Shack-Hartmann primer & Correlation wave-front sensing

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Goals of this talk

- Explanation of Shack-Hartmann sensors, probabilistic analysis of centroid algorithm
- Detailed performance analysis of correlation sensing using arbitrary images
- Comparison of centroiding to correlation for point-source applications

Why analyze algorithm performance?

- System design may be constrained
  - *e.g. read noise requirements for CCD*
- Identify important algorithmic/processing issues
  - *e.g. do we need smart background subtraction?*
- Confirm system performance via rigorous testing of predictions
  - *very useful for 'debugging' the system*
- Possible use in real-time to enhance system performance

Shack-Hartmann sensing measures the phase gradient across a lenslet

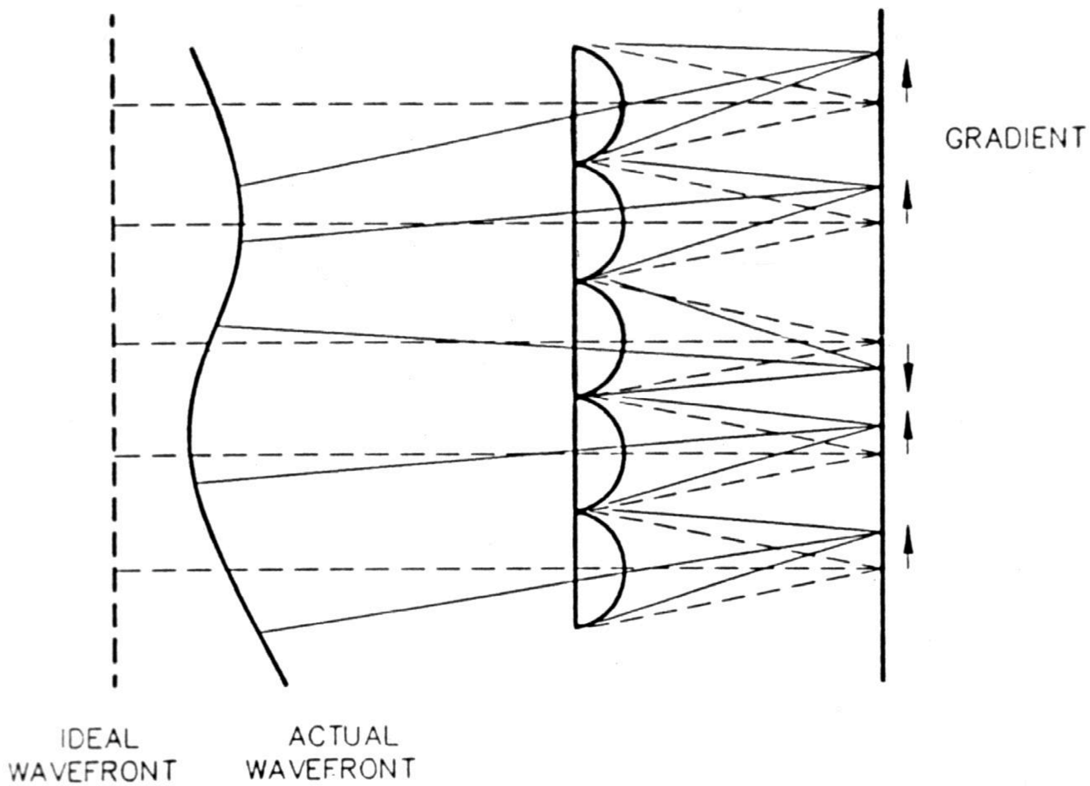
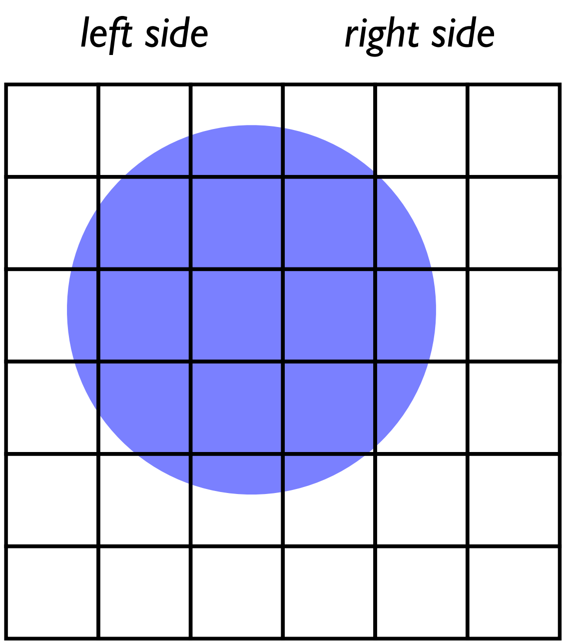


Figure from Gary Chanan

- Fourier optics principle:
  - *each lenslet forms a spot on the CCD*
  - *this spot moves with the average slope (linear phase term) in that subaperture*

Slope estimation with center-of-mass (centroid) formula

- The x-slope is based on the difference between linear combinations of the pixels on the right and left sides.



$$\hat{x} = \frac{r - l}{t}$$

Linearize formula to get tractable equation for analysis

- Each pixel value is a random variable
- Slope estimate is a non-linear function of the pixels
  - *need full probability distributions if quotient is involved*

- Linearize function around mean values of each variable using partial derivatives

$$f(x,y) \approx (x - m_x) f_x(m_x, m_y) + (y - m_y) f_y(m_x, m_y) + f(m_x, m_y)$$

Centroid analysis

- Linearize, then use basic identities to get mean, variance

$$\hat{x} \approx \frac{r}{m_t} - \frac{l}{m_t} - \frac{t(m_r - m_l)}{m_t^2} + \frac{m_r - m_l}{m_t}$$

$$\mathrm{E}[\hat{x}] = \frac{m_r - m_l}{m_t}$$

$$\mathrm{Var}(\hat{x}) = \frac{\sigma_r^2 + 2m_r^2 + \sigma_l^2 + 2m_l^2 - 4m_r m_l}{m_t^2} + \frac{\sigma_t^2 (m_r - m_l)^2}{m_t^4} + \frac{2(m_r - m_l)}{m_t^3}(\mathrm{E}[lt] - \mathrm{E}[rt])$$

