Fundamentals of estimation theory applied to wavefront reconstruction and adaptive optics control

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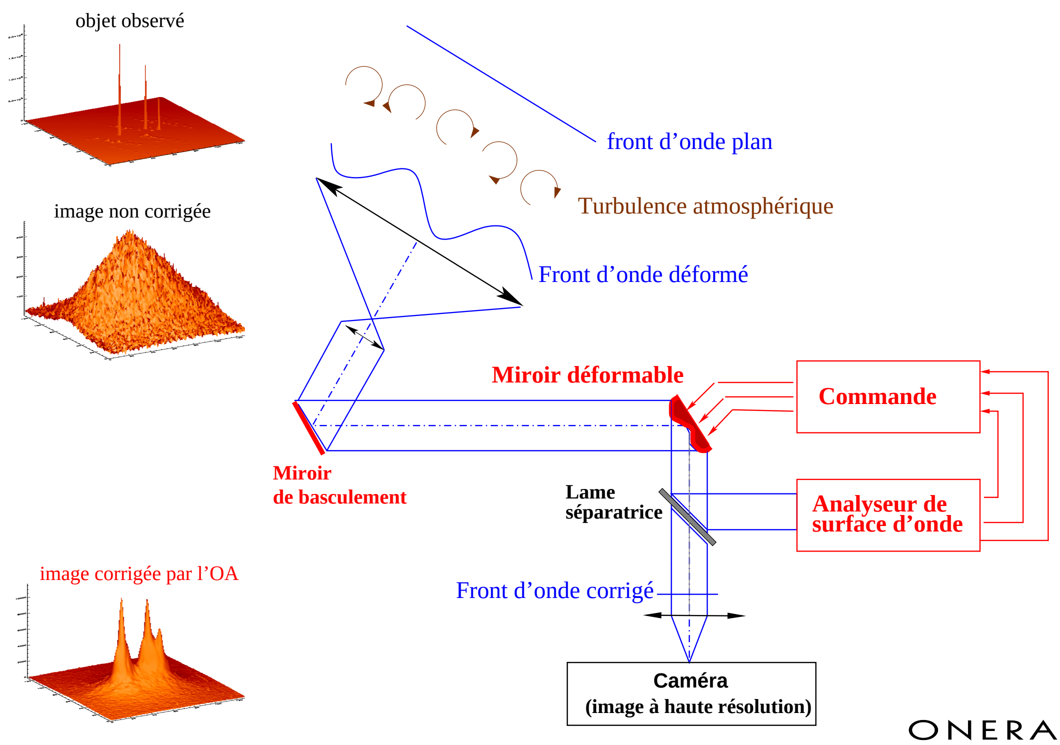
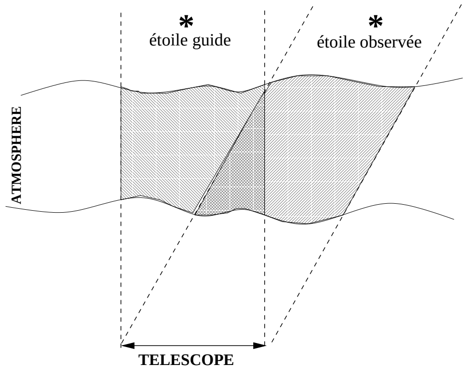
Principle of Adaptive Optics



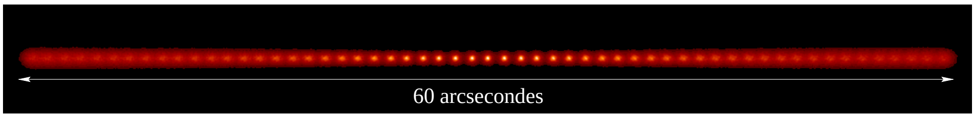
Adaptive Optics and Anisoplanatism



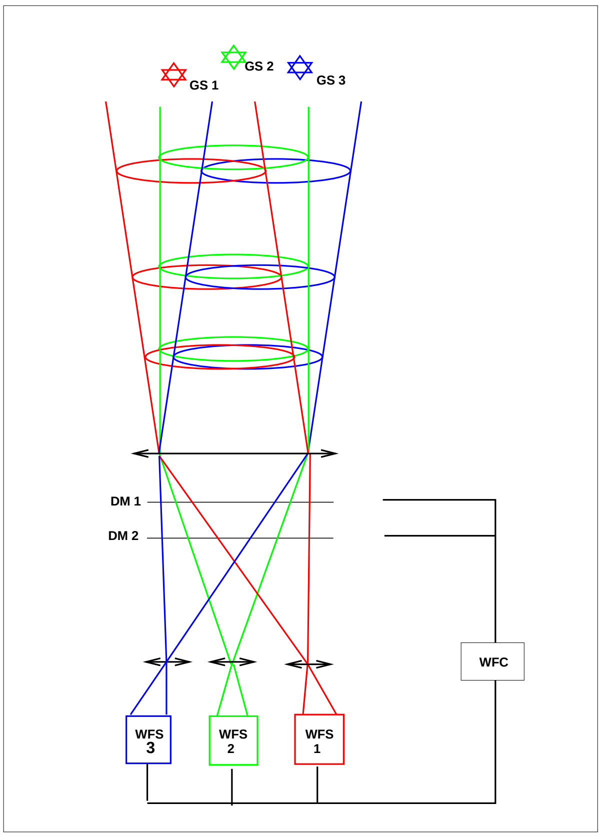
Anisoplanatism $\longrightarrow$ evolution of the turbulent phase in the field of view

