Mathematical Finance and Systemic Risk

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Outline:

Important issues: Why is modern finance mathematical? What is systemic risk?

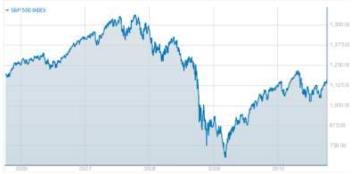
1. A brief historical overview of mathematical finance
2. The role of volatility
3. Current research directions in mathematical finance
4. Mean field models of systemic risk
5. Large deviations for mean field models
6. Concluding remarks

A historical review

- 1900: Bachelier defends thesis on "Theory of Speculation" and "invents" Brownian motion. Probability theory enters finance.
- 1929-1940: Great depression, contraction of financial markets. Qualitative "Macroeconomic Theory" dominates.
- 1973-1974: Black-Scholes-Merton theory of options pricing. Chicago Board of Options Exchange opens. Options become financial instruments with which risk (in currency exchange) can be managed. Migration of mathematicians and physicists to Investemnt Banking.
- 1987-2008: Golden age of financial mathematics. Banking, investment and finance become a quantitative and data-driven industry. Thousands of scientists, engineers and mathematicians enter the field. More that 30 top universities around the world establish degree programs in "Financial Mathematics and Engineering". Research publications on mathematical problems in investment and finance increase dramatically.
- 2008-2010: Critical reorientation of research priorities in quantitative finance with emphasis on risk. Acceleration of mathematization of investment and finance.

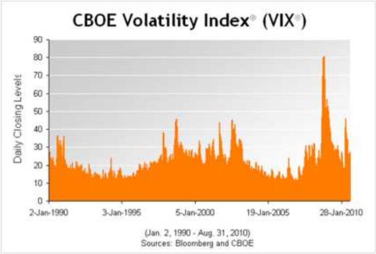
The S&P 500 Index of US Equities (Fig: 2006-10)

- S&P 500: An index that measures the value of the top 500 American companies by capitalization.
- Long term (50 year) statistics: Growth rate $11-12\%$ annualized (but negative in last 10 years), Realized Volatility $12-13\%$.
- Derivatives: Financial contracts (instruments) that are **traded** and whose value depends on another traded instrument
- Index Options (puts and calls): Derivatives that are highly liquid and can be used to control the risk of holding equities



The volatility index (VIX) (Fig: 1990-2010)

- Why is volatility important? Because risk management and the instruments used for it, such as options, depend essentially on volatility. They do not depend essentially on the gain or the loss of the value of the equity.
- Volatility is an indicator of market "health", like the temperature of the human body. VIX is a special, very important volatility index derived implicitly from options. Volatility generation and liquidity.

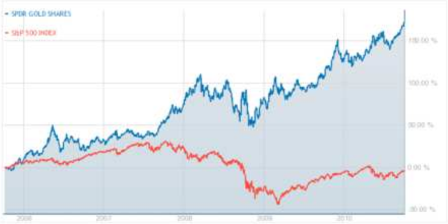


VIX and S&P 500

- When markets are healthy volatility is low and liquidity high
- When volatility is high markets can be unstable
- VIX and S&P 500 are strongly negatively correlated

Some mathematical research trends in finance

- Modeling fluctuations of prices (started with Bachelier; complexity increases)
- Pricing financial contracts (commodities, futures, insurance, etc)
- Pricing options (Black-Scholes theory)
- Managing investment portfolios
- Pricing bonds (credit instruments)
- Pricing credit default swaps (insurance against default and loss of value of bonds)



What is systemic risk and how to model it

Consider an evolving system with a large number of inter-connected components, each of which can be in a **normal** state or in a **failed** state. We want to study the probability of overall failure of the system, that is, its systemic risk.

There are three effects that we want to model and that contribute to the behavior of systemic risk:

- The intrinsic stability of each component
- The strength of external random perturbations to the system
- The degree of inter-connectedness or cooperation between components

Possible applications

- Engineering systems with a large number of interacted parts. Components can fail but the system fails only when a large number of components fail simultaneously.
- Power distribution systems. Individual components of the system are calibrated to withstand fluctuations in demand by sharing loads. But sharing also increases the probability of an overall failure.
- Banking systems. Banks cooperate and by spreading the risk of credit shocks between them can operate with less restrictive individual risk policies (capital reserves). However, this increases the risk that they may all fail, that is, the systemic risk.