Values of *r*, *l*, *t* depend on algorithm

- Weight pixels by distance for center
  - *NxN has center at value N/2 - 1/2*

$$r = \sum_{i=N/2}^{N-1} \sum_{j=0}^{N-1} (i - N/2 + 1/2) s[i,j]$$

$$l = \sum_{i=0}^{N/2-1} \sum_{j=0}^{N-1} (-i + N/2 - 1/2) s[i,j]$$

$$t = \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} s[i,j]$$

Probabilistic model for pixel values

- Each pixel in the spot has an normalized expected number of counts (poisson) scaled by exposure time + white read noise

$$s[i,j] = f \times p[i,j] + n[i,j]$$

- Mean and variance are

$$m_s[i,j] = f\lambda[i,j] \qquad \sigma_s^2[i,j] = f^2\lambda[i,j] + \sigma_n^2$$

Centroid variance is made of two independent terms

- Error due to photons is inversely related to SNR

$$\mathrm{Var}(\hat{x})_p = \frac{\sigma_r^2 + \sigma_l^2}{f m_t^2}$$

RMS error is inverse power law in SNR

- Error variance due to read noise depends on the *square* of the total number of pixels

$$\mathrm{Var}(\hat{x})_n = \sigma_n^2 \left(\frac{N^2(N^2-1)}{12 m_t^2} \right)$$

Background light reduces gain of centroid estimate

- With no background, the expected slope is

$$\mathrm{E}[\hat{x}] = \frac{m_r - m_l}{m_t}$$

- With poisson background (mean b) added in, expected slope is now

$$\mathrm{E}[\hat{x}] = \frac{m_r - m_l}{m_t + N^2 b}$$

Incorrect background subtraction, even by a few counts, can lead to significant loses in gain and increased error

Enough on centroiding for now...

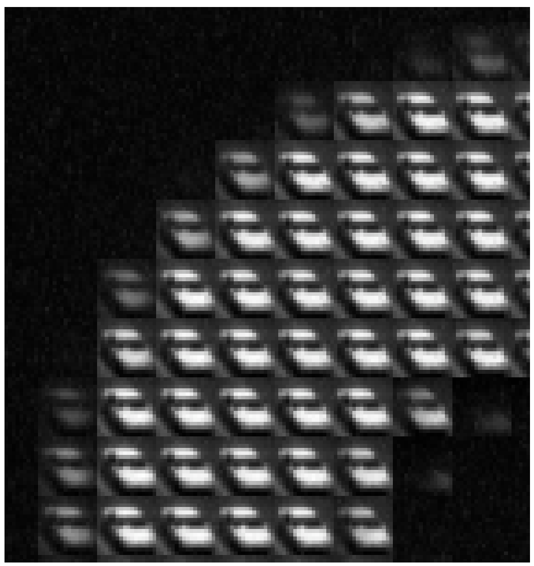
- Any questions??
- On to correlation of arbitrary images

Why limit AO to scenarios with a point source?

- Almost all AO systems use a point source to measure the phase
- Many interesting scenarios don't have a point source available
  - *observation of Earth from space with light-weight optics w/ time-varying aberrations*
  - *horizontal and slant-path imaging with small telescope*
  - *and of course solar astronomy, where AO using small images of solar granulation is successful!*

Use a Shack-Hartmann array to form subimages of the scene

WFS camera image of one quadrant of subapertures



Camera image from C. Thompson and R. Sawvel

- For pupil-conjugate phase aberration, a subimage of the scene will shift just like a point source
- Need to field stop down

How to best find image shift?

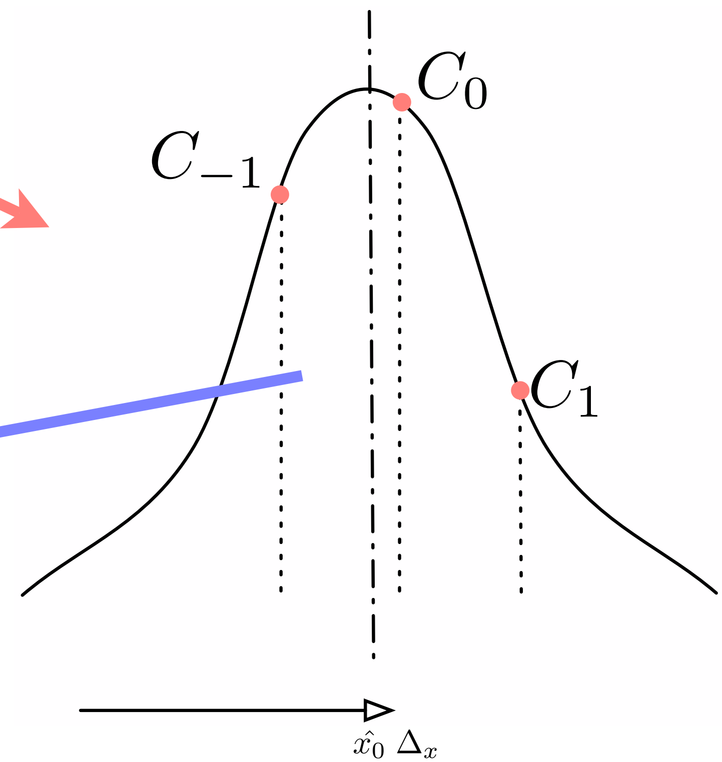
- Find shift between reference $\mathrm{r[m,n]}$ and subimage $\mathrm{s[m,n]}$
- Rich field, many possible options including
  - *non-random parameter estimation*
  - *deconvolution (linear phase fitting)*
  - *correlation implementation of MMSE metric*

$$\tilde{S}[k,l] = \tilde{R}[k,l]\mathrm{Exp}\left[\frac{-j2\pi(x_0k + y_0l)}{N}\right]$$

Best solution was 'aliased' correlation which is fastest way to get energy-normalization in this case

Estimate shift with a correlation-based algorithm

$$C[m,n] = \sum_i \sum_j r[i-m, j-n] s[i,j]$$



$$\hat{x_0} = \Delta_x + \frac{0.5(C_{-1} - C_1)}{C_{-1} + C_1 - 2C_0}$$

Parabolic interpolation

Statistical analysis of correlation formula

$$\begin{aligned}\hat{x}_0 &\approx \Delta_x + [C_{-1}(m_1 - m_0) + C_0(m_{-1} - m_1) + C_1(m_0 - m_{-1}) \\ &+0.5(m_{-1} - m_1)(m_{-1} + m_1 - 2m_0)] \times (m_{-1} + m_1 - 2m_0)^{-2}\end{aligned}$$

$$\mathrm{E}[\hat{x}_0] = \Delta_x + \frac{0.5(m_1 - m_{-1})}{m_{-1} + m_1 - 2m_0}$$

$$\begin{aligned}\sigma_x^2 &= [\sigma_{-1}^2(m_1 - m_0)^2 + \sigma_0^2(m_{-1} - m_1)^2 + \sigma_1^2(m_0 - m_{-1})^2 \\ &+2(m_1 - m_0)(m_{-1} - m_1)\sigma_{-1,0}^2 + 2(m_1 - m_0)(m_0 - m_{-1})\sigma_{-1,1}^2 \\ &+2(m_{-1} - m_1)(m_0 - m_{-1})\sigma_{0,1}^2] \times (m_{-1} + m_1 - 2m_0)^{-4}\end{aligned}$$