Classical Adaptive Optics (AO) :

- **1 deformable mirror [DM] in pupil $\longrightarrow$ ϕ_{corr} independent of position in the field**
- **1 guide star with ϕ_{corr} optimized in guide star direction**



Correction quality degrades away from the guide star

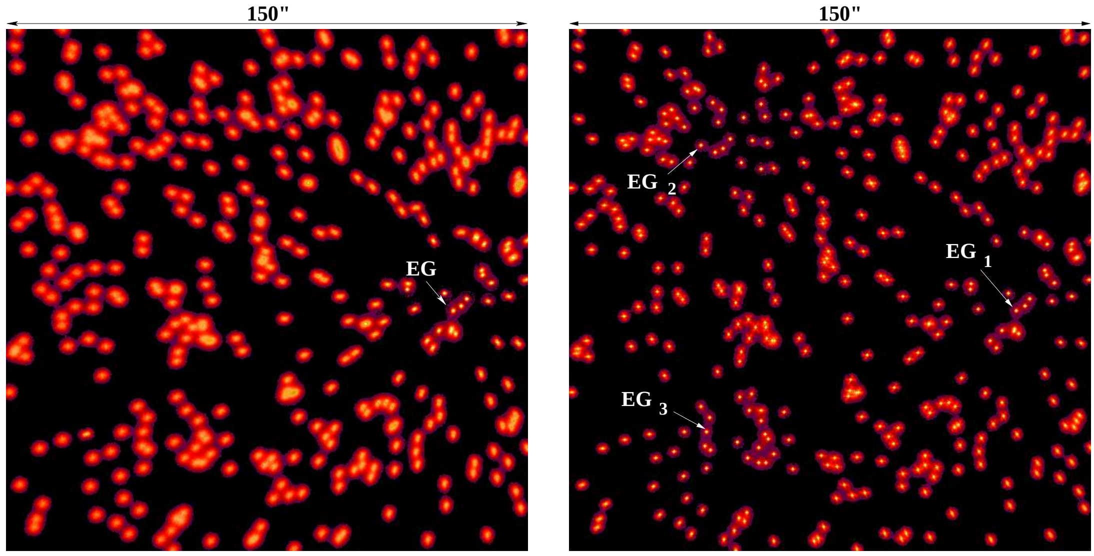


Inverse Problems in AO/MCAO

- **Wavefront Reconstruction from WFS data**
- **Optimal Mirror Control derived from WFS data**
  - **static mode : open loop, one data sample**
  - **dynamic mode : closed loop with delay, time series**
- **Corrected Image Processing : deconvolution.**

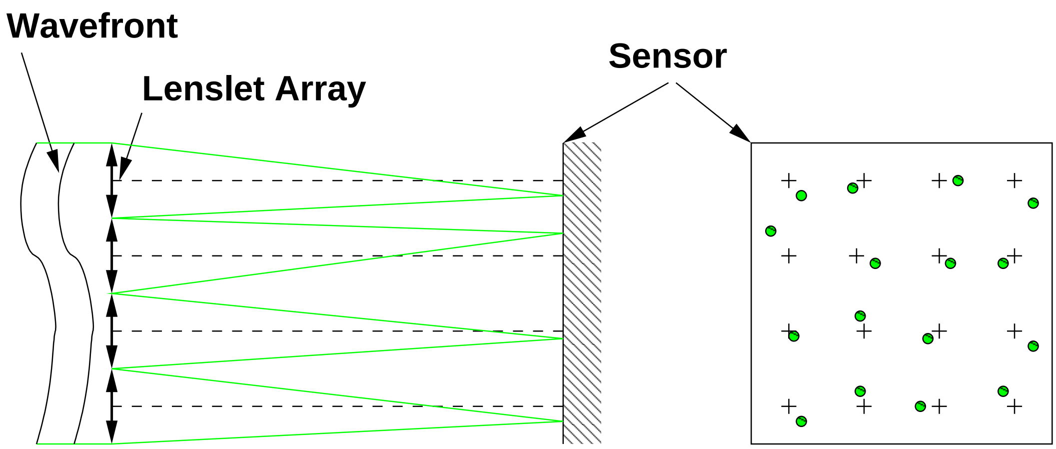
Expected Performance

- Observation at $2.2\mu m$ on a 8 m telescope.
- 3 Guide Stars, 2 Deformable Mirrors



regular AO MCAO

Principle of the Shack-Hartmann WFS



K sub-apertures $\longrightarrow 2\ K$ slopes $\{s_{x/y,i}\} = \mathrm{s}$

Shack-Hartmann Model

- **Shack-Hartmann measurements :**

$$s_{x,i} = \frac{\lambda\, f}{2\,\pi\, S_i} \int_{S_i} \frac{\partial \phi(x,y)}{\partial x} \mathrm{d}x\, \mathrm{d}y + noise$$

- **Linear model :**

$$\mathrm{s} = D\phi + \mathrm{w}$$

with for instance : $\phi = \{\phi_j\}$ **and** $\phi(x,y) = \sum_{j=2}^{j_{max}} \phi_j Z_j(x,y)$

$$D = 2K \updownarrow \begin{pmatrix} \longleftarrow j_{max} \longrightarrow \\ \cdot \\ \cdot \\ \cdot \end{pmatrix}$$

