Mathematical Challenges of Using Point Spread Function Analysis Algorithms in Astronomical Imaging

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Mathematical Challenges of Astronomical Imaging January 26-30, 2004

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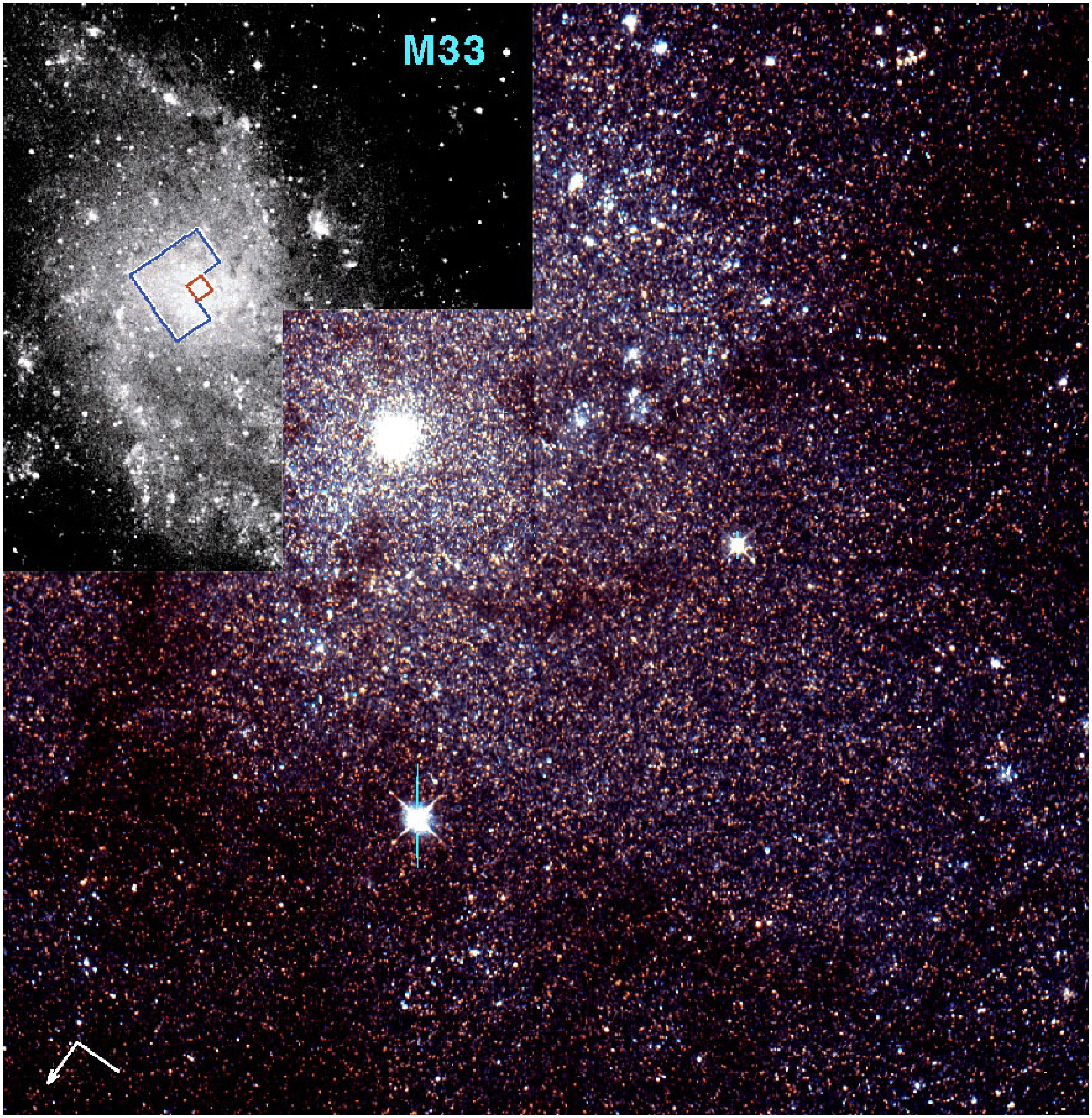
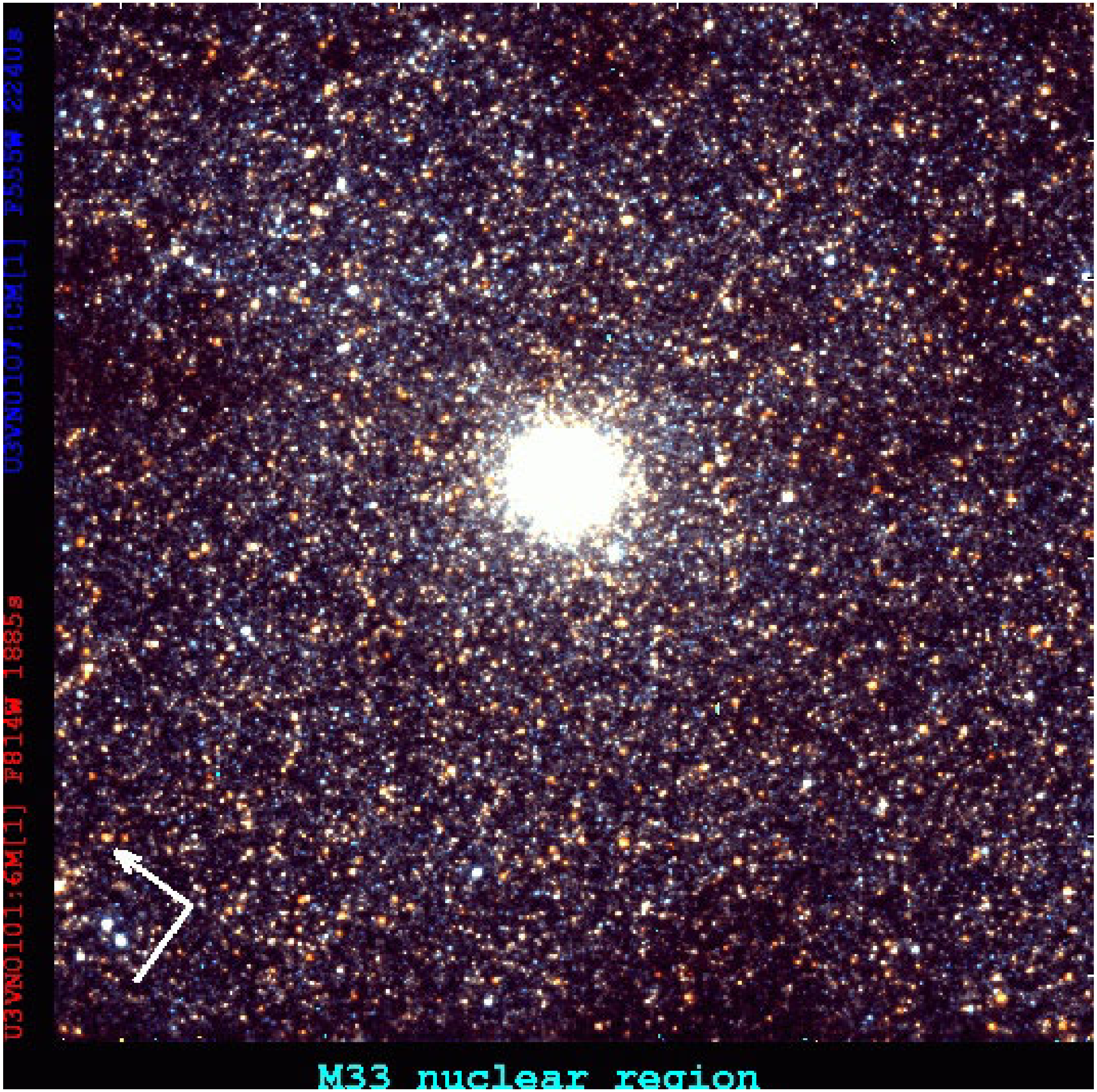
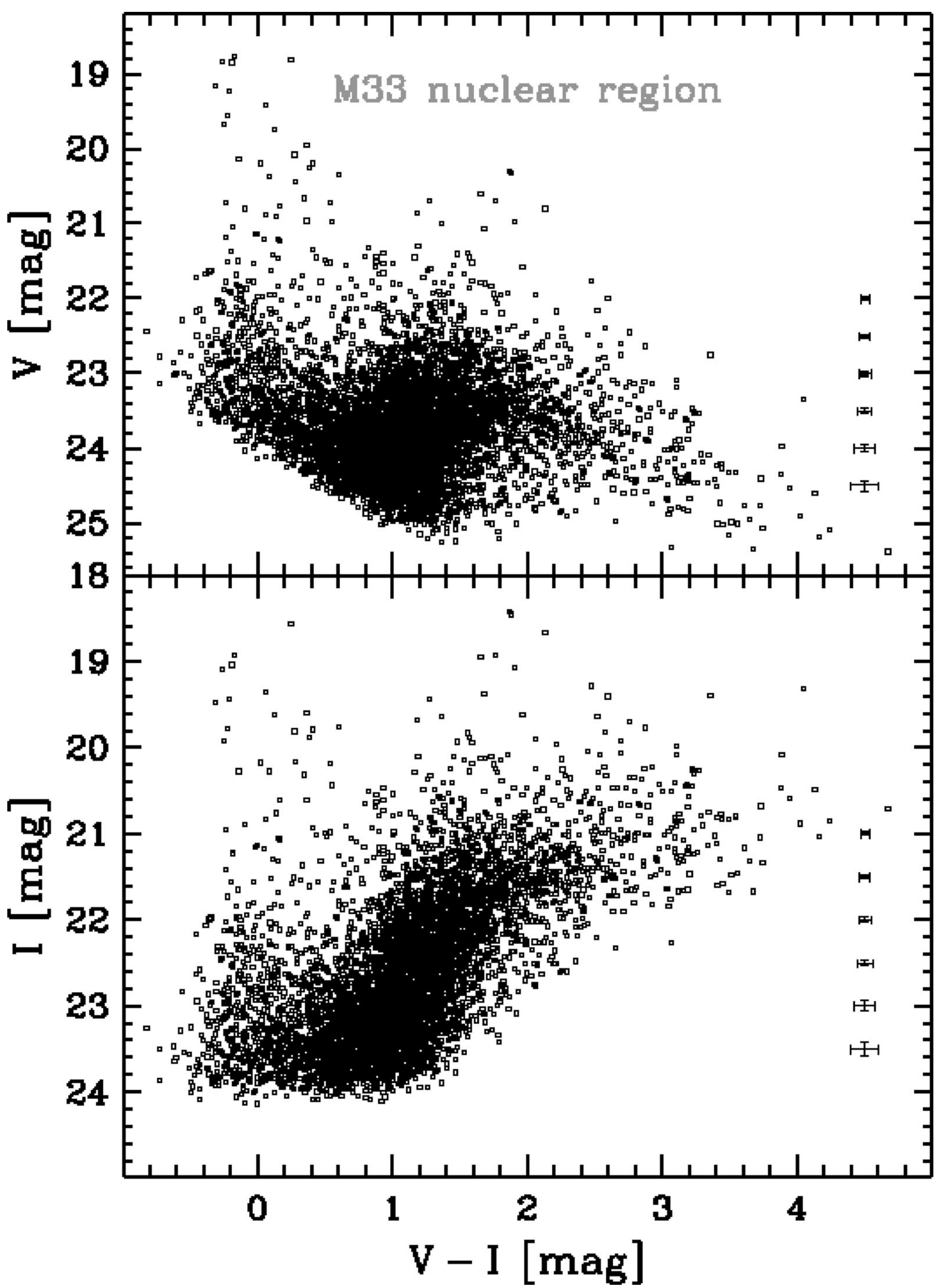
Outline