Use same statistical model for pixels

- Express means, variances of correlation points in terms of pixel models

$$\mathrm{E}[C_k] = m_k = \sum_i \sum_j m_r[i-k,j] m_s[i,j]$$

$$\mathrm{Var}(C_k) = \sigma_k^2 = \sum_i \sum_j \left(\sigma_r^2[i-k,j]\sigma_s^2[i,j] + \sigma_r^2[i-k,j] m_s^2[i,j] + m_r^2[i-k,j]\sigma_s^2[i,j]\right)$$

$$\mathrm{Covar}(C_k, C_l) = \sigma_{k,l}^2 = \sum_i \sum_j \left(\sigma_s^2[i,j] m_r[i-k,j] m_r[i-l,j] + \sigma_r^2[i,j] m_s[i+k,j] m_s[i+l,j]\right)$$

Recall that:

$$m_s[i,j] = f\lambda[i,j] \qquad \sigma_s^2[i,j] = f^2\lambda[i,j] + \sigma_n^2$$

Simplify into meaningful cases

- Combining all those equations is messy and not particularly informative in the general case
- Let's examine a few simpler cases to gain insight into the performance of correlation for slope estimation
  - *assume zero-shifted, identical reference and subimage (closed-loop case)*
  - *In that case, the means and variances of* C_{-1} , C_1 *are equal*

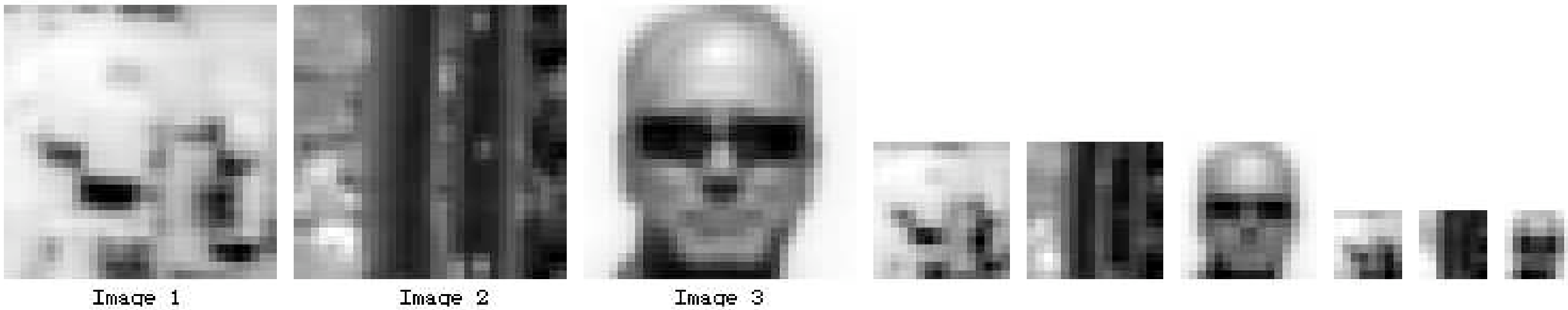
Image content directly affects variance

- Intuitively, the broader the image's band of spatial-frequency content, the better the estimation

$$\sigma_x^2 = \frac{\sigma_1^2 - \sigma_{-1,1}^2}{8(m_0 - m_1)^2}$$

sharpness of autocorrelation

Scene performance depends on frequency content



$\sigma_x = 0.013$
$\sigma_y = 0.011$

$\sigma_x = 0.009$
$\sigma_y = 0.019$

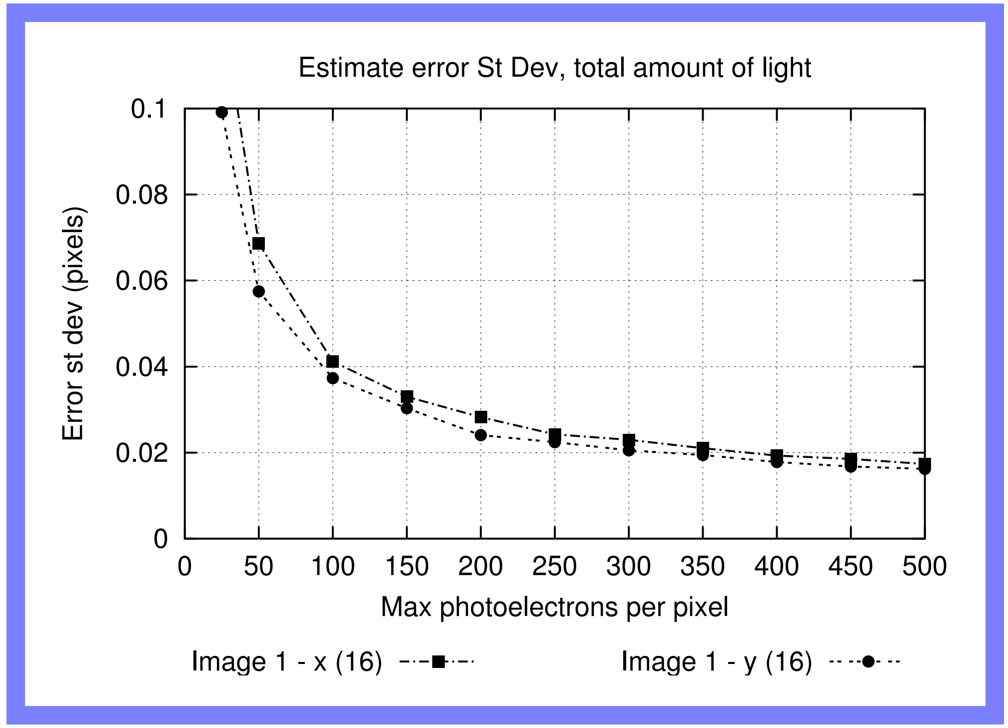
$\sigma_x = 0.013$
$\sigma_y = 0.008$

$\sigma_y = 0.087$

- Performance varies along x- or y-axis
- Need to find best field of view for expected scene scales