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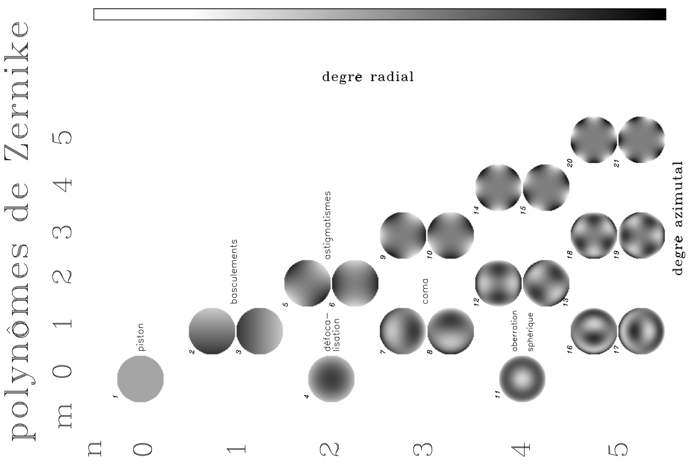
Zernike Polynomials

Decomposition on a modal basis $\{Z_i\}$:

$\phi(\mathrm{r}) = \sum_i \phi_i Z_i(\mathrm{r})$



Radial Degree n ⟺ Spatial Frequency

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WaveFront Reconstruction
Least Square Solution

- Looking for phase giving a best fit to the measurements :
$$\hat{\phi} = \arg\min_{\phi} ||\mathrm{s} - D\phi||^2$$
- Analytical solution :
$$\hat{\phi} = \left(D^t D\right)^{-1} D^t\, \mathrm{s} = R\, \mathrm{s}$$
- Noise propagation : noise covariance matrix
$$C_{\phi\ noise} = R\, C_w R^T$$

⚠ If $D^t D$ ill conditioned $\Longleftrightarrow$ badly seen modes
$\Longrightarrow$ noise amplification

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Maximum Likelihood Reconstruction
a Weighted Least Square Solution

- Likelihood : probability of the measurement (slopes) with a known phase,
$$p(\mathrm{s}|\phi) \propto \exp\left(-\frac{1}{2}(\mathrm{s} - D\phi)^t C_w^{-1} (\mathrm{s} - D\phi)\right)$$
- We look for the phase that makes the measurements the most probable ones :
$$\hat{\phi}_{\mathrm{ML}} = \arg\max_{\phi} p(\mathrm{s}|\phi) = \arg\min_{\phi} \quad (\mathrm{s} - D\phi)^t C_w^{-1} (\mathrm{s} - D\phi)$$
- Analytical solution :
$$\hat{\phi}_{\mathrm{ML}} = \left(D^t C_w^{-1} D\right)^{-1} D^t C_w^{-1}\, \mathrm{s}$$
- Particular Case $C_w = \sigma_w^2\, Id$:
$$\hat{\phi}_{\mathrm{ML}} = \arg\min_{\phi} ||\mathrm{s} - D\phi||^2 = \left(D^t D\right)^{-1} D^t\, \mathrm{s}\,,$$
least square fit on the measurements

In general, $D^t C_w^{-1} D$ badly conditioned $\Longrightarrow$ ML unacceptable
$\Longrightarrow$ “Truncated” ML : TSVD on $D^t C_w^{-1} D$
or limit the dimension of the phase space

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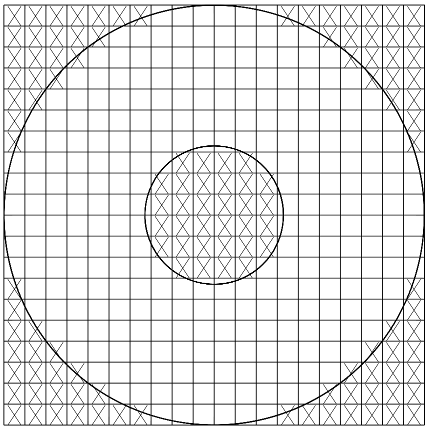
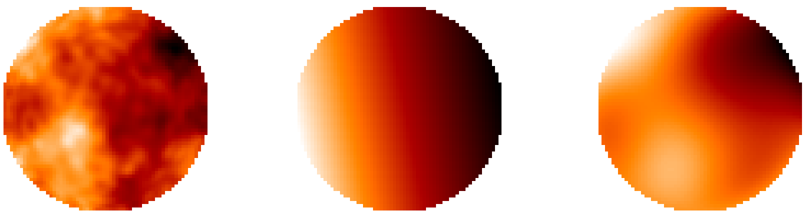
Numerical Simulation : Shack-Hartmann WFS Geometry

- 20×20 sub-apertures
- central occultation $d/D = 0.33$
- $K = 276$ valid sub-apertures
  $\Longrightarrow 2 \times 276 = 552$ slope measurements