A basic, bistable mean-field model

- The risk variable $x_j(t)$ of each component $j = 1, \ldots, N$, satisfies the SDEs

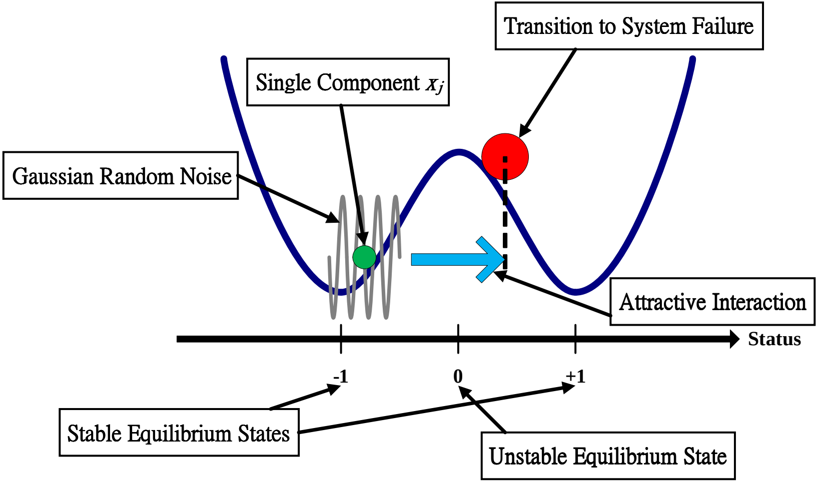
$$dx_j(t) = -h\frac{\partial}{\partial y}V\left(x_j(t)\right)dt + \theta\left(\bar{x}(t) - x_j(t)\right)dt + \sigma dw_j(t)$$

- Here $V(y)$ is a potential with two stable states. Without noise, the individual risk $x_j(t)$ stays in these states, one of which denotes the normal state and the other the failed state.
- A typical but not unique choice of $V(y)$ is $V(y) = -\frac{1}{4}y^4 + \frac{1}{2}y^2$. The parameter h controls the probility with which x_j jumps from one state to the other.
- $\{w_j\}_{j=1}^N$ are independent Brownian motions and σ is their strength.
- $\bar{x} = \frac{1}{N}\sum_{i=1}^N x_i$ is the mean-field, which we take (define) as the systemic risk, and $\theta\left(\bar{x} - x_j\right)$, with $\theta > 0$, is the cooperative interaction parameter.

Why this model?

- The three parameters, h, σ and θ control the three effects we want to study: (i) Intrinsic stability, (ii) random perturbations, and (iii) the degree of cooperation, respectively.
- Why mean field interaction? Because it is perhaps the simplest interaction that models cooperative behavior. And it can be generalized to include **diversity**, as explained later, as well as other more complex interactions such as **hierarchical** ones.
- Connection with QMU (Quantification of Margins of Uncertainty): In a cooperating or inter-connected system, individual components can be operating closer to their margin of failure, as they can benefit from the stability of the rest of the system. This, however, reduces the **overall** margin of uncertainty, that is, increases the systemic risk. **This is one of the main results of the analysis**.

Schematic for the risk of one component



The large system limit ($N \to \infty$)

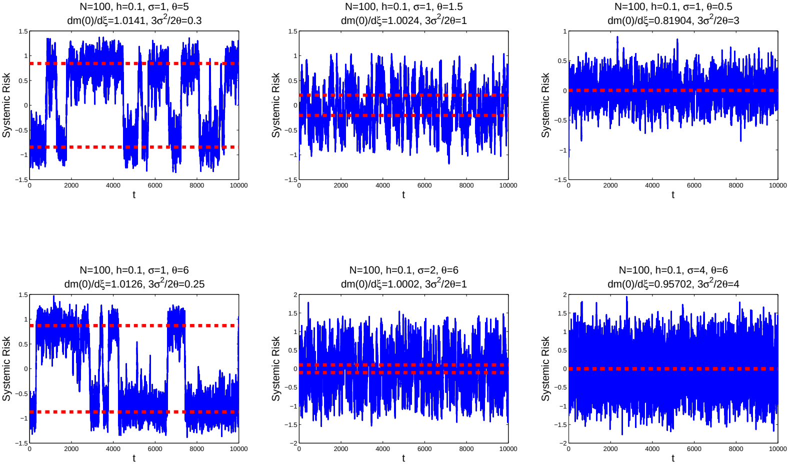
- The empirical risk density $X_N(t) := \frac{1}{N}\sum_{j=1}^{N} \delta_{x_j(t)}(\cdot)$ converges weakly, in probability, as $N \to \infty$ to $u(t, \cdot)$, the solution of the nonlinear Fokker-Planck equation :

$$\frac{\partial}{\partial t} u = h \frac{\partial}{\partial y}\left[U(y)\, u\right] + \frac{1}{2}\sigma^2 \frac{\partial^2}{\partial y^2} u - \theta \frac{\partial}{\partial y}\left\{\left[\int y u(t, dy) - y\right] u\right\}$$