Scaling laws - exposure time

- In the case of changing exposure time, the error st. dev. follows as inverse power law in SNR



$$\sigma_x(f) = \frac{1}{\sqrt{f}} \frac{(\tilde{\sigma}_1^2 - \tilde{m}_1 - \tilde{\sigma}^2_{-1,1})^{1/2}}{2\sqrt{2}(\tilde{m}_0 - \tilde{m}_1)}$$

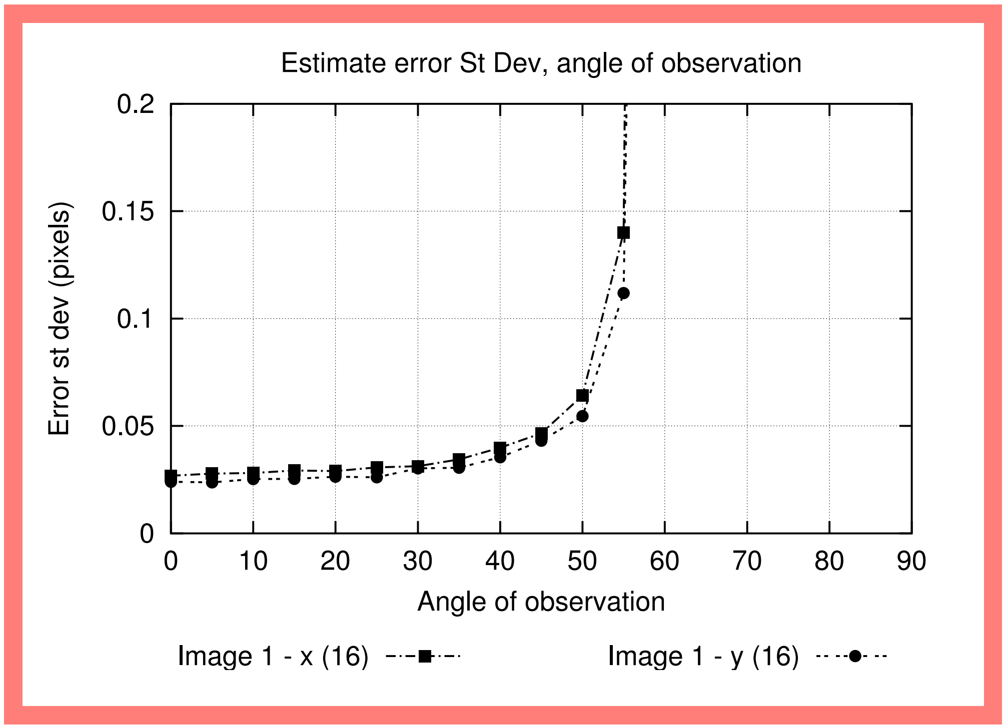
Background does not bias estimate

- Examination of the mean shift expression shows that a uniform background term cancels out entirely
- Regardless of scene, no background subtraction necessary

$$\mathrm{E}[\hat{x}_0] = \Delta_x + \frac{0.5(m_1 - m_{-1})}{m_{-1} + m_1 - 2m_0}$$

All $+b$ terms cancel

Scaling laws - angle of observation



- Generate radiometric models for background levels
- Performance falls off with too much background

$$\sigma_x(b,f) = \frac{\{Nb^2 + 2bf^2(\tilde{m}_0 - \tilde{m}_2 + \tilde{t}f^{-1}) + f^3[\tilde{\sigma}_1^2 - (f-1)f^{-1}\tilde{m}_1 - \tilde{\sigma}_{-1,1}^2]\}^{1/2}}{2\sqrt{2}f^2(\tilde{m}_0 - \tilde{m}_1)}$$

Let's take another look at point sources

- Any questions on the correlation of arbitrary images?
- Back to point sources...

How do we apply correlation?

- Use a fixed reference image
  - *e.g. ideal gaussian spot of expected system FWHM*
- Use of fixed reference significantly reduces noise
- There is some dependence on reference choice
  - *don't want reference clearly too big or too small*

For large N, correlation has significantly less read noise propagation

- Error variance due to noise does not depend of number of pixels

$$\mathrm{Var}(\hat{x})_n \propto \frac{\sigma_n^2}{8(m_0 - m_1)^2}$$

- Procedure
  - *Take 100 frames of static-aberration WFS data.*
  - *Estimate slopes for each subap, each frame*
  - *Analyze variance of slopes through time to get noise*

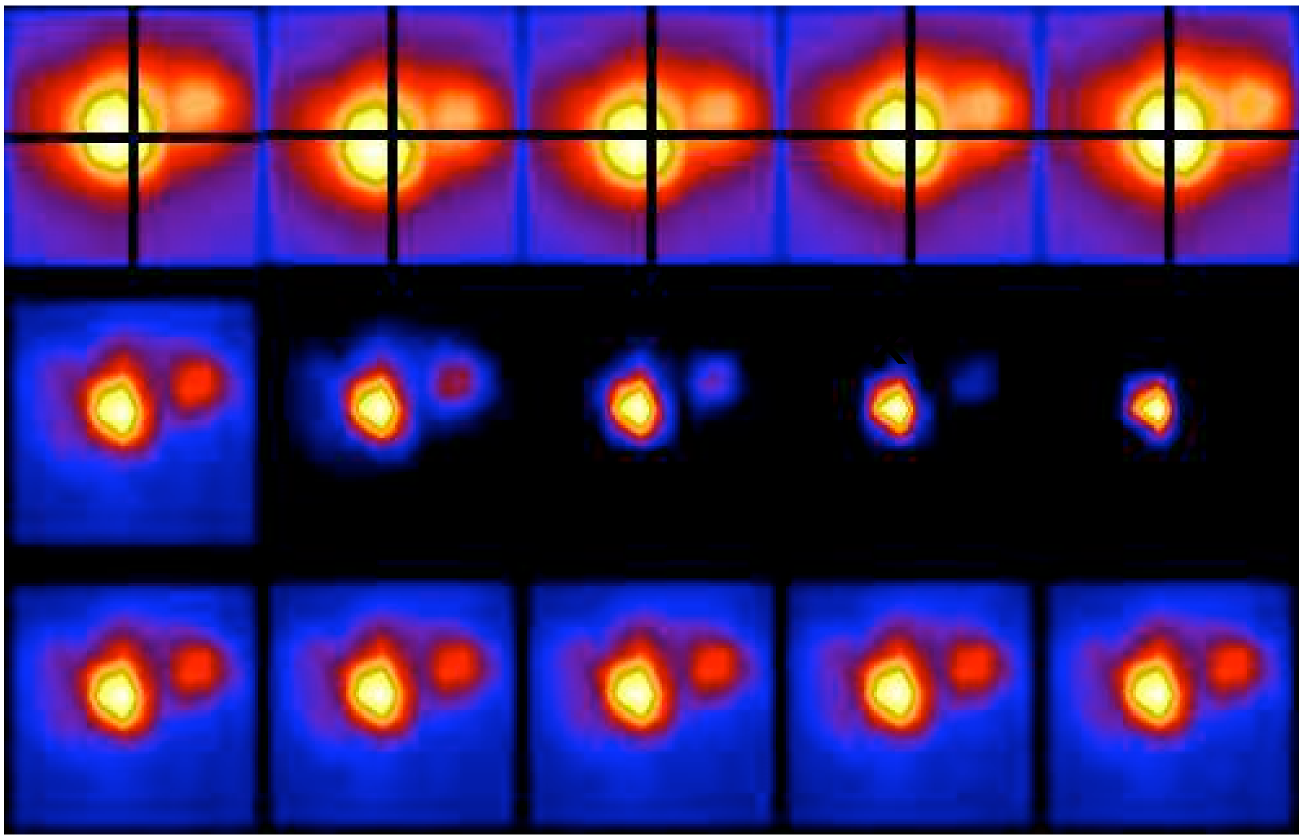
For large N, correlation has significantly less read noise propagation