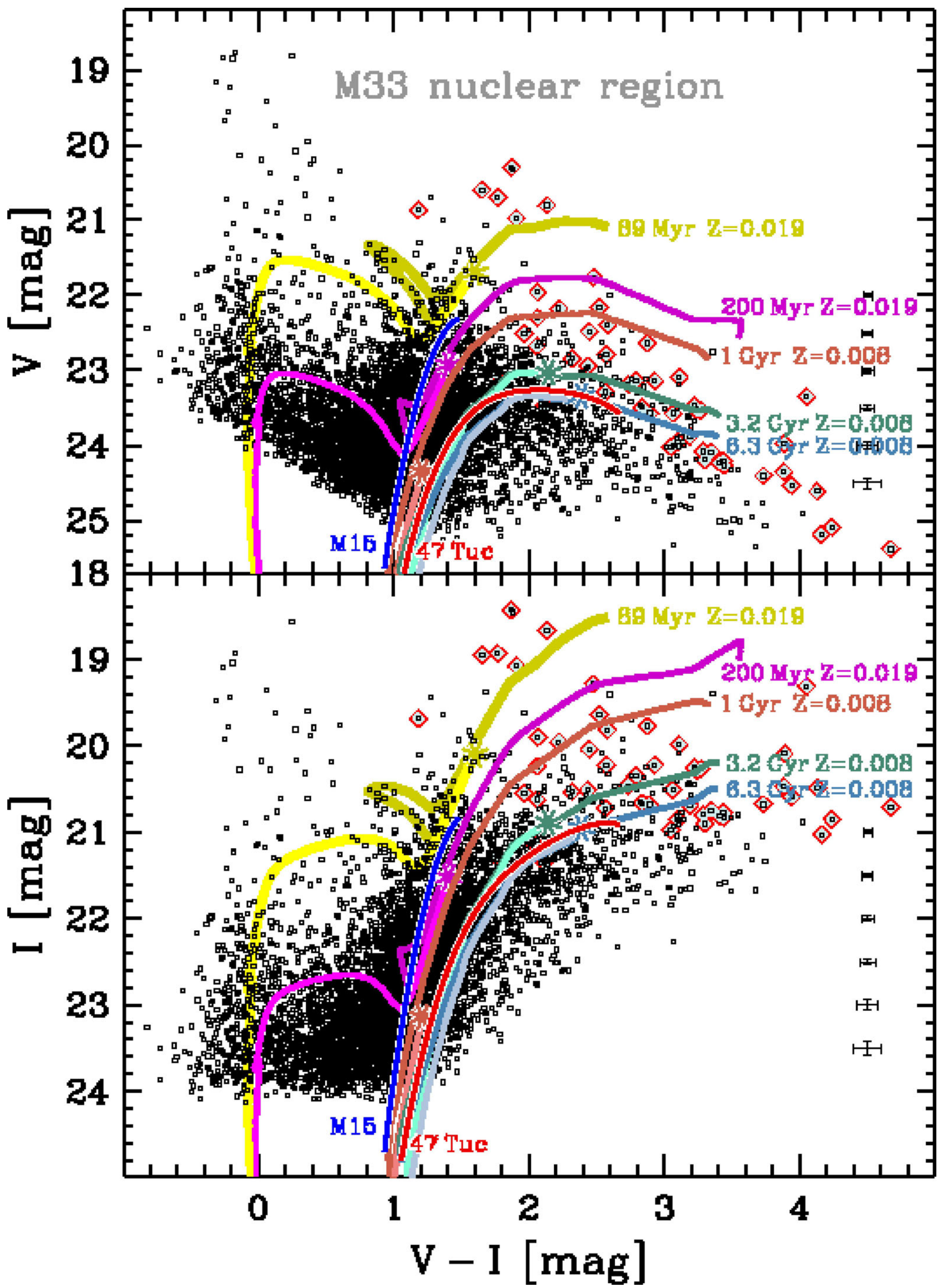
- Motivation
- Image Formation Process & Challenges
- Performance Model: Photometry
- Performance Model: Astrometry
- Analytical PSFs
- Numerical PSFs
- Ugly (real) PSFs
- Future Challenges


M33


U3VN0107:CM[1] F555W 2240s
U3VN0101:GM[1] F814W 1885s
M33 nuclear region



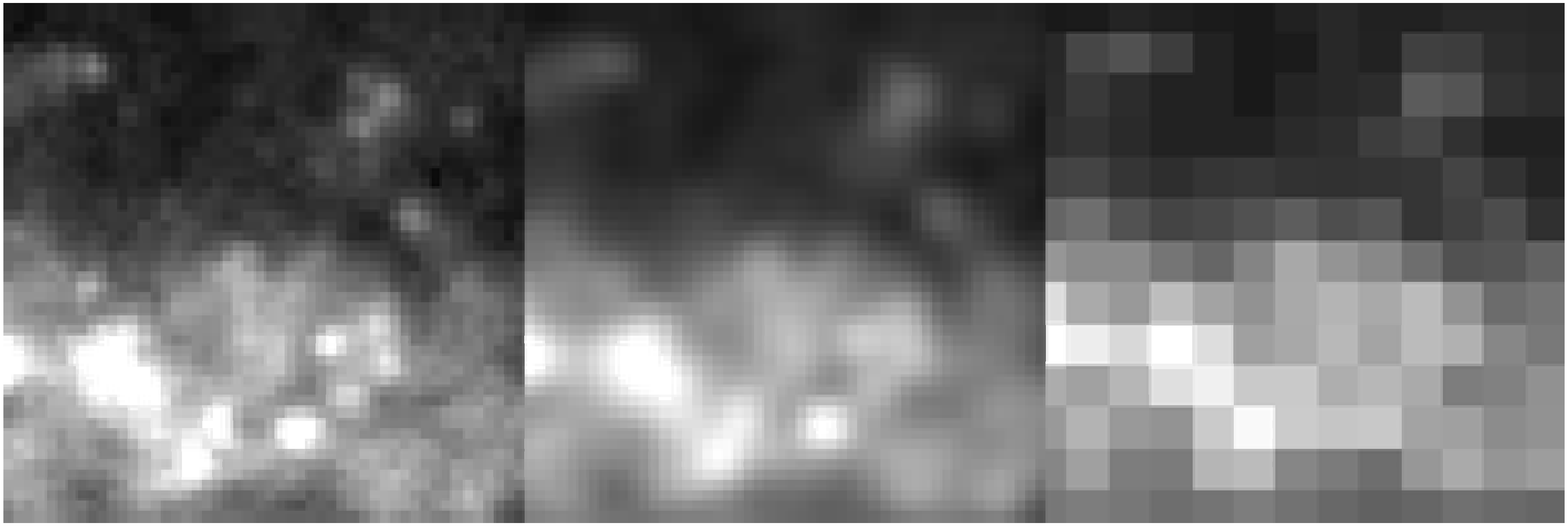
M33 nuclear region
V [mag]
I [mag]
V − I [mag]
19
20
21
22
23
24
25
18
19
20
21
22
23
24
0
1
2
3
4


M33 nuclear region
V [mag]
I [mag]
V − I [mag]
89 Myr Z=0.019
200 Myr Z=0.019
1 Gyr Z=0.008
3.2 Gyr Z=0.008
6.3 Gyr Z=0.008
M15
47 Tuc

