Nonunital simple C^*-algebras

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Joint work with Guihua Gong

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If A is simple and has weakly unperforated $K_0(A)$, then $Ell(A) = Ell(A \otimes \mathcal{Z})$.
A C^*-algebra A is $\mathcal{Z}$-stable, if $A \otimes \mathcal{Z} \cong A$.

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If A is a non-unital separable simple C^*-algebra and, if $K_0(A)_+ \neq \{0\}$, let $p \in M_n(A)$ be a nonzero projection for some integer $n \geq 1$. Then $pM_n(A)p$ is a unital separable simple C^*-algebra which is stably isomorphic to A. For those simple C^*-algebras, the classification could be easily reduced to the unital case.

Elliott, Gong, Jiang and Su, and M. Rørdam for the first part, Elliott, Gong–Niu–L for the second part, For the part (3), Gong–Niu—-L (C. R. Math. Acad. Sci. Soc. R. Canada, 42 (2020), 63-454), Elliott-Gong-Niu–L, A Tikuisis, S. White, and W. Winter (Ann. of Math. 185 (2017), 229–284)

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There is a non-unital separable simple C^*-algebra $\mathcal{Z}_0$ which is locally sub-homogeneous with a unique tracial state, and $K_0(\mathcal{Z}_0) = \mathbb{Z}$ and $K_1(\mathcal{Z}_0) = \{0\}$ with trivial paring: $K_0(\mathcal{Z}_0) = \ker \rho_{\mathcal{Z}_0}$. So $K_0(A)_+ = \{0\}$. $\mathcal{Z}_0$ is stably projectionless.

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Also $\sigma_A(\tau) = d_\tau(e_A) \in \mathrm{LAff}_+(\tilde{T}(A))$, where e_A is a strictly positive element.

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For any separable simple amenable C^*-algebra A in the UCT class, $A \otimes \mathcal{Z}_0$ is in the class that is classified in the previous theorem.

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Choose $e \in \mathrm{Ped}(A)_+ \setminus \{0\}$ and $B = \overline{eAe}$. If A is σ-unital, then $B \otimes \mathcal{K} \cong A \otimes \mathcal{K}$.

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Choose $e \in \mathrm{Ped}(A)_+ \setminus \{0\}$ and $B = \overline{eAe}$. If A is σ-unital, then $B \otimes \mathcal{K} \cong A \otimes \mathcal{K}$. One can consider $M_n(\tilde{B})$ and there is a paring $\rho_B : K_0(B) \to \mathrm{Aff}(T(B))$.
Let $x \in K_0(B) \cong K_0(A)$. Then the function $\rho_B(x)$ would depend on B.

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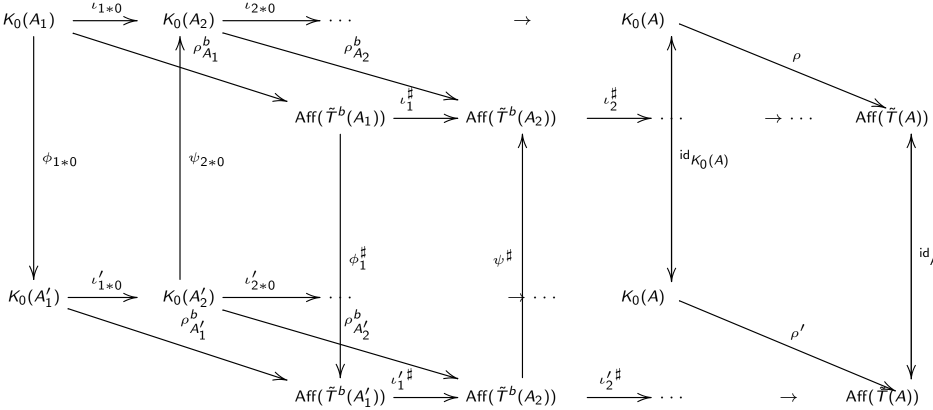
$$\tilde{T}^b(A_1) \xleftarrow{{\iota_1}_T^b} \tilde{T}^b(A_2) \xleftarrow{{\iota_2}_T^b} \tilde{T}^b(A_3) \cdots \longleftarrow \cdots \longleftarrow \tilde{T}(A).$$

Hence one also has the following commutative diagram:

$$\begin{array}{ccccccc}
K_0(A_1) & \xrightarrow{\iota_{1*o}} & K_0(A_2) & \xrightarrow{\iota_{2*o}} & K_0(A_3)) & \longrightarrow & \cdots K_0(A) \\
\downarrow{\rho_{A_1}} & & \downarrow{\rho_{A_2}} & & \downarrow{\rho_{A_3}} & & \\
\mathrm{Aff}(\tilde{T}^b(A_1)) & \xrightarrow{\iota_{1,2}^\sharp} & \mathrm{Aff}(\tilde{T}^b(A_2)) & \xrightarrow{\iota_2^\sharp} & \mathrm{Aff}(\tilde{T}^b(A_3)) & \longrightarrow & \cdots \mathrm{Aff}(\tilde{T}(A)).
\end{array}$$

Thus one obtains a homomorphism $\rho : K_0(A) \to \mathrm{Aff}(\tilde{T}(A))$.

One can show that the above does not depend on the choice of e_A by working on the following diagram:



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This strong version of weakly unperforation of paring needs to be even stronger.

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We allow $\Sigma(G) = \{0\}$. It is called unital scaled simple ordered group paring, if $\Sigma(G) = \{g \in G_+ : \rho(g) < s\} \cup \{u\}$ with $\rho(u) = s$, in which case, u is called the unit for G. Note that, in this case u is the maximum element of $\Sigma(G)$.

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or $\sigma_A(\tau) = \sup\{\tau(a) : 0 \le a \le 1, a \in A\}$.

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where $\rho_B(m)(r \cdot \tau) = (r/s)m$ for $m \in \mathbb{Z}$, $r \in \mathbb{R}$.

Theorem B

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If $G_1 = \{0\}$ and G_0 is torsion free, then A can be chosen to be an inductive limit of 1-dimensional NCCW complexes.

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Theorem **C**

Let A be a separable stably projectionless simple amenable $\mathcal{Z}$-stable C^*-algebra in the UCT class.

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Theorem C

Let A be a separable stably projectionless simple amenable $\mathcal{Z}$-stable C^*-algebra in the UCT class. Then A has generalized tracial rank one (and has stable rank one).

Theorem D

Let A and B be two (finite) separable simple amenable $\mathcal{Z}$-stable C^*-algebras in the UCT class.

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Let A and B be two (finite) separable simple amenable $\mathcal{Z}$-stable C^*-algebras in the UCT class. Then $A \cong B$ if and only if

$$Ell(A) \cong Ell(B).$$

[Gong–L, 2020] For the case that both A and B are stably projectionless.

$$((K_0(A), \Sigma(K_0(A)), \tilde{T}(A), \widehat{\langle e_A \rangle}, \rho_A), K_1(A))$$
$$\cong ((K_0(B), \Sigma(K_0(B)), \tilde{T}(B), \widehat{\langle e_B \rangle}, \rho_B), K_1(B))$$