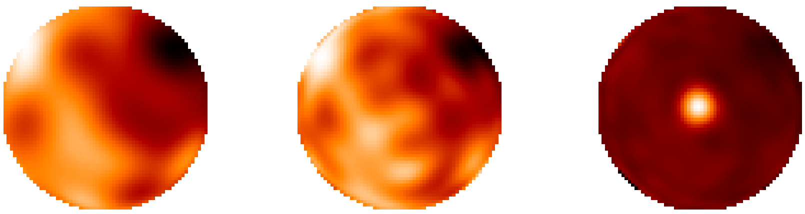


Truncated Maximum Likelihood

$SNR_{slopes} = 1.\,, K = 276$ sub-apertures



True Phase | ML $j_{max} = 3$ | ML $j_{max} = 21$



ML $j_{max} = 55$ | ML $j_{max} = 105$ | ML $j_{max} = 210$

Reconstruction quality highly dependent of j_{max}

optimum j_{max} depends on SNR_{slopes}

Optimal Reconstruction : Maximum *A Posteriori*

- Probability *A Posteriori*: probability of the phase for a given measurement

$$\begin{aligned} p(\phi|\mathrm{s}) &\propto p(\mathrm{s}|\phi) \times p(\phi) \\ &\propto \exp\left(-\frac{1}{2}(\mathrm{s}-D\phi)^t C_n^{-1}(\mathrm{s}-D\phi)\right) \times \exp\left(-\frac{1}{2}\phi^t C_\phi^{-1}\phi\right) \end{aligned}$$

- MAP solution : most probable phase for a given measurement

$$\begin{aligned} \hat{\phi}_{\text{MAP}} &= \arg\max_{\phi}\ p(\phi|\mathrm{s}) = \arg\min_{\phi}\ -2.\ln(p(\phi|\mathrm{s})) \\ &= \arg\min_{\phi}\ (\mathrm{s}-D\phi)^t C_n^{-1}(\mathrm{s}-D\phi) + \phi^t C_\phi^{-1}\phi \\ &= \left(D^t C_n^{-1} D + C_\phi^{-1}\right)^{-1} D^t C_n^{-1}\mathrm{s} = C_\phi D^t \left(D C_\phi D^t + C_n\right)^{-1}\mathrm{s} \end{aligned}$$

If Gaussian priors justified, MAP solution is also the MMSE solution $\hat{\phi}_{mmse}$:

$$\left\langle \|\phi - \hat{\phi}_{mmse}\|^2 \right\rangle_{\phi,w} = \left\langle \int_{Pup} \left[\phi(x,y) - \hat{\phi}_{mmse}(x,y)\right]^2 \mathrm{d}x\ \mathrm{d}y \right\rangle_{\phi,w}$$

minimum mean square error on the phase!

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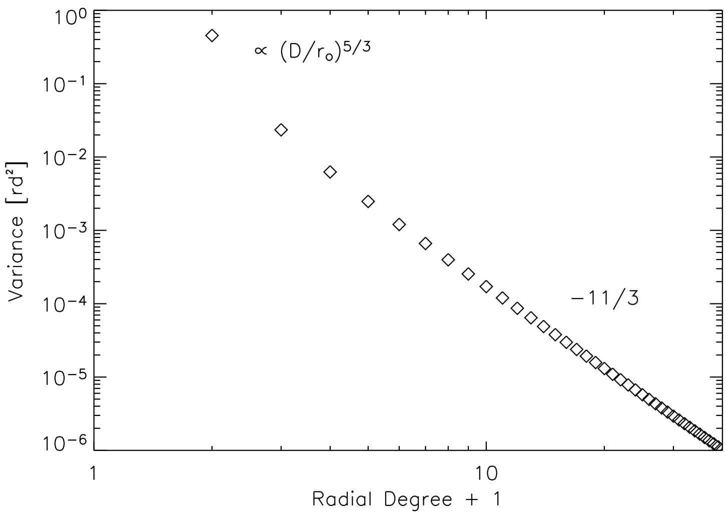
Statistical Characteristics of the Turbulent Phase

- Kolmogorov spectrum $\longrightarrow$ covariance matrix C_ϕ.
- $C_\phi(j,j) = \sigma^2_{\phi_j}$, energy distribution on Zernikes :



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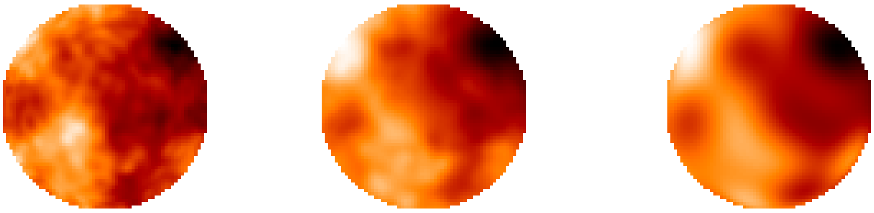
Optimal Reconstruction

$SNR_{slopes} = 1.\,, K = 276$ sub-apertures



True Phase	MAP/MMSE	ML $j_{max} = 55$
$[\sigma^2_\phi = 3.0\, rd^2]$	$[\sigma^2_{error} = 0.7\, rd^2]$	$[\sigma^2_{error} = 0.8\, rd^2]$

Optimal estimation : minimal phase error variance

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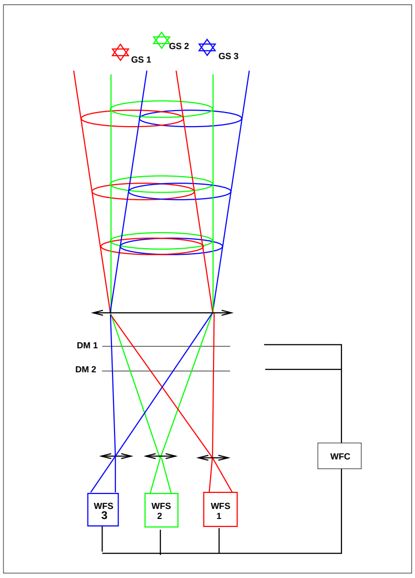
Phase Reconstruction in MCAO