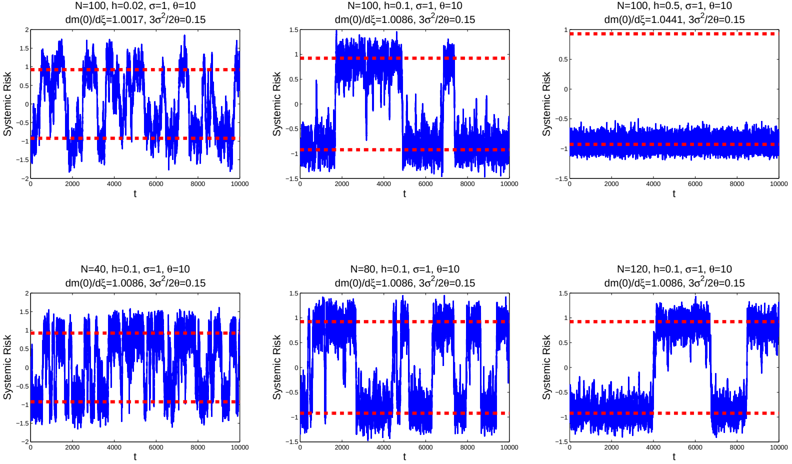
$$U(y) = \frac{d}{dy} V(y).$$

- Existence of bi-stable equilibrium states in the limit: Given θ and h, there exists a critical value σ_c such that u has one stable equilibrium for $\sigma \geqslant \sigma_c$, and has two stable equilibria for $\sigma < \sigma_c$.
- Simplification: If h is small, then u have the bi-stable states if and only if $3\sigma^2 < 2\theta$.
- Explanation: The condition $2\theta \geqslant 3\sigma^2$ means that the system interaction dominates the noise, and therefore component cooperation dominates. In contrast, with strong noise forces, all x_j's act more as independent components and roughly one half are in one state and the rest are in the other state.

Simulation 1 - Impact of increasing θ (stabilizing) and σ (destabilizing)



Simulation 2 - Impact of increasing h(stabilizing) and N



Transition to failure and its probability

- For $\sigma < \sigma_c$ (or $3\sigma^2 < 2\theta$ for small h), the value of the systemic risk remains around $\bar{x} \approx \pm\xi_b$.
- Because of the randomness, the transition in $(0, T)$ (or the system collapse in the risk sense):

$$\bar{x}(0) \approx -\xi_b, \qquad \bar{x}(T) \approx \xi_b$$

  happens with nonzero probability.
- Question: What is the probability of this happening?

Large deviation principle

- The asymptotic probabilities, for large N, can be computed through a large deviation principle.
- [Dawson & Gartner, 1987] $M_1(\mathbb{R})$ is the space of probability measures on $\mathbb{R}$, and A is a set of $M_1(\mathbb{R})$-valued continuous process on $[0, T]$. Then

$$\mathbf{P}(X_N \in A) \approx \exp\left(-N \inf_{\phi \in A} I_h(\phi)\right)$$

where

$$I_h(\phi) = \frac{1}{2\sigma^2} \int_0^T \sup_{f : \left\langle \phi, \left(\frac{\partial}{\partial y} f\right)^2 \right\rangle \neq 0} \frac{\left\langle \frac{\partial}{\partial t}\phi - \mathcal{L}_\phi^* \phi - h\mathcal{M}^* \phi, f \right\rangle^2}{\left\langle \phi, \left(\frac{\partial}{\partial y} f\right)^2 \right\rangle} dt$$

$$\mathcal{L}_\psi^* \phi = \frac{1}{2}\sigma^2 \frac{\partial^2}{\partial y^2}\phi - \theta \frac{\partial}{\partial y}\left\{\left[\int y\psi(t, dy) - y\right]\phi\right\}$$

$$\mathcal{M}^* \phi = \frac{\partial}{\partial y}\left[\left(y^3 - y\right)\phi\right].$$