Skill Sets for Precision Stellar Photometry and Astrometry

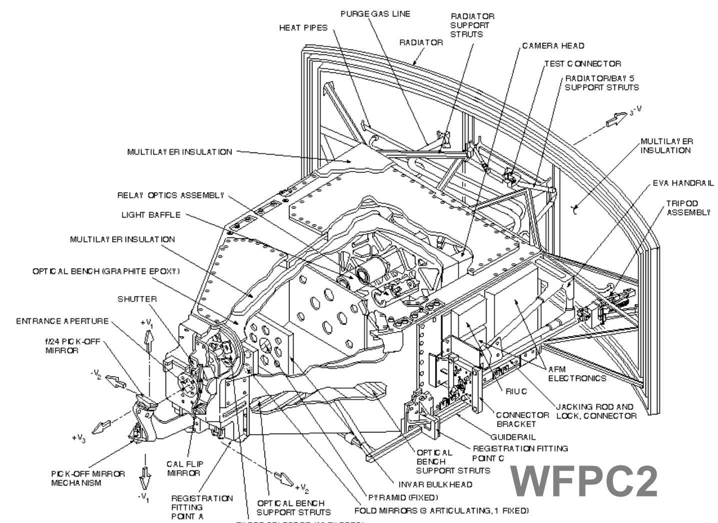
- Image Analysis **(how bright is the star? where is it located?)**
- Error Analysis **(what is the error of those measurements?)**
- Numerical Analysis **(non-linear least-squares minimization)**
- Computational Methods **(precise & robust computations)**
- Signal Processing
  - Low Signal vs. High Noise **(finding the needle in the haystack)**
- Detector Technology
  - Solid-State Physics **(conversion process of photons to electrons)**
  - Fabrication Techniques **(how detectors made and read)**
- Optics **(blurring due to telescope & camera optics)**
- Atmospheric Physics
  - Turbulence Theory **(blurring due to the atmosphere)**
- Stellar Physics **(photon statistics & pulsation theory)**

Image Formation Process



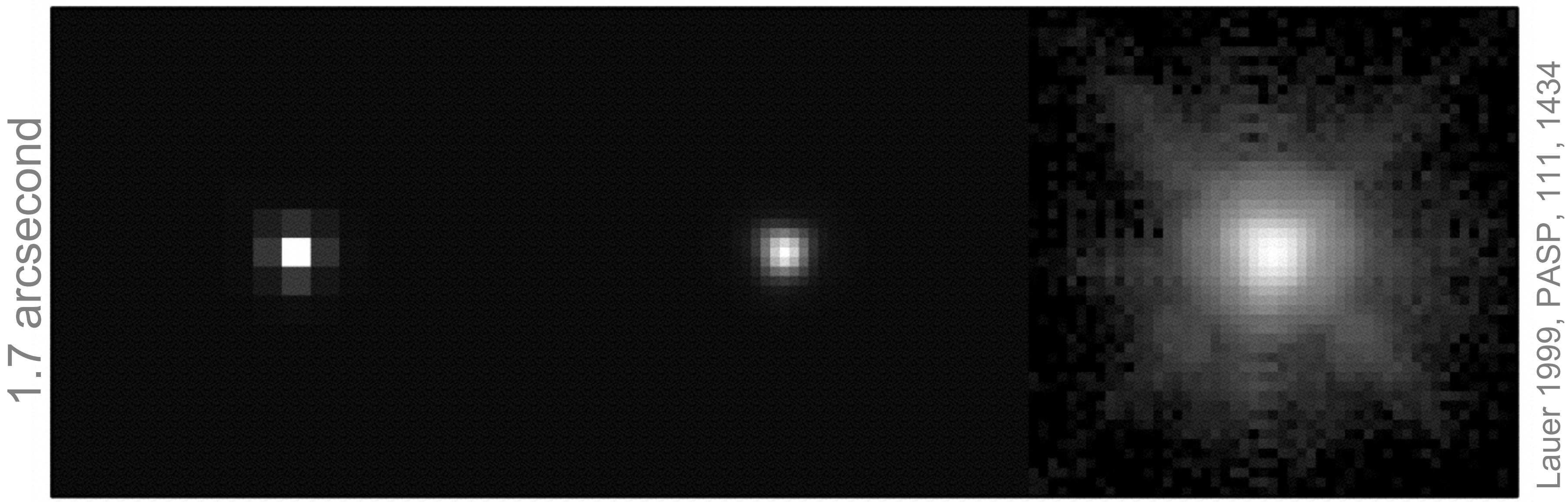
Fruchter & Hook 2000 (ADASS IX)

SKY $\Rightarrow$ TELESCOPE $\Rightarrow$ DETECTOR



Under-Sampled Point Spread Functions

HST WFPC2: WFC F555W 0.10 arcsecond pixels

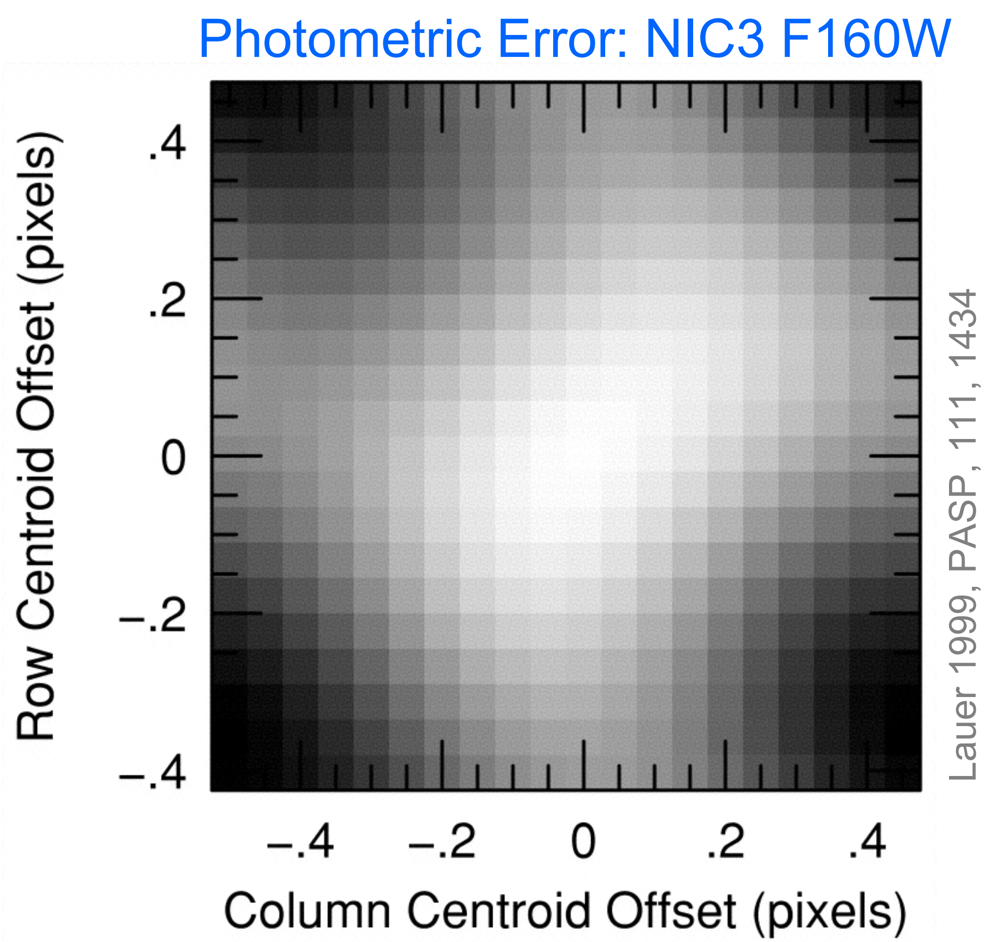


original sampling
linear stretch

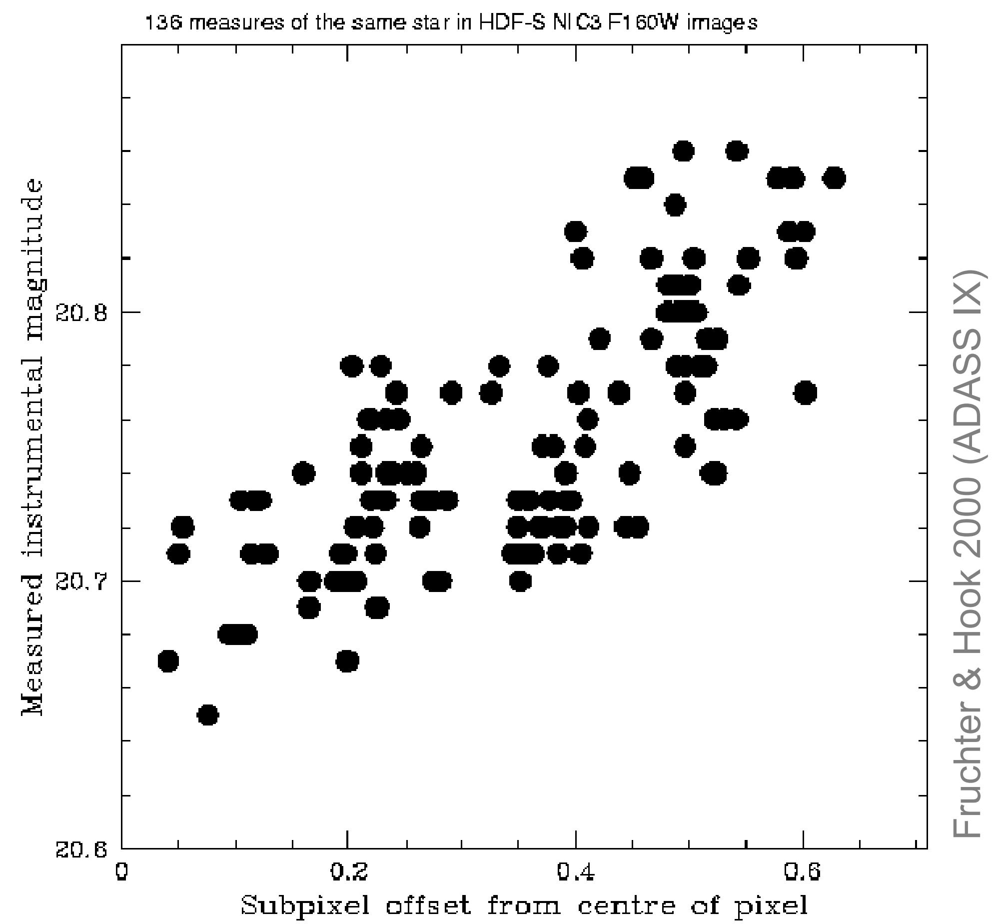
3x3 supersampled
linear stretch

3x3 supersampled
logarithmic stretch

Non-Uniform Pixel Response Function



white=0.12 mag excess, black=0.09 mag deficit



Point Spread Function ▼ ▼ Photon Distribution Function

$$\Psi \equiv \phi * \mathsf{DRF}$$

▲ **D**etector **R**esponse **F**unction

$$\Psi_i(x_i, y_i) \equiv \int_{x_i-0.5}^{x_i+0.5} \int_{y_i-0.5}^{y_i+0.5} \phi(x,y)\,dx\,dy \qquad (\text{for an ideal } \mathsf{DRF})$$

$$\mathrm{V} \equiv \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Psi\,dx\,dy \lesssim 1$$