- Based on spot size from data, predict noise propagator using known formula
- Compare to actual rms noise

	Noise prop	σ_x	σ_y
Correlation	1.58×10^{-5}	3.8×10^{-3}	4.23×10^{-3}
Centroider	212×10^{-5}	19.6×10^{-3}	20.9×10^{-3}

LLNL/UC Davis Vision system data from Abdul Awwal

Spot deformities and background bias the centroid estimate



Centroid estimate

After thresholding

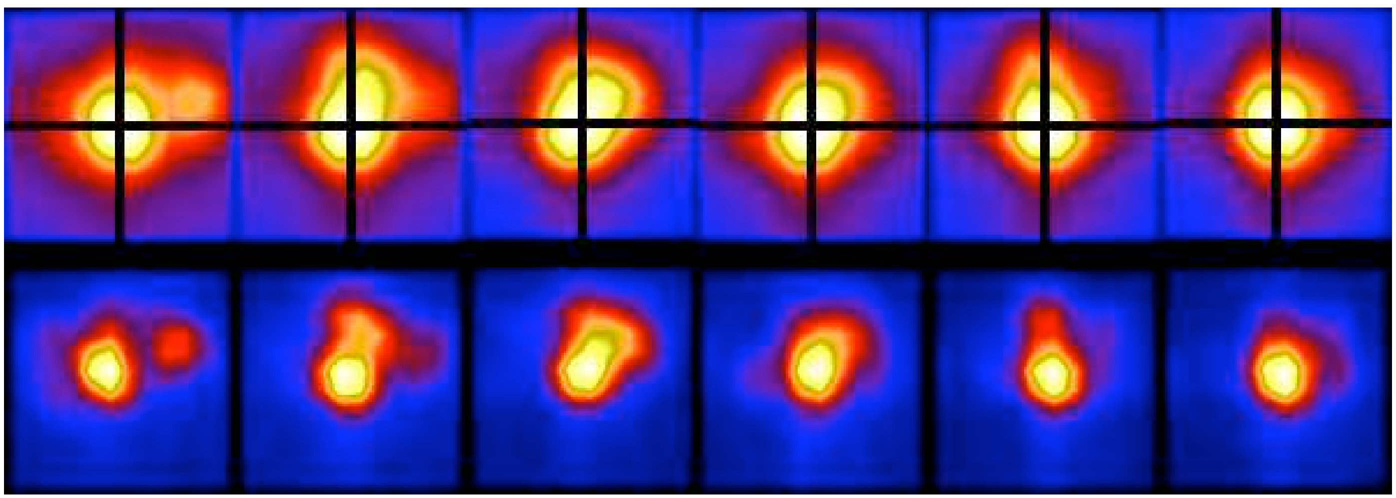
Interpolated WFS data

Increasing threshold -> ->

LLNL SSHCL system data from Kai LaFortune

Correlation insensitive to deformities and background

- Used same Gaussian reference for all frames
- Since N = 12, also read noise improvement



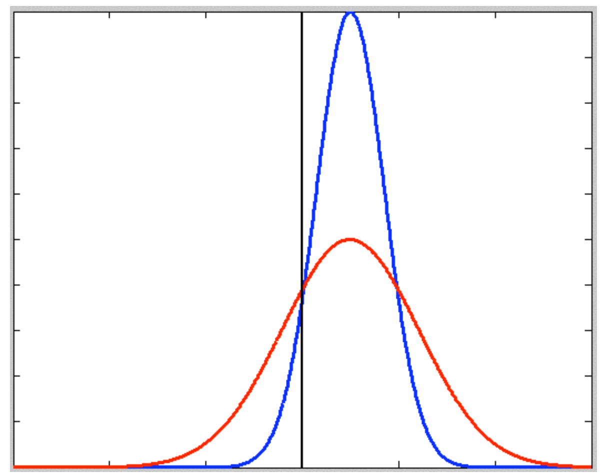
Correlation estimate

Interpolated WFS data

Same subaperture, six different time steps

The quad-cell method suffers from gain changes with spot size

- Quad-cell is 2 x 2 centroid
- Lost light off edge of pixel also biases estimate
- As spot size changes, so does gain
  - *leads to increased residual error if there are non-common-path errors and references*