- N_{gs} GS in direction $\beta = \{\beta_i\}$
- N_{gs} WFS measurements S $= \{s_i\}$
- L layers for volumic phase $\varphi = \{\varphi_k\}$
- Phase covariance matrix C_φ :
  bloc diagonal, independent Kolmogorov layers
  with $\left[C_n^2 \delta h\right]_k \quad k \in \{1, L\}$
- Resulting phase in a given direction α :
  $\phi_\alpha = M_\alpha^L \varphi$

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Optimal Reconstruction in MCAO

- Measurement Equation :

$$\mathbf{s} \;=\; D\,M^{L}_{\beta}\,\varphi \;+\; w \;=\; D'\varphi \;+\; w$$

- MMSE of the phase in the volume :

$\hat{\varphi}_{mmse} = G(\mathbf{s})$ so that $\left\langle \|\varphi - \hat{\varphi}_{mmse}\|^2 \right\rangle_{\varphi,w}$ minimal :

$$\begin{aligned}\hat{\varphi}_{mmse} &= \left(D'^{\,t}C_w^{-1}D' + C_\varphi^{-1}\right)^{-1} D'^{\,t}C_w^{-1}\;\mathbf{s}\\ &= R_{mmse}\;\mathbf{s}\end{aligned}$$

- MMSE of the phase in a given direction α :

$$\phi_\alpha \;=\; M^{L}_{\alpha}\;\hat{\varphi}_{mmse}$$

Optimal estimation of the phase in the volume

hence best interpolation of the phase between GSs

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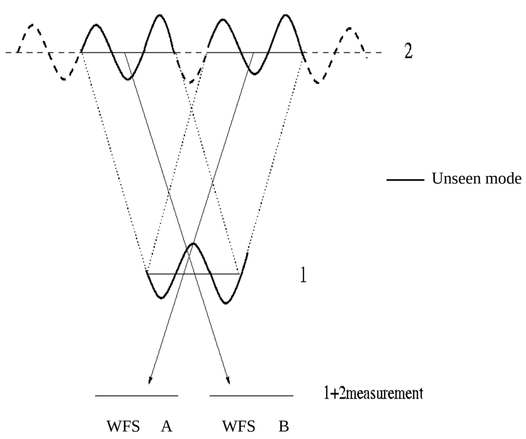
Unseen modes in MCAO



Good interpolation of the resulting phase between GSs

=> estimation of unseen modes

=> MMSE approach using statistical priors

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Estimation/Control in AO

- **DM model :**

$$\phi_{cor} = \mathbf{N}\mathrm{u}$$

- **Control problem : find** $\mathrm{u} = F(s)$ **that minimizes the stochastic criterion**

$$\left\langle \left\| \phi - \mathbf{N}\mathrm{u} \right\|^2 \right\rangle_{\phi,w}$$

- **Separation Principle :** u **also minimizes the deterministic criterion :**

$$\left\| \hat{\phi}_{mmse} - \mathbf{N}\mathrm{u} \right\|^2$$

- **Analytical solution :**

$$\mathrm{u} = \mathbf{P} \; \hat{\phi}_{mmse} = \mathbf{P} \; R_{mmse} \; s$$

$$\text{with} \quad \mathbf{P} = \left(\mathbf{N}^t\mathbf{N}\right)^+ \mathbf{N}^t$$

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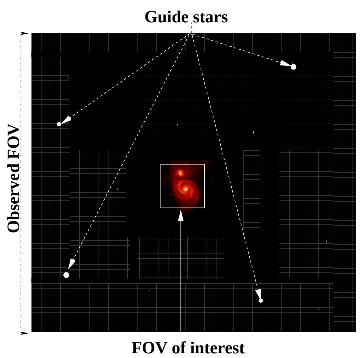
Estimation/Control in MCAO



Objective : estimate u **to optimize the mean performance in the FoV of Interest** $\{\alpha_i\}$

- **criterion to be minimized :**

$$\left\langle \sum_i \left\| \phi_{\alpha_i} - \phi_{\alpha_i}^{cor} \right\|^2 \right\rangle_{\phi,w} = \left\langle \sum_i \left\| \mathbf{M}_{\alpha_i}^{L} \varphi - \mathbf{M}_{\alpha_i}^{DM} \mathbf{N}\mathrm{u} \right\|^2 \right\rangle_{\varphi,w}$$

- **solution :** $\mathrm{u} = \mathbf{P} \; \hat{\varphi}_{mmse} = \mathbf{P} \; R_{mmse} \; \mathbf{S}$

$$\text{with} \quad \mathbf{P} = \left(\sum_i \left(\mathbf{M}_{\alpha_i}^{DM}\mathbf{N}\right)^t \mathbf{M}_{\alpha_i}^{DM}\mathbf{N} \right)^+ \left(\sum_i \left(\mathbf{M}_{\alpha_i}^{DM}\mathbf{N}\right)^t \mathbf{M}_{\alpha_i}^{L} \right)$$

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Standard Reconstruction in MCAO

- Measurement Equation : identification of turbulence space and DM space :

$$\mathbf{s} \;=\; D\, M_{\beta}^{L}\, \mathbf{N}\mathbf{u} \;+\; w \;=\; D_{inter}\mathbf{u} \;+\; w$$