Volume ▲

$$\mathsf{sharpness} \equiv \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{\Psi}^2\,dx\,dy \equiv \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(\frac{\Psi}{\mathrm{V}}\right)^2 dx\,dy$$

▲ Normalized PSF

sharpness of a Gaussian PSF:

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[\frac{1}{2\pi\mathcal{S}^2} \exp\left(-\frac{x^2+y^2}{2\mathcal{S}^2}\right)\right]^2 dx\,dy = \frac{1}{4\pi\mathcal{S}^2}$$

standard deviation ▲

$$\chi^2(\mathbf{p}) \equiv \sum_{i=1}^{N} \frac{1}{\sigma_i^2}\left(z_i - m_i \right)^2$$

parameter vector ▲ ▲data ▲model

▲measurement error

$$\chi^2(\mathbf{p}+\boldsymbol{\delta}) = \sum_{n=0}^{\infty} \frac{1}{n!}(\boldsymbol{\delta}\cdot\nabla)^n \chi^2(\mathbf{p})$$

correction vector ▲

$$\approx \chi^2(\mathbf{p}) + \boldsymbol{\delta}\cdot\nabla\chi^2(\mathbf{p}) + \frac{1}{2}\boldsymbol{\delta}\cdot\boldsymbol{H}\cdot\boldsymbol{\delta}$$

▲Hessian matrix

$$[\mathbf{H}]_{\mathbf{jk}} \equiv \frac{\partial^{\mathbf{2}}}{\partial \mathbf{a_j} \partial \mathbf{a_k}} \chi^2(\mathbf{p})$$

$$\boldsymbol{H}\cdot\boldsymbol{\delta} = -\nabla\chi^2(\mathbf{p})$$

$$\mathbf{p}' = \mathbf{p}+\boldsymbol{\delta}$$

$$\sigma_{\mathbf{j}} \approx \sqrt{[\mathbf{H}^{-\mathbf{1}}]_{jj}} = \left[\sum_{i=1}^{N} \frac{1}{\sigma_i^2} \left(\frac{\partial m_i}{\partial p_j} \right)^2 \right]^{-1/2}$$

measurement error ▲

Hessian matrix

Standard definition for model $\boldsymbol{m} = \boldsymbol{\mathcal{E}}\,\boldsymbol{\Psi}(\boldsymbol{\mathcal{X}}, \boldsymbol{\mathcal{Y}}) + \boldsymbol{\mathcal{B}}$ and N data values:

$$
\boldsymbol{H} \equiv \begin{bmatrix}
\sum_{i=1}^{N} \frac{1}{\sigma_i^2} \frac{\partial^2 m_i}{\partial\mathcal{E}\partial\mathcal{E}} & \sum_{i=1}^{N} \frac{1}{\sigma_i^2} \frac{\partial^2 m_i}{\partial\mathcal{E}\partial\mathcal{X}} & \sum_{i=1}^{N} \frac{1}{\sigma_i^2} \frac{\partial^2 m_i}{\partial\mathcal{E}\partial\mathcal{Y}} & \sum_{i=1}^{N} \frac{1}{\sigma_i^2} \frac{\partial^2 m_i}{\partial\mathcal{E}\partial\mathcal{B}} \\
\sum_{i=1}^{N} \frac{1}{\sigma_i^2} \frac{\partial^2 m_i}{\partial\mathcal{X}\partial\mathcal{E}} & \sum_{i=1}^{N} \frac{1}{\sigma_i^2} \frac{\partial^2 m_i}{\partial\mathcal{X}\partial\mathcal{X}} & \sum_{i=1}^{N} \frac{1}{\sigma_i^2} \frac{\partial^2 m_i}{\partial\mathcal{X}\partial\mathcal{Y}} & \sum_{i=1}^{N} \frac{1}{\sigma_i^2} \frac{\partial^2 m_i}{\partial\mathcal{X}\partial\mathcal{B}} \\
\sum_{i=1}^{N} \frac{1}{\sigma_i^2} \frac{\partial^2 m_i}{\partial\mathcal{Y}\partial\mathcal{E}} & \sum_{i=1}^{N} \frac{1}{\sigma_i^2} \frac{\partial^2 m_i}{\partial\mathcal{Y}\partial\mathcal{X}} & \sum_{i=1}^{N} \frac{1}{\sigma_i^2} \frac{\partial^2 m_i}{\partial\mathcal{Y}\partial\mathcal{Y}} & \sum_{i=1}^{N} \frac{1}{\sigma_i^2} \frac{\partial^2 m_i}{\partial\mathcal{Y}\partial\mathcal{B}} \\
\sum_{i=1}^{N} \frac{1}{\sigma_i^2} \frac{\partial^2 m_i}{\partial\mathcal{B}\partial\mathcal{E}} & \sum_{i=1}^{N} \frac{1}{\sigma_i^2} \frac{\partial^2 m_i}{\partial\mathcal{B}\partial\mathcal{X}} & \sum_{i=1}^{N} \frac{1}{\sigma_i^2} \frac{\partial^2 m_i}{\partial\mathcal{B}\partial\mathcal{Y}} & \sum_{i=1}^{N} \frac{1}{\sigma_i^2} \frac{\partial^2 m_i}{\partial\mathcal{B}\partial\mathcal{B}}
\end{bmatrix}
$$

Hessian matrix

Robust approximation for model $\boldsymbol{m} = \boldsymbol{\mathcal{E}}\,\boldsymbol{\Psi}(\boldsymbol{\mathcal{X}},\boldsymbol{\mathcal{Y}}) + \boldsymbol{\mathcal{B}}$ and N data values:

$$\boldsymbol{H} \approx \begin{bmatrix}
\sum_{i=1}^{N} \frac{1}{\sigma_i^2} \frac{\partial m_i}{\partial \mathcal{E}} \frac{\partial m_i}{\partial \mathcal{E}} & \sum_{i=1}^{N} \frac{1}{\sigma_i^2} \frac{\partial m_i}{\partial \mathcal{E}} \frac{\partial m_i}{\partial \mathcal{X}} & \sum_{i=1}^{N} \frac{1}{\sigma_i^2} \frac{\partial m_i}{\partial \mathcal{E}} \frac{\partial m_i}{\partial \mathcal{Y}} & \sum_{i=1}^{N} \frac{1}{\sigma_i^2} \frac{\partial m_i}{\partial \mathcal{E}} \frac{\partial m_i}{\partial \mathcal{B}} \\
\sum_{i=1}^{N} \frac{1}{\sigma_i^2} \frac{\partial m_i}{\partial \mathcal{X}} \frac{\partial m_i}{\partial \mathcal{E}} & \sum_{i=1}^{N} \frac{1}{\sigma_i^2} \frac{\partial m_i}{\partial \mathcal{X}} \frac{\partial m_i}{\partial \mathcal{X}} & \sum_{i=1}^{N} \frac{1}{\sigma_i^2} \frac{\partial m_i}{\partial \mathcal{X}} \frac{\partial m_i}{\partial \mathcal{Y}} & \sum_{i=1}^{N} \frac{1}{\sigma_i^2} \frac{\partial m_i}{\partial \mathcal{X}} \frac{\partial m_i}{\partial \mathcal{B}} \\
\sum_{i=1}^{N} \frac{1}{\sigma_i^2} \frac{\partial m_i}{\partial \mathcal{Y}} \frac{\partial m_i}{\partial \mathcal{E}} & \sum_{i=1}^{N} \frac{1}{\sigma_i^2} \frac{\partial m_i}{\partial \mathcal{Y}} \frac{\partial m_i}{\partial \mathcal{X}} & \sum_{i=1}^{N} \frac{1}{\sigma_i^2} \frac{\partial m_i}{\partial \mathcal{Y}} \frac{\partial m_i}{\partial \mathcal{Y}} & \sum_{i=1}^{N} \frac{1}{\sigma_i^2} \frac{\partial m_i}{\partial \mathcal{Y}} \frac{\partial m_i}{\partial \mathcal{B}} \\
\sum_{i=1}^{N} \frac{1}{\sigma_i^2} \frac{\partial m_i}{\partial \mathcal{B}} \frac{\partial m_i}{\partial \mathcal{E}} & \sum_{i=1}^{N} \frac{1}{\sigma_i^2} \frac{\partial m_i}{\partial \mathcal{B}} \frac{\partial m_i}{\partial \mathcal{X}} & \sum_{i=1}^{N} \frac{1}{\sigma_i^2} \frac{\partial m_i}{\partial \mathcal{B}} \frac{\partial m_i}{\partial \mathcal{Y}} & \sum_{i=1}^{N} \frac{1}{\sigma_i^2} \frac{\partial m_i}{\partial \mathcal{B}} \frac{\partial m_i}{\partial \mathcal{B}}
\end{bmatrix}$$