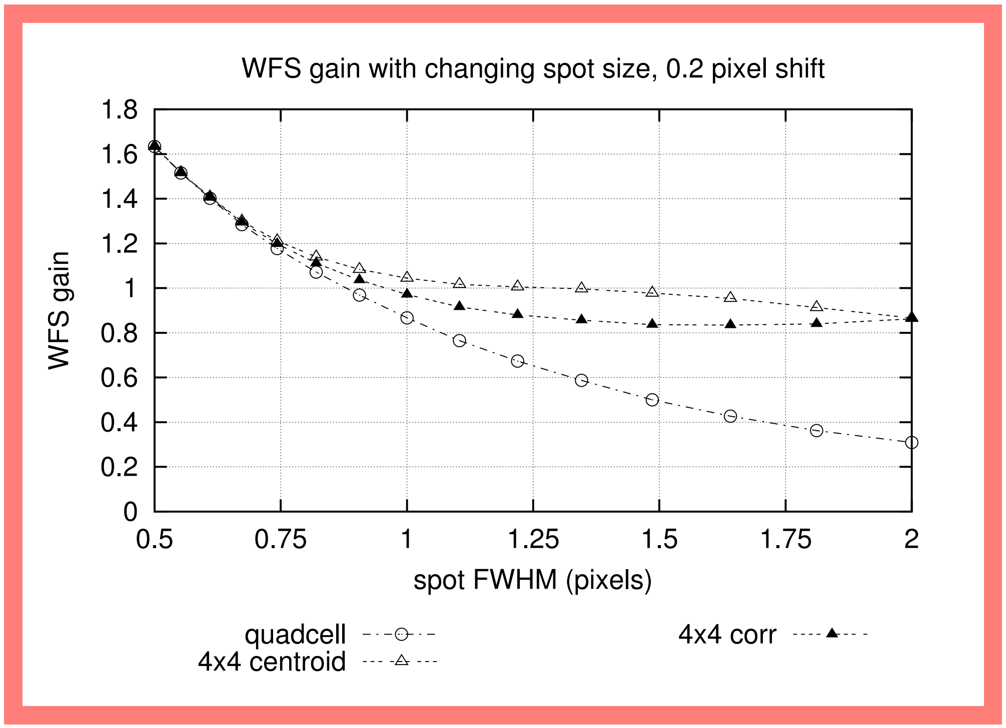


Est is 0.28

Est is 0.14

Figure from Marcos van Dam

Using 4x4 pixels ameliorates the gain problem



Plot data based on algorithm statistical performance and spot models

- 4x4 centroid has a more regular response with spot size than quad-cell.
- 4x4 correlation is even smoother

Testing at Lick of correlation on 4 x 4

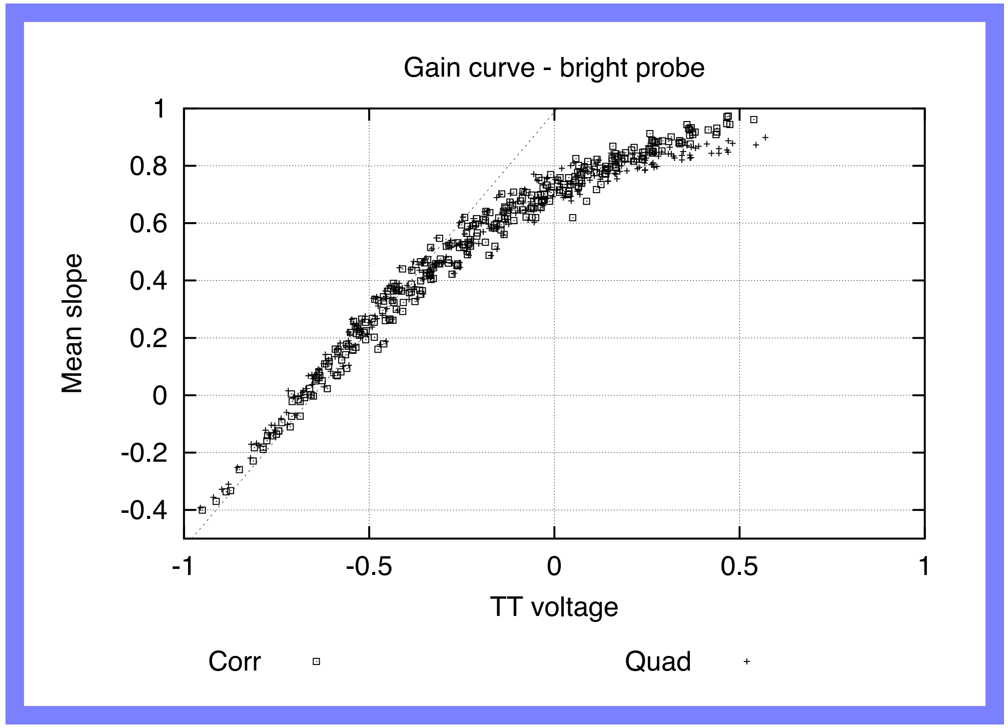
- Pixels are ~9 times under-Nyquist sampled for diffraction limited spots
- However, laser spot is very large - it extends beyond quad cell area

- Test white light probe, larger red probe, and laser on the sky, estimate system gain for both algorithms
- Take 4K frames telemetry from system to analyze

Lick Tests done with Dave Palmer, Elinor Gates and Tony Misch

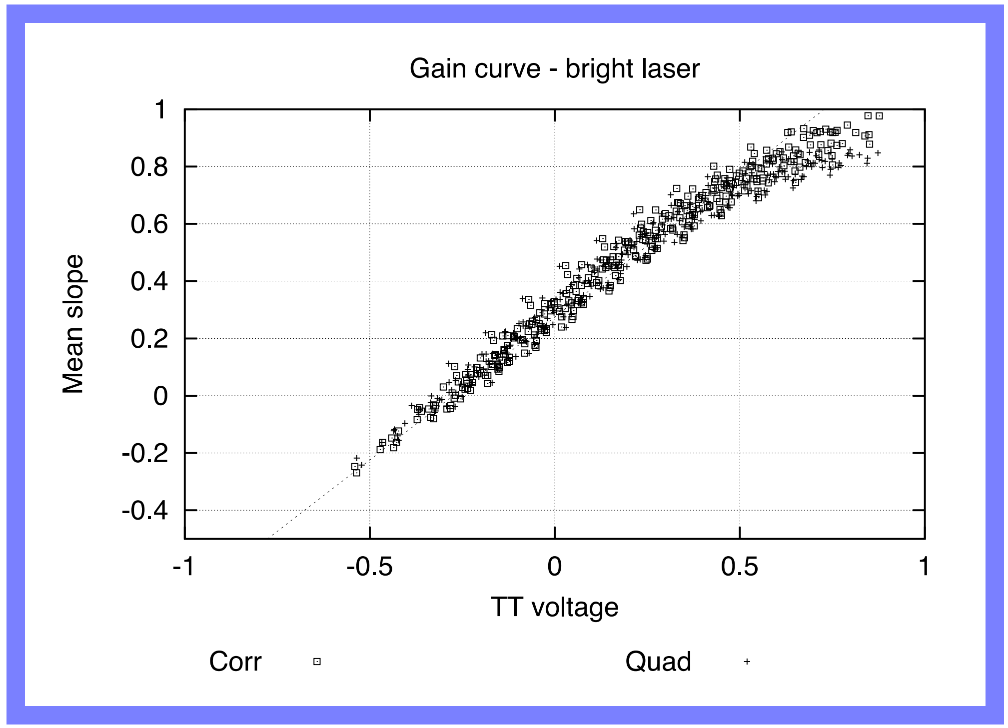
Gain high for white probe

- Gain is 1.52 from TT steering mirror to slope estimate
- Both algorithms have same gain in linear region
- Correlation is noisier



