alpha_g$ is conjugate to $\beta$, i.e. there exists $\theta \in \text{Aut}(R)$ such that $\beta_g = \theta \circ (\text{Ad } w_g) \circ \alpha_g \circ \theta^{-1}$.

In that case, there is a natural $\ast$-isomorphism $\psi : R \rtimes_\alpha \Gamma \rightarrow R \rtimes_\beta \Gamma$ with $\psi|_R = \theta$.

Question: when are two actions cocycle conjugate?
Theorem (Ocneanu, 1985)

Let $\Gamma$ be an amenable group.

- If $\Gamma \acts_{\alpha, \nu} R$ is a free cocycle action on the hyperfinite $II_1$ factor, then $\nu$ is a coboundary.
  
  Thus: cohomology vanishing holds for all free actions of amenable groups on $R$.

- Up to cocycle conjugacy, there is a unique free action of $\Gamma$ on $R$.

- For cyclic groups: Connes, 1976.

- For finite groups (and then, uniqueness up to conjugacy): Jones, 1979.
Examples – constructions

Let $\Gamma$ be an infinite group.

- **Bernoulli action.** Take $(A_0, \tau)$ with $A_0 = M_k(\mathbb{C})$ or $A_0 = \mathbb{R}$.

  Then consider $\Gamma \curvearrowright R_1 = \bigotimes_{\Gamma}(A_0, \tau)$ by shifting.

- **Stabilized version.** Consider $\Gamma \curvearrowright^\beta R_1 \otimes R_2 : \beta_g = \alpha_g \otimes \text{id}$.

- **Connes-Størmer Bernoulli action.** Let $\varphi$ be a state on $M_k(\mathbb{C})$.

  Then consider $\Gamma \curvearrowright (P, \varphi) = \bigotimes_{\Gamma}(M_k(\mathbb{C}), \varphi)$ by shifting.

We have $R \cong P^\varphi$ and $\Gamma \curvearrowright R$.

These are all free actions on $R$. None of them are “obviously” cocycle conjugate.
A digression to subfactors

- **Jones index** of $N \subset M$ is defined as $[M : N] = \dim_N L^2(M)$.

- Let $\Gamma \actson^\alpha N$ be a free action on a II$_1$ factor $N$. Let $g_1, \ldots, g_n \in \Gamma$. Define the diagonal subfactor $N \hookrightarrow \rightarrow M_n(\mathbb{C}) \otimes N : a \mapsto \left( \begin{array}{ccc} \alpha_{g_1}(a) & & \\ & \ddots & \\ & & \alpha_{g_n}(a) \end{array} \right)$

- We can “encode” the action $\Gamma \actson^\alpha N$ as a finite index subfactor.

- We can view a finite index subfactor $N \subset M$ as “quantum symmetries” of $N$.

- The group $\Gamma$ is replaced by the **standard invariant** of $N \subset M$.

- **Popa (1992)**: for hyperfinite subfactors with amenable standard invariant, this is a complete invariant.

- In particular: a subfactor proof of Ocneanu’s theorem.
Popa’s cohomology vanishing theorem

Theorem (Popa, 2018)

Let $\Gamma$ be an amenable group and $\Gamma \curvearrowright^{\alpha, \nu} N$ a free cocycle action on any $\text{II}_1$ factor with separable predual.

Then, $\nu$ is a coboundary: $\nu(g, h) = w_g \alpha_g(w_h) w_{gh}^*$. 

Notation: $\Gamma \in \mathcal{VC}$.

Open problem: characterize this class $\mathcal{VC}$.

- If $\Gamma_1, \Gamma_2 \in \mathcal{VC}$, then $\Gamma_1 \ast \Gamma_2 \in \mathcal{VC}$. Also $\Gamma_1 \ast_K \Gamma_2 \in \mathcal{VC}$ if $K < \Gamma_i$ is a finite subgroup.
- $\Gamma \notin \mathcal{VC}$ if $\Gamma$ admits an infinite subgroup $\Lambda < \Gamma$ with the relative property (T).
- $\Gamma \notin \mathcal{VC}$ if $\Gamma$ admits an infinite subgroup $\Lambda < \Gamma$ with nonamenable centralizer $C_\Gamma(\Lambda)$.
- Speculation: $\mathcal{VC}$ is closely related to the class of treeable groups.
Connes-Jones cocycles

Let $\Gamma$ be a countable group and $\pi : F_\infty \to \Gamma$ a surjective group homomorphism.

- Let $\Lambda = \text{Ker } \pi$.

- We canonically have $L(F_\infty) = L(\Lambda) \rtimes_{\alpha, \nu} \Gamma$ for some cocycle action.

- If cohomology vanishing holds, we find $L(\Gamma) \hookrightarrow L(F_\infty)$.

- Thus: if $\Gamma \in \mathcal{VC}$, then $L(\Gamma)$ is embeddable in a free group factor.

- Many group von Neumann algebras are known to be non-embeddable into $L(F_\infty)$.

- It is **wide open** to characterize the groups $\Gamma$ such that $L(\Gamma)$ is embeddable into $L(F_\infty)$.

  Also here: possible relation to treeable groups.
Proof of Popa’s cohomology vanishing theorem

The proof of the theorem is really a subfactor proof.

**Theorem (Popa, 2018)**

Let \( \Gamma \) be an amenable group and \( \Gamma \curvearrowright^{\alpha, \nu} N \) a free cocycle action on any II\(_1\) factor with separable predual.