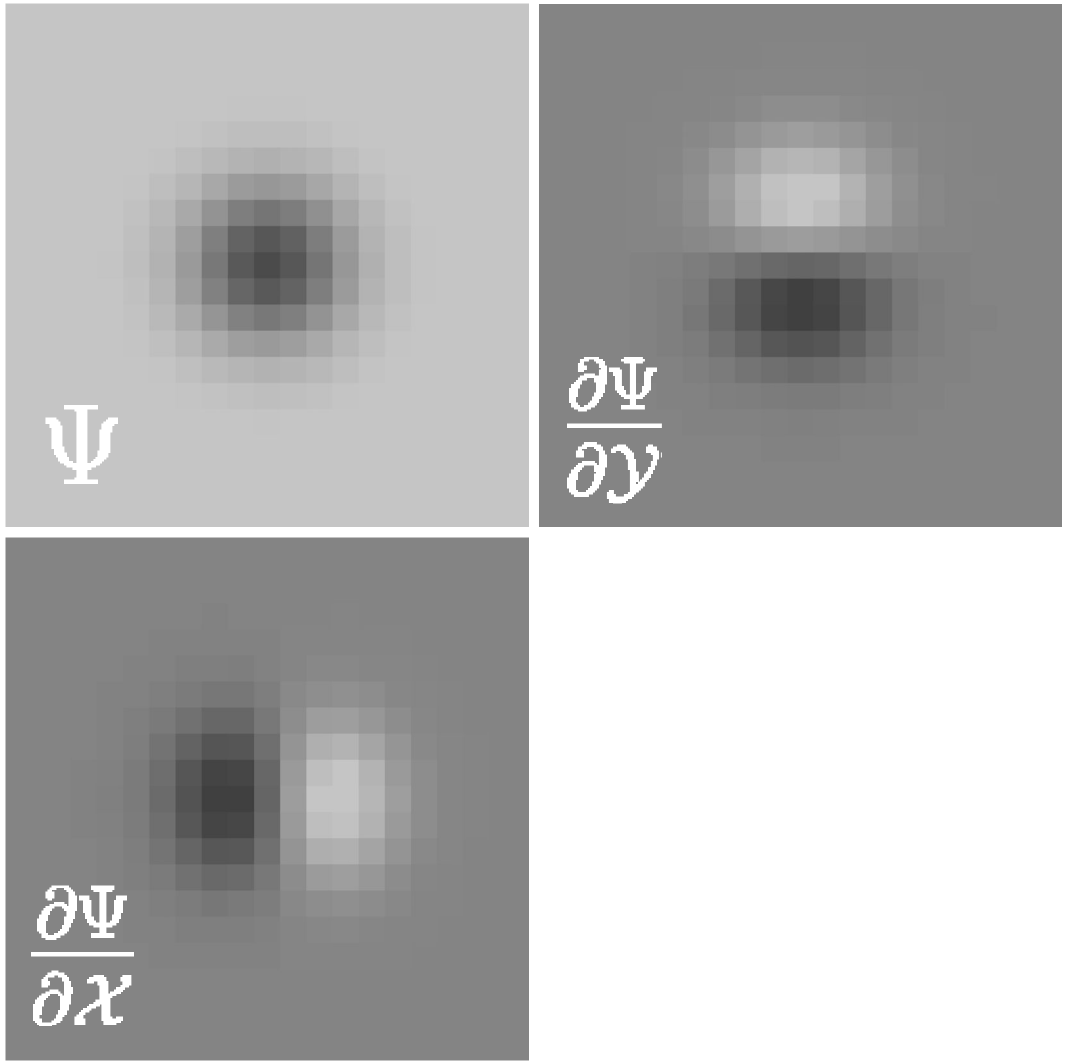
Jacobian matrix

For model $m_i \equiv \mathcal{E}\,\Psi_i(\mathcal{X},\mathcal{Y}) + \mathcal{B}$ with N data values,

$$
\boldsymbol{J} \equiv \begin{bmatrix}
\frac{\partial m_1}{\partial \mathcal{E}} & \frac{\partial m_1}{\partial \mathcal{X}} & \frac{\partial m_1}{\partial \mathcal{Y}} & \frac{\partial m_1}{\partial \mathcal{B}} \\
\frac{\partial m_2}{\partial \mathcal{E}} & \frac{\partial m_2}{\partial \mathcal{X}} & \frac{\partial m_2}{\partial \mathcal{Y}} & \frac{\partial m_2}{\partial \mathcal{B}} \\
\frac{\partial m_3}{\partial \mathcal{E}} & \frac{\partial m_3}{\partial \mathcal{X}} & \frac{\partial m_3}{\partial \mathcal{Y}} & \frac{\partial m_3}{\partial \mathcal{B}} \\
\vdots & \vdots & \vdots & \vdots \\
\frac{\partial m_N}{\partial \mathcal{E}} & \frac{\partial m_N}{\partial \mathcal{X}} & \frac{\partial m_N}{\partial \mathcal{Y}} & \frac{\partial m_N}{\partial \mathcal{B}}
\end{bmatrix}
$$

Each column forms an image

Derivatives of the PSF

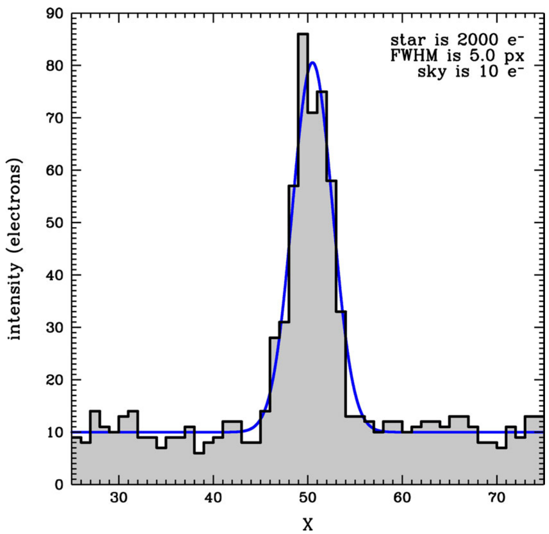


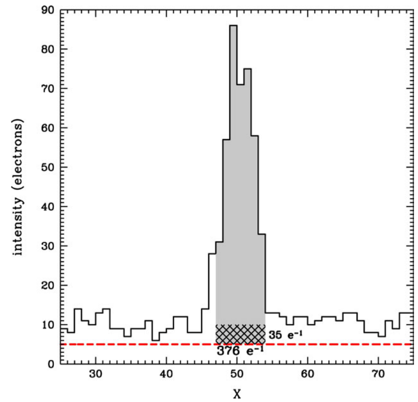
$$
\begin{aligned}
\sigma^2_{\mathcal{E}:\,\mathrm{bright}} &\approx \left[\sum_{i=1}^{N} \frac{1}{\sigma_i^2} \left(\frac{\partial\, m_i}{\partial \mathcal{E}} \right)^2 \right]^{-1} \\
&\approx \left[\sum_{i=1}^{N} \frac{1}{\mathcal{E}\Psi_i} \left(\frac{\partial}{\partial \mathcal{E}} \left[\, \mathcal{E}\Psi_i + \mathcal{B} \,\right] \right)^2 \right]^{-1} \\
&= \left[\frac{1}{\mathcal{E}} \sum_{i=1}^{N} \frac{1}{\Psi_i} \left(\Psi_i\right)^2 \right]^{-1} \\
&= \left[\frac{1}{\mathcal{E}} \sum_{i=1}^{N} \Psi_i \right]^{-1} \\
&\approx \mathcal{E} \left[\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \Psi \, dx \, dy \right]^{-1} \\
&\equiv \frac{\mathcal{E}}{\mathrm{V}}
\end{aligned}
$$

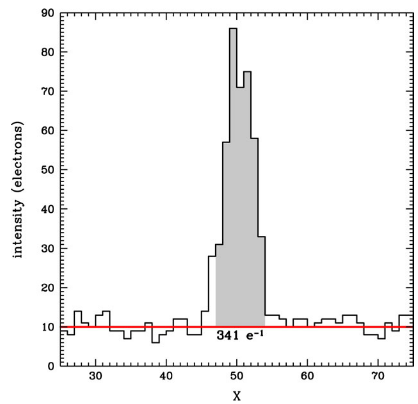
$$\sigma_{\rm rms} \equiv \sqrt{\frac{1}{N}\sum_{i=1}^{N}\sigma_i^2} \approx \sqrt{\mathcal{B}+\sigma_{\rm RON}^2}$$

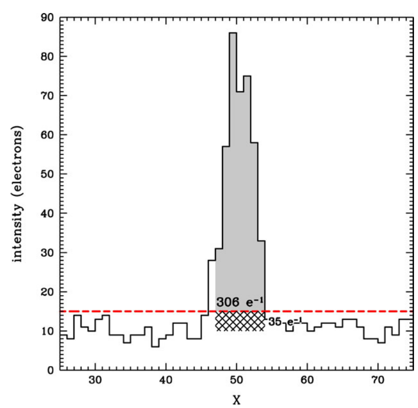
$$\begin{aligned}
\sigma^2_{\mathcal{E}:\,\rm faint} &\approx \left[\sum_{i=1}^{N}\frac{1}{\sigma_i^2}\left(\frac{\partial m_i}{\partial\mathcal{E}}\right)^2\right]^{-1} \\
&\approx \left[\sum_{i=1}^{N}\frac{1}{\sigma_{\rm rms}^2}\left(\frac{\partial}{\partial\mathcal{E}}\left[\,\mathcal{E}\Psi_i+\mathcal{B}\,\right]\right)^2\right]^{-1} \\
&\equiv \sigma_{\rm rms}^2\left[\sum_{i=1}^{N}\left(\Psi_i\right)^2\right]^{-1} \\
&\approx \sigma_{\rm rms}^2\left[\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\Psi^2\,dx\,dy\right]^{-1} \\
&\equiv \sigma_{\rm rms}^2\,\beta
\end{aligned}$$