- To compute the transition probability, A is the set of all continuous transition paths:

$$A = \left\{\phi : [0, T] \to M_1(\mathbb{R}), \mathbb{E}_{\phi(0)}\mathbf{X} = -\xi_b, \mathbb{E}_{\phi(T)}\mathbf{X} = \xi_b\right\}.$$

Small h (intrinsic stability) analysis

- Why consider small h?
  - The problem is nonlinear and infinite-dimensional, and is generally intractable.
  - If h is small then the problem can be reduced to a finite-dimensional problem. For $V(y) = -\frac{1}{4}y^4 + \frac{1}{2}y^2$, it is a four-dimension problem.
  - Numerical simulations show that the probability of transitions is almost zero even for moderate h.
- When $h = 0$, the systemic risk is effectively a Brownian motion:

$$\bar{x}(t) = \frac{\sigma}{\sqrt{N}}\bar{w}(t).$$

We expect, therefore, that for small h, a transition path of empirical densities is Gaussian with a small perturbation:

$$A = \left\{\phi = p + hq : p(t,y) = \frac{1}{\sqrt{2\pi b^2(t)}} \exp\left[\frac{(y-a(t))^2}{2b^2(t)}\right], a(0) = -\xi_b, a(T) = \xi_b\right\}.$$

Small h analysis (with J. Garnier and T-W Yang)

- For h small, the large deviation problem is solvable approximately by a "Chapman-Enskog" expansion, and the transition probability is

$$\mathbf{P}\left(X_N \in A\right) \approx \exp\left(-\frac{N}{\sigma^2 T}\left[2\left(1-3\frac{\sigma^2}{2\theta}\right)+\frac{24h}{\sigma^2}\left(\frac{\sigma^2}{2\theta}\right)^2\left(1-2\frac{\sigma^2}{2\theta}\right)+O(h^2)\right]\right).$$

- Here are some comments of this result:
  - A large system is more stable than a small system.
  - In the long run (T large), a transition will happen.
  - Increase of the intrinsic stabilization parameter h reduces systemic risk.
  - Mean transition times are simply related to transition probabilities in this approximation (use for this M. Williams '82)

Modeling of diversity in cooperative behavior

- The cooperative behavior of components can be different across groups:

$$dx_j(t) = -h\kappa \frac{\partial}{\partial y} V(x_j(t))\,dt + \sigma dw_j(t) + \theta_j(\bar{x}(t) - x_j(t))\,dt.$$

- The components are partitioned into K groups. In group k, the components have cooperative parameter Θ_k.
- In the limit $N \to \infty$ the empirical densities of each groups converge to the solution of the joint Fokker-Planck equations:

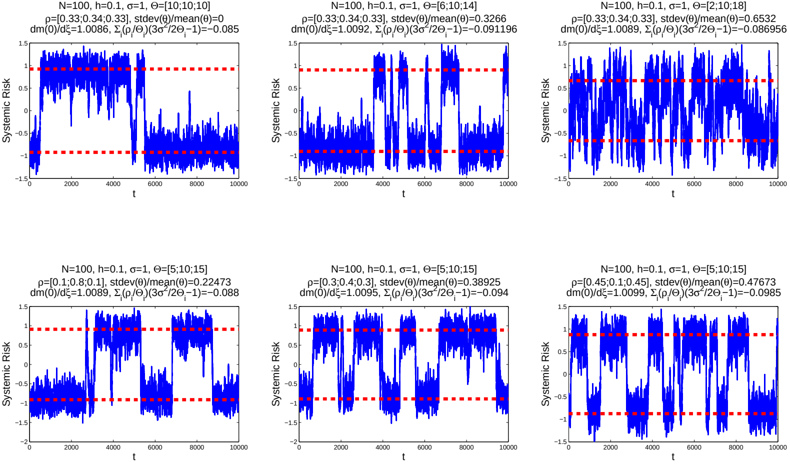
$$\begin{aligned}
\frac{\partial}{\partial t} u_1 &= h\kappa \frac{\partial}{\partial y}[U(y)u_1] + \frac{1}{2}\sigma^2 \frac{\partial^2}{\partial y^2} u_1 - \Theta_1 \frac{\partial}{\partial y}\left\{\left[\int y \sum_{k=1}^{K} \rho_k u_k(t,dy) - y\right] u_1\right\} \\
&\vdots \\
\frac{\partial}{\partial t} u_K &= h\kappa \frac{\partial}{\partial y}[U(y)u_K] + \frac{1}{2}\sigma^2 \frac{\partial^2}{\partial y^2} u_K - \Theta_K \frac{\partial}{\partial y}\left\{\left[\int y \sum_{k=1}^{K} \rho_k u_k(t,dy) - y\right] u_K\right\}
\end{aligned}$$

where $\rho_k N$ is the size of group k.