There exists a copy of \( R \subset N \) and an inner perturbation \( \beta_g = (\text{Ad} \ w_g) \circ \alpha_g \) with corresponding \( \nu'(g, h) \) such that

- \( \beta_g(R) = R \) and the restriction of \( \beta \) to \( R \) is a free action,
- \( \nu'(g, h) \in R \) for all \( g, h \in \Gamma \).

Next, apply Ocneanu’s theorem to \( \Gamma \curvearrowright^{\beta, \nu'} R \).

The above property characterizes amenability.
Approximate vanishing of 1-cohomology

- For the (unique, up to cocycle conjugacy) free action $\Gamma \curvearrowright^\alpha R$ of an infinite amenable group, the space of 1-cocycles $(w_g)$ is huge.

Recall: $w_g \alpha_g(w_h) = w_{gh}$. We say $w \sim w'$ if $w'_g = a w_g \alpha_g(a^*)$ for some $a \in U(R)$.

- That is “hidden” behind $\Gamma$ having free actions with very different ergodic theoretic properties: compact, mixing, invariant elements or not, etc.

**Theorem (Popa – Shlyakhtenko – V, 2018)**

Let $\Gamma$ be amenable and $\Gamma \curvearrowright^\alpha N$ a free action on any II$_1$ factor with separable predual.

Any 1-cocycle $(w_g)$ is an approximate coboundary: there exists a sequence $a_n \in U(N)$ such that $\lim_n \|w_g - a_n \alpha_g(a_n^*)\|_2 = 0$ for all $g \in \Gamma$.

Also this is a characterization of amenability.
Beyond amenability: no-go theorems

Let $\Gamma$ be a nonamenable group.

- (Jones, 1982) The group $\Gamma$ admits at least two free actions on $R$ that are not cocycle conjugate (and not even outer conjugate).

  **Invariant:** existence of a nontrivial central sequence $a_n \in \mathcal{U}(R)$ with $\lim_n \|\alpha_g(a_n) - a_n\|_2 = 0$ for all $g \in \Gamma$.

  The Bernoulli action $\Gamma \acts^\alpha R_1$ does not admit this.

  The stabilized action $\alpha_g \otimes \text{id}$ on $R_1 \overline{\otimes} R_2$ admits this.

- (Popa, 2001) If $\Gamma$ admits an infinite normal subgroup with the relative property (T), then $\Gamma$ admits uncountably many free actions on $R$ that are not outer conjugate.

  **Invariant:** the fundamental group of a free action $\Gamma \acts^\alpha R$. 
Popa’s fundamental group of a free action

Let $\Gamma \curvearrowright \alpha R$ be a free action. Let $0 < t < 1$.

- Pick a projection $p \in R$ with $\tau(p) = t$.
- For every $g \in \Gamma$, $\alpha_g(p) \sim p$. Choose $w_g \in R$ with $w_g^*w_g = \alpha_g(p)$ and $w_gw_g^* = p$.
- Define the cocycle action $\alpha^t : \Gamma \curvearrowright pRp : \alpha^t_g(a) = w_g\alpha_g(a)w_g^*$.
- Similarly, $\alpha^t$ for all $t > 0$. Well defined up to cocycle conjugacy.
- Define $F(\alpha) = \{ t > 0 \mid \alpha^t \text{ and } \alpha \text{ are outer conjugate} \}$.
- Then, $F(\alpha) \subset \mathbb{R}_+^*$ is a subgroup. It is an outer conjugacy invariant.

**Theorem (Popa, 2001)**

When $\Gamma$ admits an infinite normal subgroup with the relative property (T) and $\Gamma \curvearrowright \alpha R$ is a Connes-Størmer Bernoulli action with eigenvalues $t_i$, then $F(\alpha) = \langle t_i/t_j \rangle$. 
Beyond amenability: no-go theorems

Theorem (Brothier - V, 2013)

A nonamenable group $\Gamma$ admits uncountably many non outer conjugate free actions on $R$. There even is a concrete list of them.

- Let $\mathcal{G}$ be the class of amenable groups $\Lambda$ with the following property: if $g \in \Lambda \setminus \{e\}$ has a finite conjugacy class, then $g$ has infinite order.

- Whenever $\Gamma$ is nonamenable and $\Lambda \in \mathcal{C}$, realize

$$R = (M_2(\mathbb{C})_{\Gamma} \times \Lambda \otimes M_2(\mathbb{C})^\Lambda) \rtimes \Lambda.$$ 

- We naturally have $\alpha_\Lambda : \Gamma \curvearrowright R$.

- **Theorem:** $\alpha_\Lambda$ and $\alpha_{\Lambda'}$ are outer conjugate if and only if $\Lambda \cong \Lambda'$. 
And what about subfactors

Many, many open problems!

- Which standard invariants $G$ come from hyperfinite subfactors?

- Recall: any countable group acts freely on $R$. So, is there any restriction on $G$ at all?

- Even open for: Temperley-Lieb-Jones standard invariant!

- And when a nonamenable standard invariant $G$ “acts freely” on $R$, are there infinitely/uncountably many non “outer conjugate” actions?

- Very few results: essentially Bisch-Haagerup type subfactors $R^H \subset R \rtimes K$, based on non outer conjugacy of actions $\langle H, K \rangle \actson R$.

Their standard invariant “is” the countable group $\langle H, K \rangle$. 