- Truncated ML estimation :

$$\begin{aligned}\hat{\mathbf{u}}_{ml} &= \left(D_{inter}^{t} D_{inter}\right)^{-} D_{inter}^{t}\;\mathbf{s} \\ &= R_{ml}\;\mathbf{s}\end{aligned}$$

  – no explicit turbulence estimation
  – brutal filtering of unseen modes
  – no account of Kolmogorov statistics and turbulence profile
  – no specification of a FoV of interest

=> poor performance between guide stars

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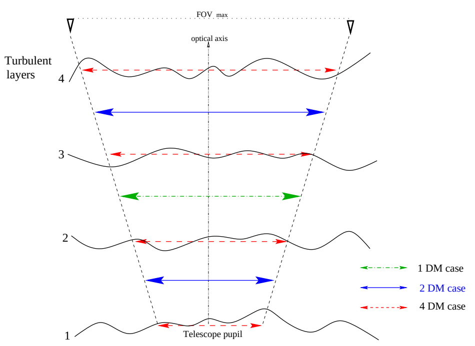
MCAO Simulation Conditions

2 DM 3 GS Case

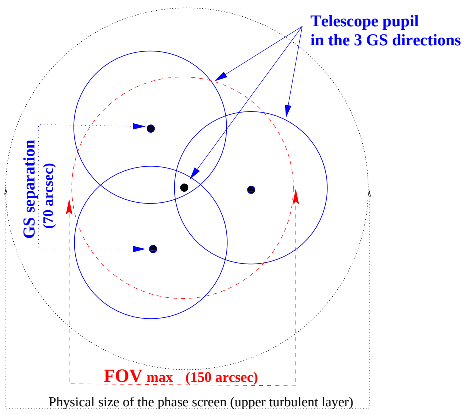
- Telescope diameter = 4 m
- 4 layer C_n^2 profile, $h = [\,0.\,,\; 2.5\,,\; 5.\,,\; 7.5$ km $]$, FoV = $150''$

$D/r_0(@2.2\mu m) = 6.8$



(a) DM and layer positions

$h = [\,1.25\,,\; 6.75$ km $]$



(b) beam section in upper layer

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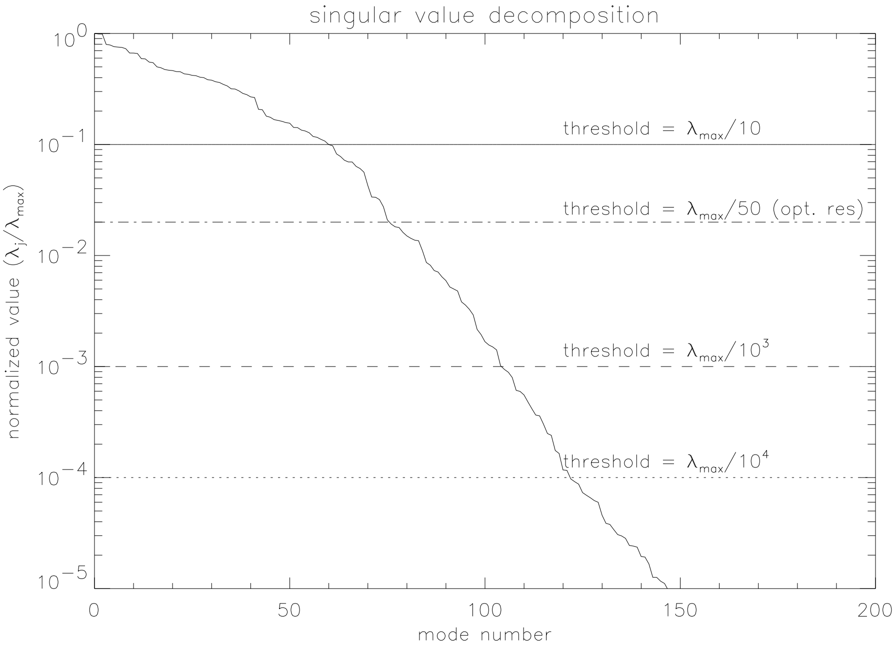
Truncated Maximum Likelihood :

choice of the threshold



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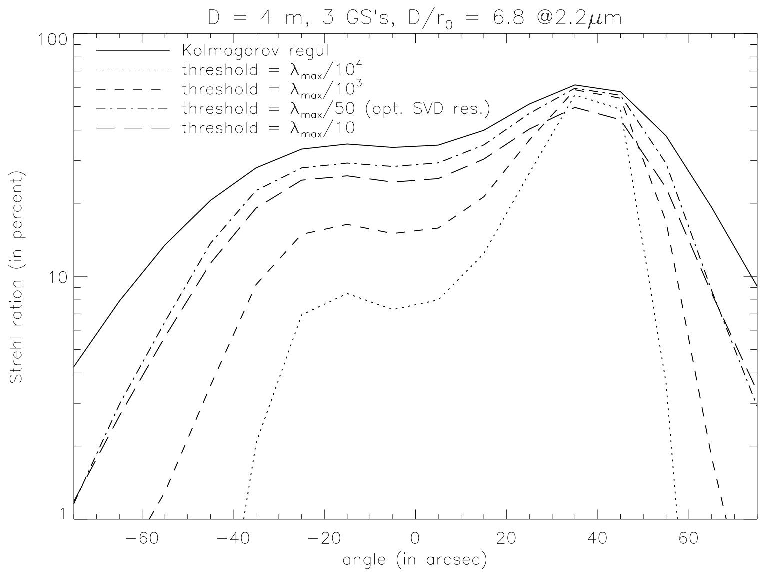
Performance with Truncated ML and Optimal Control