Impact of component diversity on the systemic risk

- Why is the diversity interesting?
  - The model is more realistic and more widely applicable.
  - Diversity significantly affects the system stability by reducing it.
- Impact from the diversity:
  - Analytical and numerical studies show that even with the same parameters and with $\{\theta_j\}$ whose average equals θ the system still changes significantly.

Simulation 3 - Impact of diversity, change of Θ_k and ρ_k



Analysis in the diversity case

- The critical value of the system fluctuation is lower:
  - The critical value σ_c^{homo} of the homogeneous case is $\sqrt{\frac{2}{3}\theta}$, and σ_c^{div} is

$$\left[\left(\sum_{k=1}^{K}\frac{\rho_k}{\Theta_k}\right)\Big/\left(\sum_{k=1}^{K}\frac{3\rho_k}{2\Theta_k^2}\right)\right]^{1/2}.$$

  - If $\theta = \sum_{k=1}^{K}\rho_k\Theta_k$, then $\sigma_c^{\text{homo}} \geqslant \sigma_c^{\text{div}}$ always.
- System with diversity have larger transition probabilities:
  - We show that when h is zero, the average of θ_j is θ, and the diversity of θ_j is small, then the system has a higher transition probability.
  - Mathematically, if $h = 0$, $\Theta_k = \theta(1+\delta\alpha_k)$ with $\delta \ll 1$, and $\sum_{k=1}^{K}\rho_k\alpha_k = 0$, then

$$\mathbf{P}\left(X_N^{\text{div}} \in A\right) \approx \exp\left\{-\frac{N}{\sigma^2 T}\left[2\left(1-\frac{3\sigma^2}{2\theta}\right) - 2\delta^2\left(\sum_k \rho_k\alpha_k^2\right)\left(\frac{3\sigma^2}{2\theta} + \frac{1}{T}\int_0^T\left(1-e^{-\theta s}\right)^2 ds\right)\right]\right\}.$$

A Hierarchical Model of Systemic Risk

Here we consider a hierarchical model with a central agent:

$$dx_0 = \sigma_0 dw_0 - h_0 U_0(x_0)dt - \theta_0\Big(x_0 - \frac{1}{N}\sum_{j=1}^{n} x_j\Big)dt$$

$$dx_j = \sigma dw_j - hU(x_j)dt - \theta\big(x_j - x_0\big)dt, \qquad j = 1, \ldots, N$$

- X_0 models the central stable agent. It is intrinsically stable ($h_0 > 0$), and not subjected to external fluctuations ($\sigma_0 = 0$). It can be destabilized through a mean field interaction with the other agents.
- X_j, $j = 1, \ldots, N$ model individual agents that are subjected to external fluctuations. They are ($h > 0$) or are not ($h = 0$) intrinsically stable. They are stabilized with an interaction with the stable agent X_0.

On-going and future work, analysis

- Compute the transition probability of the diversity case for small h.
  - Show that it is unconditionally true that systems with diversity are less stable than homogeneous ones.
- Study models with diversities everywhere:
  - The most general case (when the limit exists) is that V, σ and θ in each group can be different, i.e. if x_j is in the group k, then the SDE is

$$dx_j(t) = -h\frac{\partial}{\partial y}V_k(x_j(t))\,dt + \sigma_k dw_j(t) + \Theta_k(\bar{x}(t) - x_j(t))\,dt.$$

  - Combine hierarchical and diversity models. How general should our models be to deal with realistic problems?

Research directions for systemic risk

Interconnected financial systems have many sources of instability. Instabilities are "everywhere" in the financial world. They are a consequence of the dual market variables: volatility and liquidity.

- Using the analysis as a guide, design importance sampling algorithms for computing efficiently (very) small systemic failure probabilities
- Can dynamic control mechanisms reduce systemic risk? What if the controllers must rely on imperfect information?
- Can transaction fees (Tobin tax) stabilize markets in the systemic risk sense? (It is not known if they increase or decrease volatility).
- Is statistical arbitrage destabilizing? (Hedging derivatives increases volatility but portfolio optimization decreases it).