Gain lower for red laser probe

- Gain is 0.997 from TT steering mirror to slope estimate
- Both algorithms have same gain
- Correlation is noisier



Both gain and noise results make sense

Pixel SNRs

Subap 24, slope = 0

1.2	2.3	2.3	1.3
3.0	13.5	24.7	2.7
2.5	22.7	15.3	2.6
1.2	3.2	3.0	1.2

Subap 24, slope = 0.5

1.5	2.3	3.1	1.2
3.0	21.9	27.6	3.2
1.6	12.4	13.5	2.2
1.2	2.2	2.5	1.1

Subap 4, slope = 0

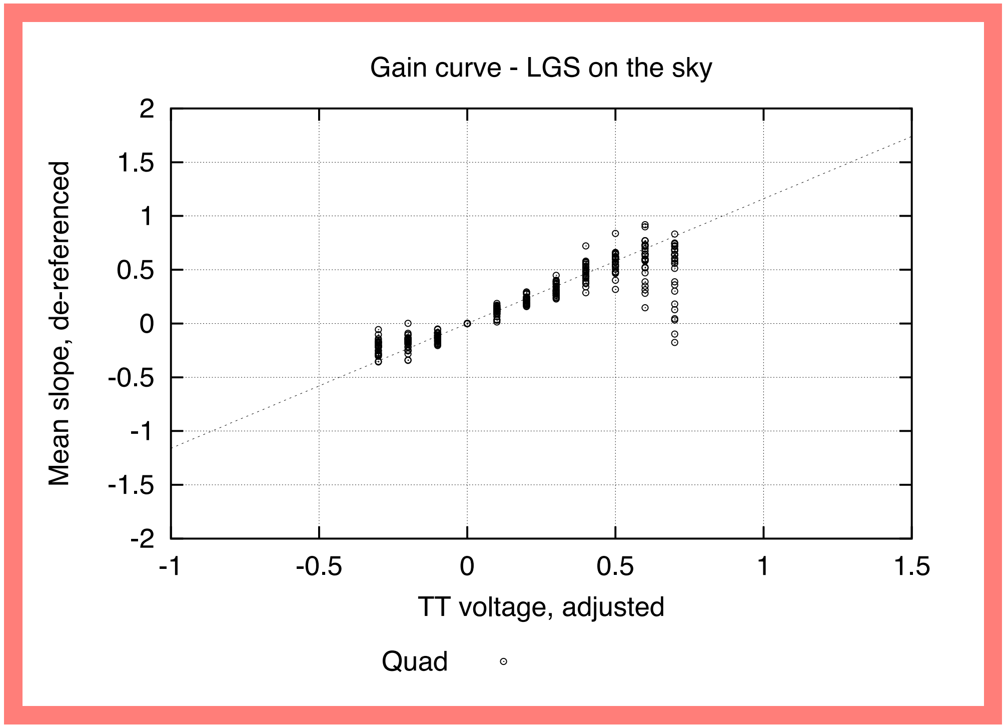
1.8	2.6	2.5	1.9
2.7	14.2	21.7	3.6
2.7	14.4	18.5	3.5
1.8	2.8	2.9	1.7

Subap 4, slope = 0.5

1.8	2.9	2.7	2.2
3.2	16.9	29.3	4.7
1.9	9.6	12.8	2.7
1.9	2.8	2.7	2.0

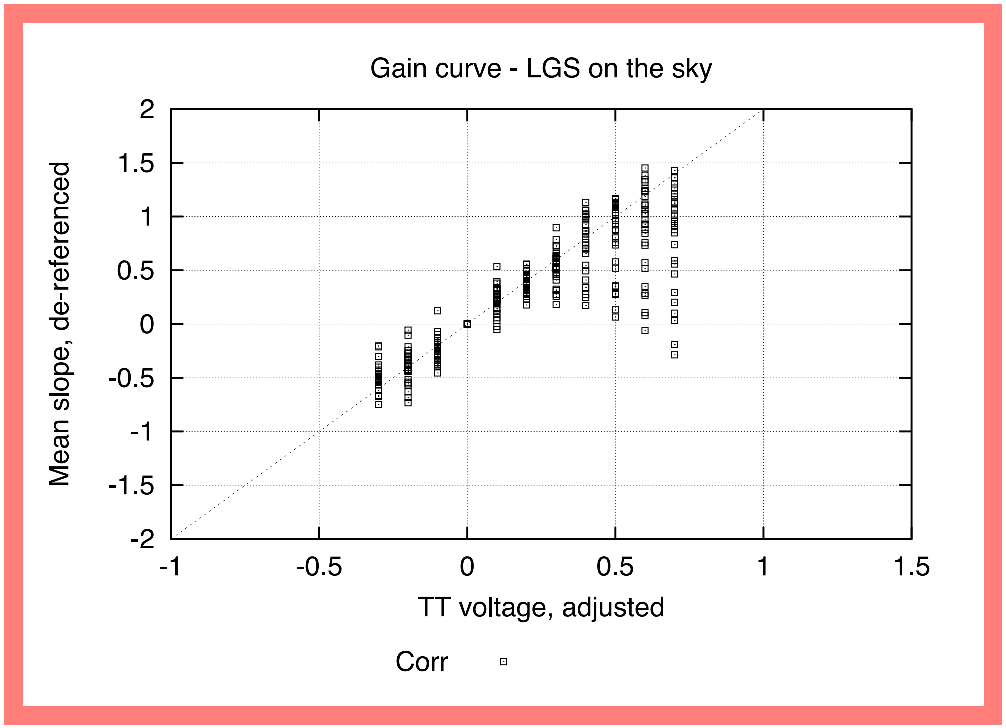
- Both probes are small enough that extra pixels have very low signal
  - *using those pixels leads to more noise*
- Gain should go down as spot gets bigger

With LGS, quadcell had low gain



- Moved LGS with uplink mirror
- Used best-centered voltage to reference other trials
- Gain is 1.16