Optimal Reconstruction : significant gain, no ad-hoc threshold

[Fusco et al., "Optimal wavefront...", JOSA A, 18, 2527–253, 2001]

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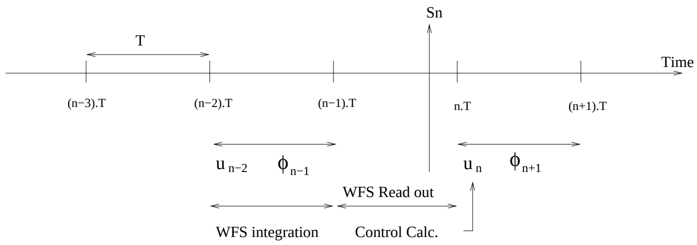
Dynamic Control : Basic Equations



- Measurement Equation [closed loop, no mirror dynamics, two frame delay] :

$$\mathbf{s}_\eta = D \left(M_\beta^L \varphi_{\eta-1} - M_\beta^{DM} \mathbf{N}\mathrm{u}_{\eta-2} \right) + w$$

- Control problem : find $\mathrm{u}_\eta = F(\mathbf{s}_\eta, \mathbf{s}_{\eta-1} \ldots)$ that minimizes the stochastic criterion :

$$\left\langle \sum_i \left\| \mathbf{M}_{\alpha_i}^L \varphi_{\eta+1} - \mathbf{M}_{\alpha_i}^{DM} \mathbf{N}\mathrm{u}_\eta \right\|^2 \right\rangle_{\varphi, w}$$

- solution : $\mathrm{u}_\eta = \mathbf{P} \; \hat{\varphi}_{\eta+1}$
  back to an MMSE estimation problem : $\hat{\varphi}_{\eta+1/\eta} = G(\mathbf{s}_\eta, \mathbf{s}_{\eta-1} \ldots)$ so that $\left\langle \| \varphi_{\eta+1} - \hat{\varphi}_{\eta+1/\eta} \|^2 \right\rangle_{\varphi, w}$ minimal

Dynamic Control : Optimal Estimation of the Turbulent Phase

- Temporal and spatial priors on turbulence :

$$\varphi_{\eta+1} = \mathcal{F}\left[\varphi_\eta, \varphi_{\eta-1}, \varphi_{\eta-2}, \ldots\right] + \nu_n$$

ν white noise, covariance matrix $\mathbf{C}_\nu$ imposes Kolmogorov statistics

$\mathcal{F}$ linear function describing temporal behavior [Taylor].

- State space representation : $\mathrm{X}_\eta^t = \{\varphi_{\eta+1}, \varphi_\eta, \ldots, \mathrm{u}_{\eta-1}, \mathrm{u}_{\eta-2} \ldots\}$

$$\mathrm{X}_{\eta+1} = \mathcal{A}\mathrm{X}_\eta + \mathcal{B}\mathrm{u}_\eta + \mathrm{V}_\eta$$
$$\mathbf{s}_\eta = \mathcal{C}\mathrm{X}_\eta + \mathrm{w}_\eta,$$

- Recursive Kalman estimator :

$$\hat{\mathrm{X}}_{\eta+1/\eta} = \mathcal{A}\hat{\mathrm{X}}_{\eta/\eta-1} + \mathcal{B}\mathrm{u}_\eta + \mathcal{A}\mathrm{H}_\eta(\mathbf{s}_\eta - \mathcal{C}\hat{\mathrm{X}}_{\eta/\eta-1}),$$

H_η, the observer gain, is deduced from $\mathbf{C}_\mathrm{w}$ and $\mathbf{C}_\mathrm{v}$.

$$\begin{pmatrix} \hat{\varphi}_{\eta+2/\eta} \\ \hat{\varphi}_{\eta+1/\eta} \\ \hat{\varphi}_{\eta/\eta} \end{pmatrix} = \mathrm{A} \begin{pmatrix} \hat{\varphi}_{\eta+1/\eta-1} \\ \hat{\varphi}_{\eta/\eta-1} \\ \hat{\varphi}_{\eta-1/\eta-1} \end{pmatrix} + \mathrm{H}\left[\mathbf{s}_\eta - \left(D \left(M_\beta^L \hat{\varphi}_{\eta-1/\eta-1} - M_\beta^{DM} \mathbf{N}\mathrm{u}_{\eta-2}\right)\right)\right]$$

Control in Adaptive Optics : Standard Approach

- **Control increment from residual measurements :**

$$\delta \hat{\mathbf{u}}_\eta = \left(D_{inter}^t D_{inter}\right)^{-} D_{inter}^t \; \mathbf{s}_\eta$$
$$= R_{ml} \; \mathbf{s}_\eta$$

- **Integrator control law :**

$$\mathbf{u}_\eta = \mathbf{u}_{\eta-1} + g\,\delta\hat{\mathbf{u}} = \mathbf{u}_{\eta-1} + g\,R_{ml}\;\mathbf{s}_\eta$$

g **loop gain, can be optimized globally or mode per mode.**

  - **no explicit turbulence estimation**
  - **brutal filtering of unseen modes**
  - **no global optimization of the control law using Kolmogorov/Taylor statistics**
  - **no specification of a FoV of interest**