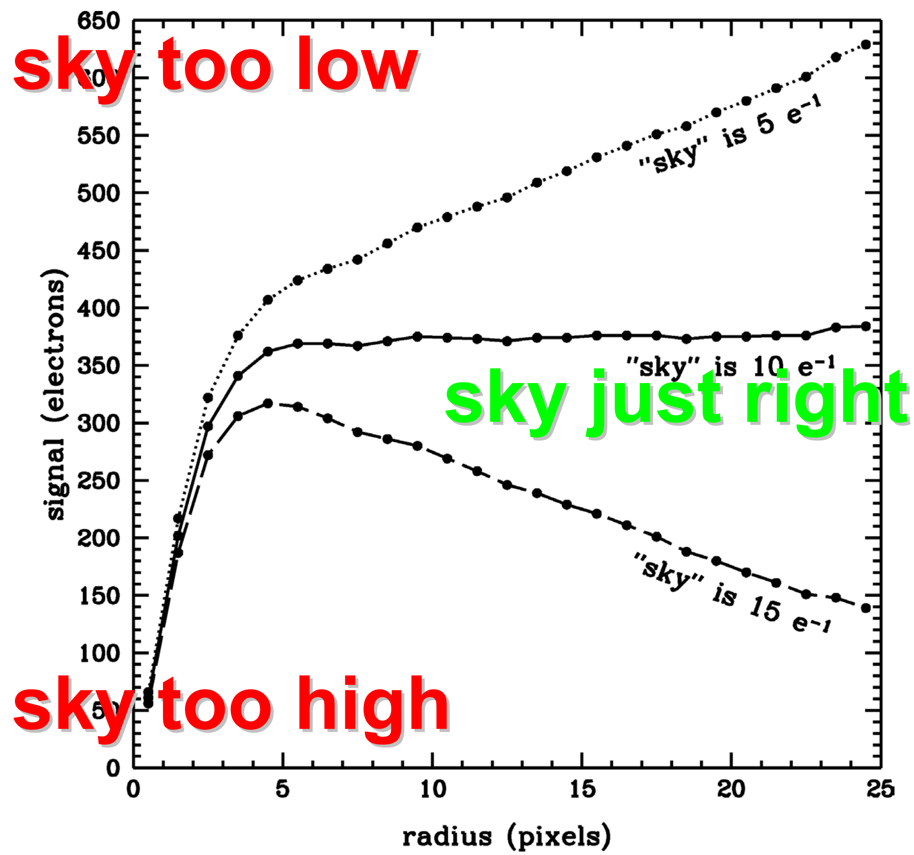
Systematic Error: Poor Sky Measurement











$$\sigma_{\mathcal{E}:\,\mathrm{faint}} \approx \sigma_{\mathrm{rms}}\sqrt{\beta} + \underline{\boldsymbol{\sigma_{\mathcal{B}}}\,\beta}$$

$$\begin{aligned}
\boldsymbol{\sigma_{\mathcal{B}}} &\approx \left[\sum_{i=1}^{N}\frac{1}{\sigma_i^2}\left(\frac{\partial m_i}{\partial\mathcal{B}}\right)^2\right]^{-1/2} \\
&\approx \left[\sum_{i=1}^{N}\frac{1}{\sigma_{\mathrm{rms}}^2}\left(\frac{\partial}{\partial\mathcal{B}}\left[\,\mathcal{E}\Psi_i + \mathcal{B}\,\right]\right)^2\right]^{-1/2} \\
&= \sigma_{\mathrm{rms}}\left[\sum_{i=1}^{N}(1)^2\right]^{-1/2} \\
&= \frac{\sigma_{\mathrm{rms}}}{\sqrt{N}} \\
&\Rightarrow \sqrt{\frac{\mathcal{B}}{N}}\,, \ \text{as} \ \ \sigma_{\mathrm{RON}}^2 \Rightarrow 0
\end{aligned}$$

$$\mathsf{SNR} \lesssim \frac{\mathcal{E}}{\sigma_{\mathcal{E}}}$$

$$\approx \frac{\mathcal{E}}{\sqrt{\sigma^2_{\mathcal{E}:\,\mathrm{bright}} + \sigma^2_{\mathcal{E}:\,\mathrm{faint}}}}$$

$$\Delta m = \frac{5\,/\,\ln(100)}{\mathsf{SNR}}$$

$$= \frac{2.5\log(e)}{\mathsf{SNR}}$$

$$\approx \frac{1.0857}{\mathsf{SNR}}$$

$$\mathsf{g}(x,y;\mathcal{X},\mathcal{Y},\mathcal{S}) \equiv \frac{1}{2\pi\mathcal{S}^2} e^{-\left[\frac{(x-\mathcal{X})^2+(y-\mathcal{Y})^2}{2\mathcal{S}^2}\right]}$$

$$\mathsf{g}_i \equiv \mathsf{g}(x_i,y_i;\mathcal{X},\mathcal{Y},\mathcal{S})$$

$$\mathsf{G}_i \equiv \int_{x_i-0.5}^{x_i+0.5}\int_{y_i-0.5}^{y_i+0.5} \mathsf{g}(x,y;\mathcal{X},\mathcal{Y},\mathcal{S})\,dx\,dy$$

$$\mathsf{G}_i \approx \mathsf{g}_i \qquad (\text{for } \mathcal{S} \gg 1\ )$$

$$m_i \equiv \mathcal{E}\mathrm{V}\mathsf{G}_i + \mathcal{B}$$

$$
\begin{aligned}
\sigma^2_{\mathcal{X}:\,\mathrm{bright}} &\approx \left[\sum_{i=1}^{N} \frac{1}{\sigma_i^2}\left(\frac{\partial m_i}{\partial \mathcal{X}}\right)^2\right]^{-1} \\
&\approx \left[\sum_{i=1}^{N} \frac{1}{\mathcal{E}\mathrm{V}\mathsf{G}_i}\left(\frac{\partial}{\partial \mathcal{X}}\left[\,\mathcal{E}\mathrm{V}\mathsf{G}_i + \mathcal{B}\,\right]\right)^2\right]^{-1} \\
&\approx \left[\sum_{i=1}^{N} \frac{1}{\mathcal{E}\mathrm{V}\mathrm{g}_i}\left(\mathcal{E}V\frac{\partial}{\partial \mathcal{X}}\mathrm{g}_i\right)^2\right]^{-1} \\
&= \left[\sum_{i=1}^{N} \frac{1}{\mathcal{E}\mathrm{V}\mathrm{g}_i}\left(\mathcal{E}V\mathrm{g}_i\frac{x_i - \mathcal{X}}{\mathcal{S}^2}\right)^2\right]^{-1} \\
&\approx \frac{\mathcal{S}^4}{\mathcal{E}\mathrm{V}}\left[\iint_{-\infty}^{\infty} \mathrm{g}(x,y;\mathcal{X},\mathcal{Y},\mathcal{S})\,(x-\mathcal{X})^2\,dx\,dy\right]^{-1} \\
&= \frac{\mathcal{S}^2}{\mathcal{E}\mathrm{V}} \\
&= \frac{1}{\mathcal{E}\mathrm{V}}\left(\frac{\beta}{4\pi}\right)
\end{aligned}
$$

▲ ≡ 1 **if critically sampled**

$$
\begin{aligned}
\sigma^2_{\mathcal{X}:\,\mathrm{faint}} &\approx \left[\sum_{i=1}^{N} \frac{1}{\sigma_i^2}\left(\frac{\partial m_i}{\partial \mathcal{X}}\right)^2\right]^{-1} \\
&\approx \left[\sum_{i=1}^{N} \frac{1}{\sigma_{\mathrm{rms}}^2}\left(\frac{\partial}{\partial \mathcal{X}}\left[\mathcal{E}\mathrm{V}\mathsf{G}_i + \mathcal{B}\right]\right)^2\right]^{-1} \\
&\approx \sigma_{\mathrm{rms}}^2 \left[\sum_{i=1}^{N} \left(\mathcal{E}V\frac{\partial}{\partial \mathcal{X}}\mathsf{g}_i\right)^2\right]^{-1} \\
&= \sigma_{\mathrm{rms}}^2 \left[\sum_{i=1}^{N} \left(\mathcal{E}V\mathsf{g}_i\frac{x_i - \mathcal{X}}{\mathcal{S}^2}\right)^2\right]^{-1} \\
&\approx \sigma_{\mathrm{rms}}^2 \frac{S^4}{\mathcal{E}^2\mathrm{V}^2}\left[\iint_{-\infty}^{\infty} \left[\mathsf{g}(x,y;\mathcal{X},\mathcal{Y},\mathcal{S})(x-\mathcal{X})\right]^2 dx\,dy\right]^{-1} \\
&= \sigma_{\mathrm{rms}}^2 \frac{8\pi S^4}{\mathcal{E}^2\mathrm{V}^2} \\
&= \sigma_{\mathrm{rms}}^2 \frac{1}{2\pi\mathcal{E}^2\mathrm{V}^2}\beta^2 \;=\; \sigma_{\mathrm{rms}}^2 8\pi \left(\sigma^2_{\mathcal{X}:\,\mathrm{bright}}\right)^2
\end{aligned}
$$

$$\sigma_{\mathcal{X}} \approx \sqrt{\sigma^2_{\mathcal{X}:\,\mathrm{bright}} + \sigma^2_{\mathcal{X}:\,\mathrm{faint}}}$$