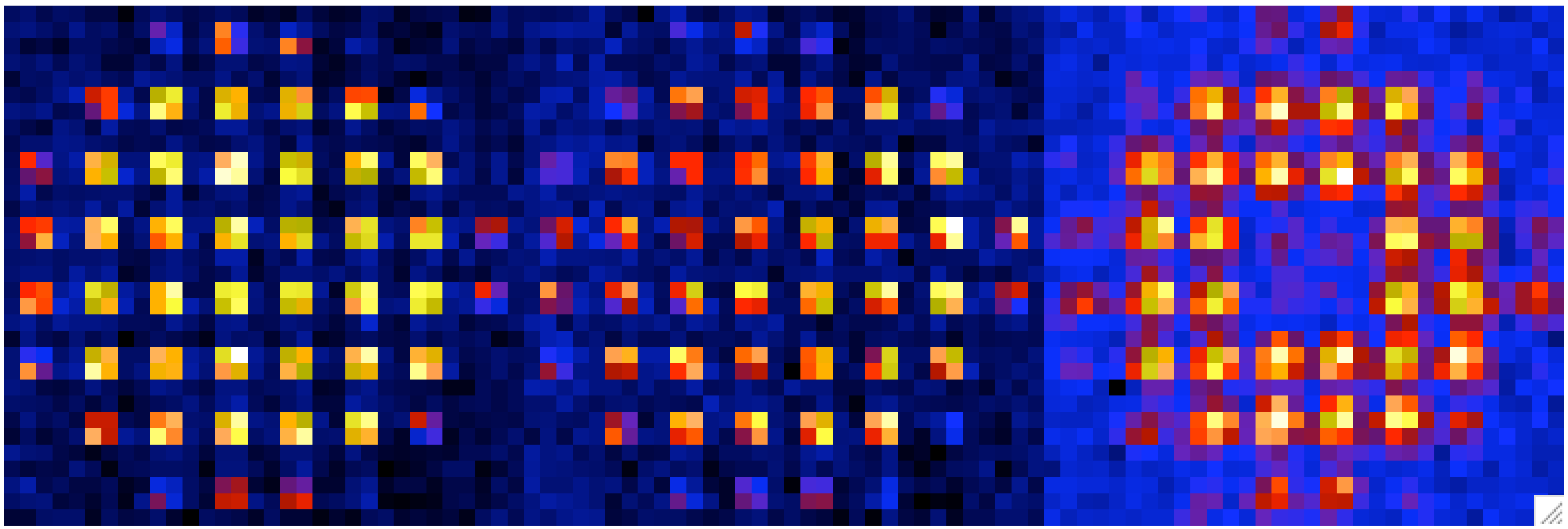
With LGS, correlation had high gain



- Low values are due to truncation in code of output, not algorithm error
- Gain-normalized, noise is equivalent to quadcell
- Gain is 2.0

LGS spot size clearly visible in CCD

(independent color scales)



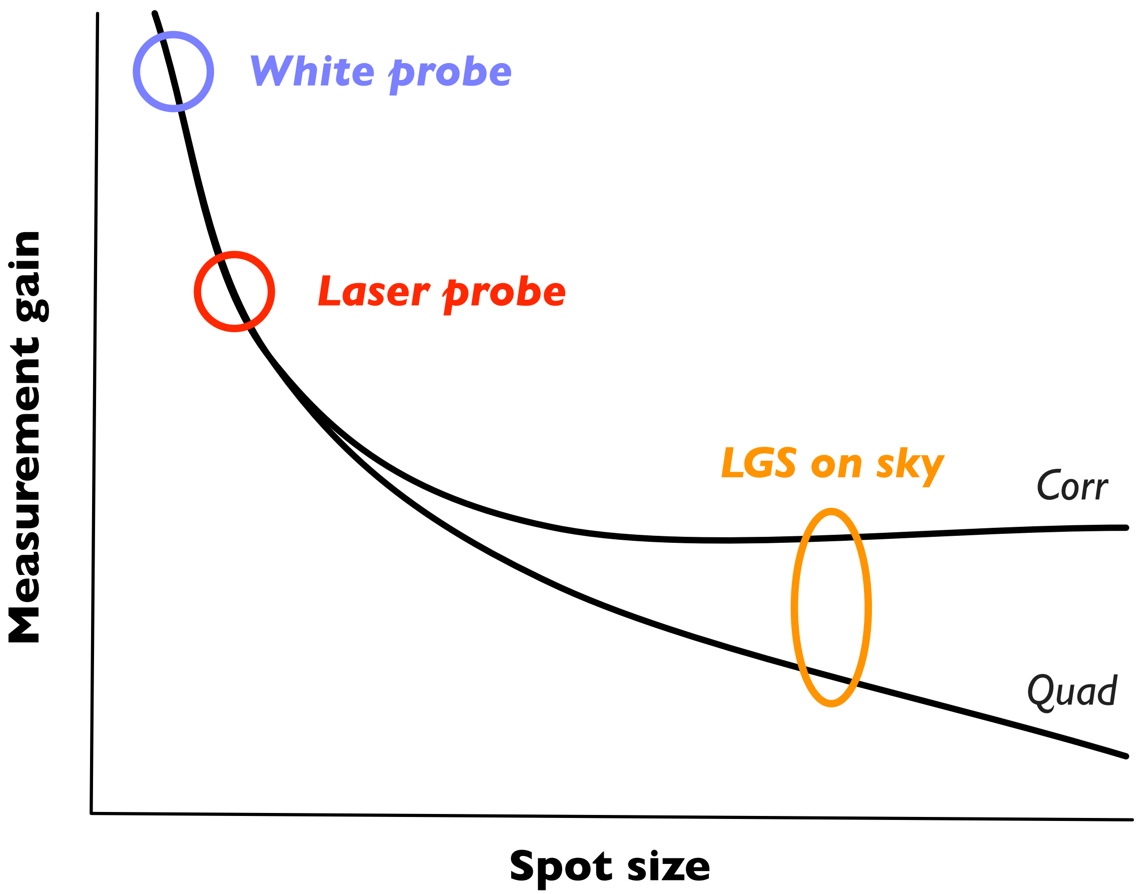
White probe, dim — Frame 55

Laser probe, dim — Frame 44

LGS on sky — Frame 03

- SNR of each pixel shown, 100 frames of CCD data

We have probed spot size curve



- Need to calibrate relationships between probes, LGS
- How much will spot size vary?

There are other options for algorithms

- Thresholding of background pixels
- ‘Shrinking box’ (Vision AO) - necessary to use fractional pixels
- Optimal estimation strategies
  - *see van Dam and Lane (JOSA A* **17***), Sallberg et al (JOSA A* **14***), etc.*

Analyze the slope estimation algorithm!

- Can do detailed probabilistic analysis of algorithm
- This leads to
  - *better understanding of its behavior*
  - *ability to design and test system to certain performance specifications*
- Correlation is preferable to centroiding in certain situations, especially
  - *many pixels per subaperture with read noise*
  - *time-varying background*