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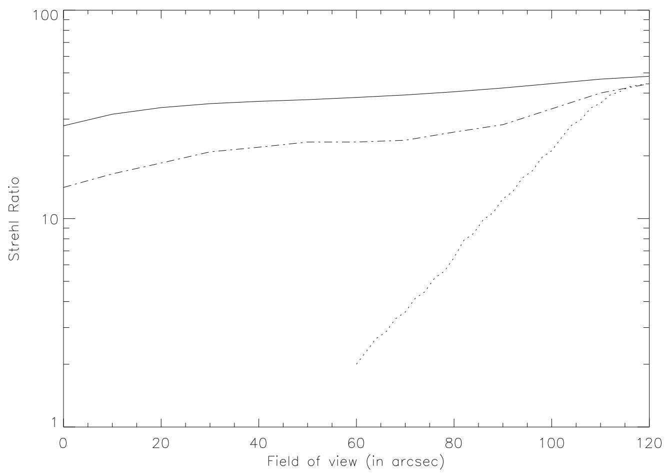
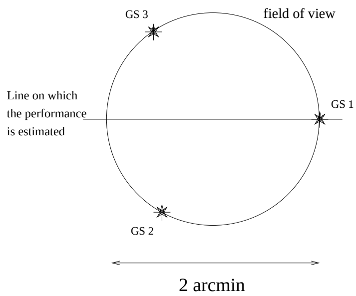
Performance with Optimal Kalman Control





- 2 **turbulent layers : [0.5,5 km]**
  $C_n^2 = [75\%, 25\%]$ **; overall** $D/r_o = 6.8$ **;** $V/D = 2$ **Hz**
- 2 **DMs and** 3 **GSs with** 1 **arcmin separation ;** $\mathcal{M} \approx 15$
  100 **Hz sampling frequency**

[Le Roux et al., “Optimal control law...”, JOSA A, submitted]

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Image Deconvolution / Reconstruction and Control

Similarities and Differences

- **s1: Linear direct problem :** $i \quad = \quad H\,o + noise$
- **s2: Badly conditioned problem : regularization is required!**
- **s3: Even if object o not really outcome of a stochastic process, the regularization corresponds to an implicit statistical prior**
- **d1: Operator H often circulant :** $i \quad = \quad h \star o + noise$
  all matrix expressions can be expressed with FFTs
- **d2: Regularization (positivity, edge preserving) non equivalent to Gaussian prior**
- **d3: Operator H sometimes not perfectly known (myopic deconvolution)**
- **d2 and d3 lead to non analytical linear solution**
  => iterative minimization of a multi-term criterion

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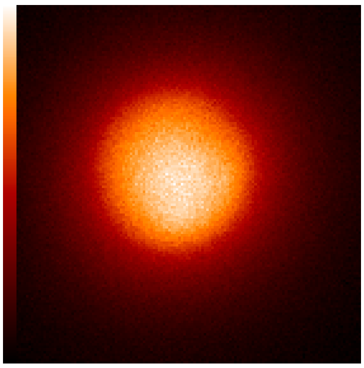
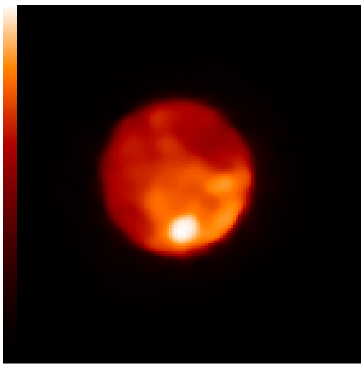
Myopic Deconvolution of AO Corrected Images

Ganymede deconvolved with our algorithm MISTRAL



AO corrected



myopic deconvolution

JPL data
courtesy : NASA/JPL/Caltech

$\lambda = 0.85\mu m$, $D/r_o \simeq 23$, $SR \simeq 5\%$, **Field =** $3.80''$

Exposure Time = 100 sec., Total Flux $\simeq 8.10^7$ **photons**

(28/09/1997, 20:18 UT) Observatoire de Haute Provence, 1.52 m telescope

[Mugnier et al., “MISTRAL: a myopic...”, JOSA A, accepted]

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Onera Activities on MCAO/ELTs

- AO/MCAO [4 PhDs, 2 PostDocs] :
  - study of optimal control in MCAO [collab. L2TI]
  - comparison of multi-object WFS strategies [star/layer oriented, MFOV...]
  - FALCON : MCAO for large field spectro-imaging [collab. Obs. Paris]
  - experimental validation :
    MonoCAO, off-axis optimization at Onera
    collab. ESO: test control algorithm on MAD
  - Planet Finder : design study of XAO for VLT
    collab. Obs. Grenoble/Marseille/Nice... [ESO contract]
  - AGN study with NAOS/MISTRAL data
- ELTs [1 PhD, 1 PostDoc] :
  - AO/MCAO for ELTs : AO/MCAO/GLAO design and simulations
    for 20 m class telescope: NGCFHT study with Obs. Marseille
    for 50 m and larger : collab. ESO/Arcetri [EC contract]
  - turbulence characterization on large scales: collab. Arcetri [EC contract]
    Paranal campaigns with NAOS/VLTI/Scidar/Balloons + specific WFS expe. on VLT
    two campaigns planned in 2005/2006

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