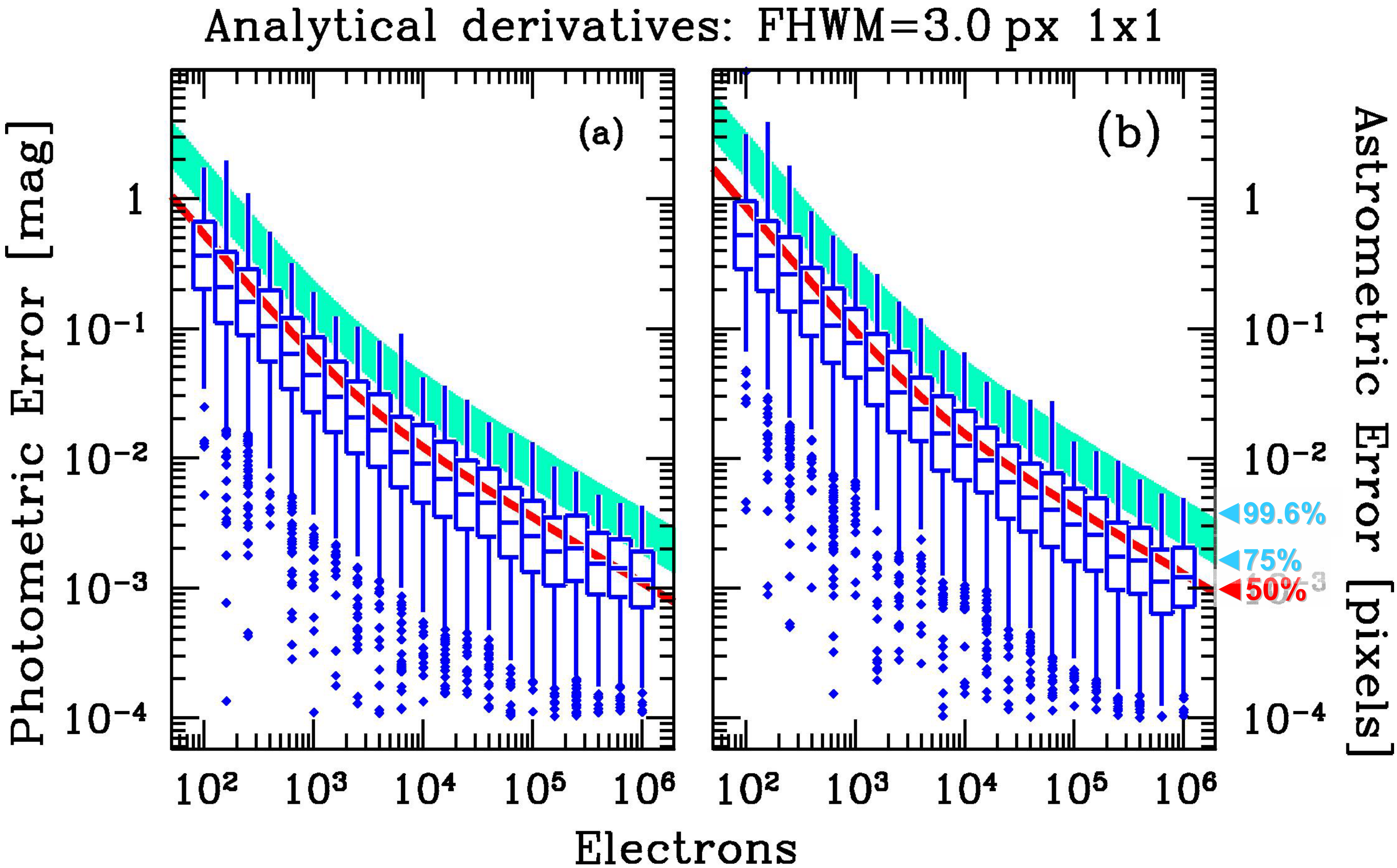
Bright stars: $$\boxed{\frac{\sigma_{\mathcal{X}}}{\sqrt{\beta/\left(4\pi\right)}} = \frac{\sigma_{\mathcal{E}}}{\mathcal{E}}}$$

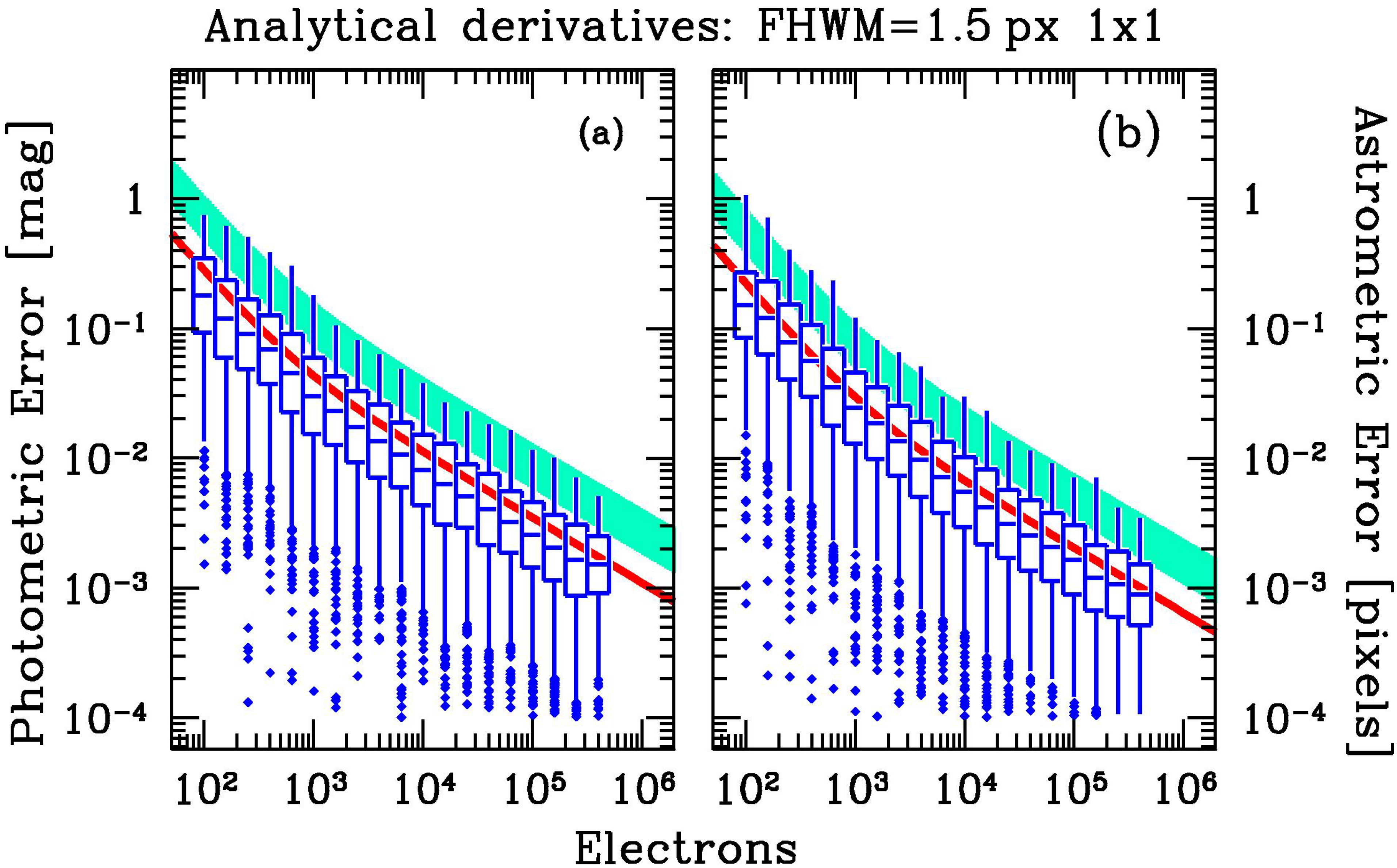
Analytical Derivatives

$$
\begin{aligned}
\Psi_i(x_i, y_i; \mathcal{E}, \mathcal{X}, \mathcal{Y}, \mathcal{S}) &\equiv \int_{x_i-0.5}^{x_i+0.5} \int_{y_i-0.5}^{y_i+0.5} \phi(x, y; \mathcal{E}, \mathcal{X}, \mathcal{Y}, \mathcal{S})\, dx\, dy \qquad \text{(ideal DRF)} \\
&\approx \frac{1}{\eta^2} \sum_{j=1}^{\eta} \sum_{k=1}^{\eta} \phi(x_i - \Delta + j\delta,\ y_i - \Delta + k\delta; \mathcal{E}, \mathcal{X}, \mathcal{Y}, \mathcal{S}) \\
&\quad \text{where} \quad \Delta \equiv \frac{\eta+1}{2\eta} \quad \text{and} \quad \delta \equiv \frac{1}{\eta}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial}{\partial \mathcal{X}} \Psi_i &\equiv \frac{\partial}{\partial \mathcal{X}} \int_{x_i-0.5}^{x_i+0.5} \int_{y_i-0.5}^{y_i+0.5} \phi(x, y; \mathcal{E}, \mathcal{X}, \mathcal{Y}, \mathcal{S})\, dx\, dy \qquad \text{(ideal DRF)} \\
&\approx \frac{1}{\eta^2} \sum_{j=1}^{\eta} \sum_{k=1}^{\eta} \frac{\partial}{\partial \mathcal{X}} \phi(x_i - \Delta + j\delta,\ y_i - \Delta + k\delta; \mathcal{E}, \mathcal{X}, \mathcal{Y}, \mathcal{S})
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial}{\partial \mathcal{Y}} \Psi_i &\equiv \frac{\partial}{\partial \mathcal{Y}} \int_{x_i-0.5}^{x_i+0.5} \int_{y_i-0.5}^{y_i+0.5} \phi(x, y; \mathcal{E}, \mathcal{X}, \mathcal{Y}, \mathcal{S})\, dx\, dy \qquad \text{(ideal DRF)} \\
&\approx \frac{1}{\eta^2} \sum_{j=1}^{\eta} \sum_{k=1}^{\eta} \frac{\partial}{\partial \mathcal{Y}} \phi(x_i - \Delta + j\delta,\ y_i - \Delta + k\delta; \mathcal{E}, \mathcal{X}, \mathcal{Y}, \mathcal{S})
\end{aligned}
$$


Analytical derivatives: FHWM=3.0 px 1x1
(a)
(b)
Photometric Error [mag]
Astrometric Error [pixels]
1
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^2
10^3
10^4
10^5
10^6
Electrons
99.6%
75%
50%


Analytical derivatives: FHWM=1.5 px 1x1
(a)
(b)
Photometric Error [mag]
Astrometric Error [pixels]
1
10⁻¹
10⁻²
10⁻³
10⁻⁴
10²
10³
10⁴
10⁵
10⁶
Electrons

Why does this work so well?

We analyzed the volume integral of the photon distribution function rather than sample it in the middle of a pixel.

But remember that moderation in all things is good.

Failure is inevitable with extreme undersampling when all of the light from a star can fall within a single pixel; photometry gets diluted and astrometry suffers as information about the location of the star within the pixel is lost.

Numerical Derivatives

The mathematics of determining the position partial derivatives of the observational model with respect to the x and y direction vectors is exactly the same with analytical or numerical PSFs. The implementation methodology, however, is significantly different. The position partial derivatives of numerical PSFs can be determined using numerical differentiation techniques on the numerical PSF. Numerical experiments have shown that the following five-point differentiation formula works well with numerical PSFs:

$$f'(x_i) \approx \frac{1}{12}\left[f(x_{i-2}) - 8f(x_{i-1}) + 8f(x_{i+1}) - f(x_{i+2})\right]$$

How does one move a PSF?

Analytical PSFs: Just compute the PSF at the desired location in the observational model.

Numerical PSFs: Take the reference numerical PSF and shift it to the desired location using a perfect 2-d interpolation function. OK… but how is that done in practice?

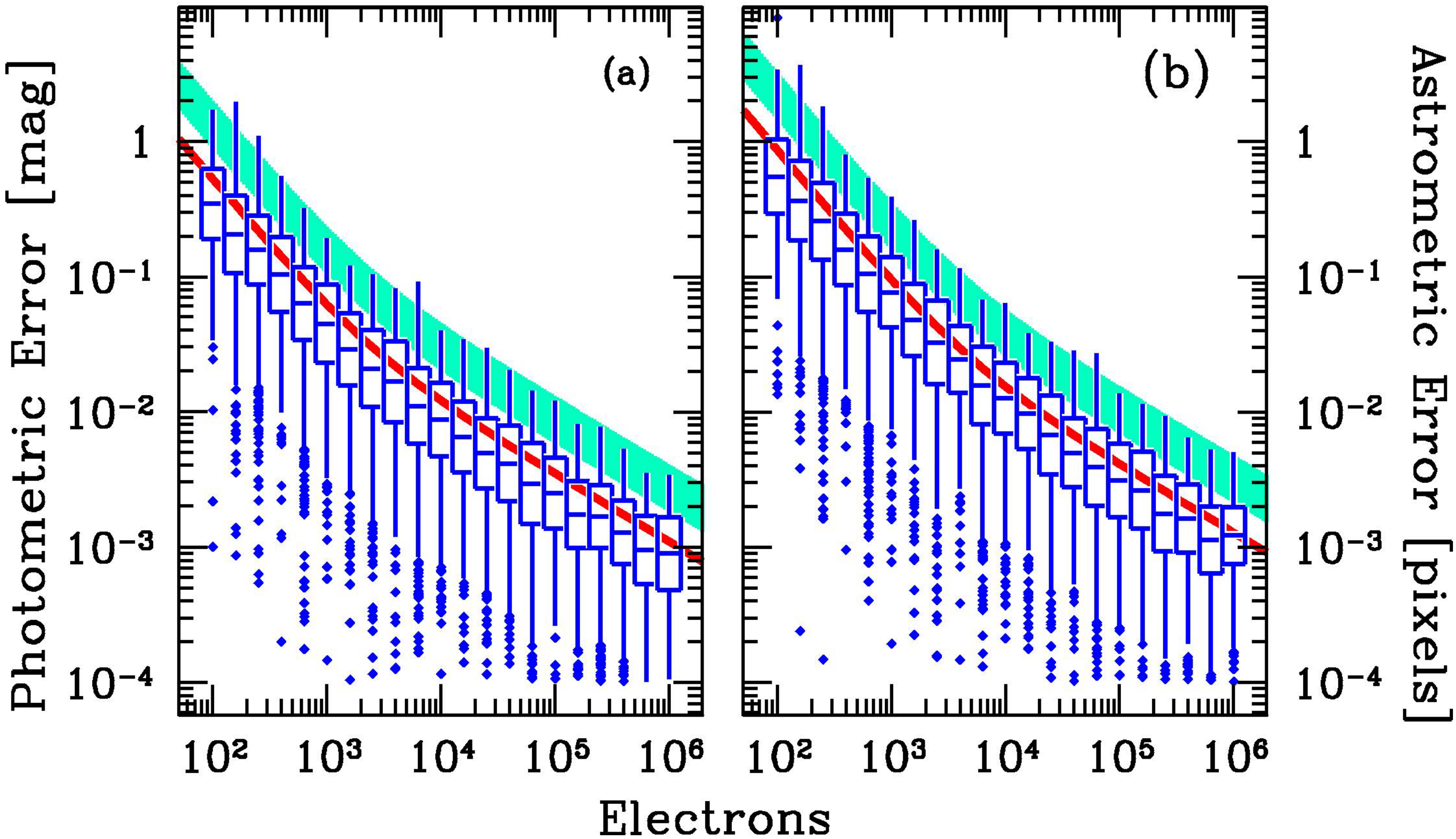
Solution

Use the following 21-pixel-wide damped sinc function:

$$f^{\text{shifted}}(\delta x) \equiv \sum_{i=-10}^{10} f(x_i) \frac{\sin\left(\pi(x_i - \delta x)\right)}{\sin(x_i - \delta x)} \exp\left(-\frac{(x_i - \delta x)^2}{(3.25)^2}\right)$$

Note: The 2-d sinc function is separable in x and y.

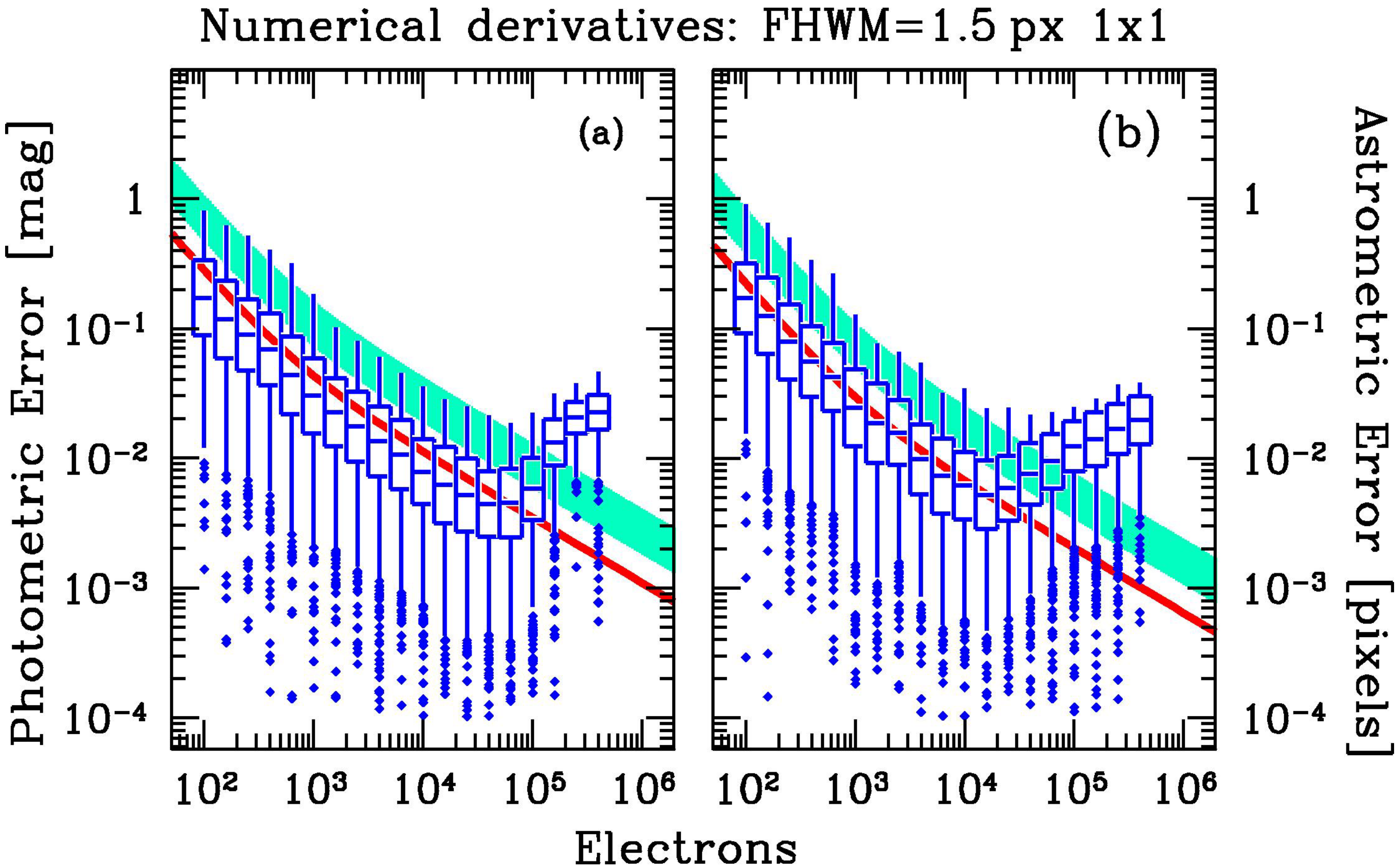
Numerical derivatives: FHWM=3.0 px 1x1

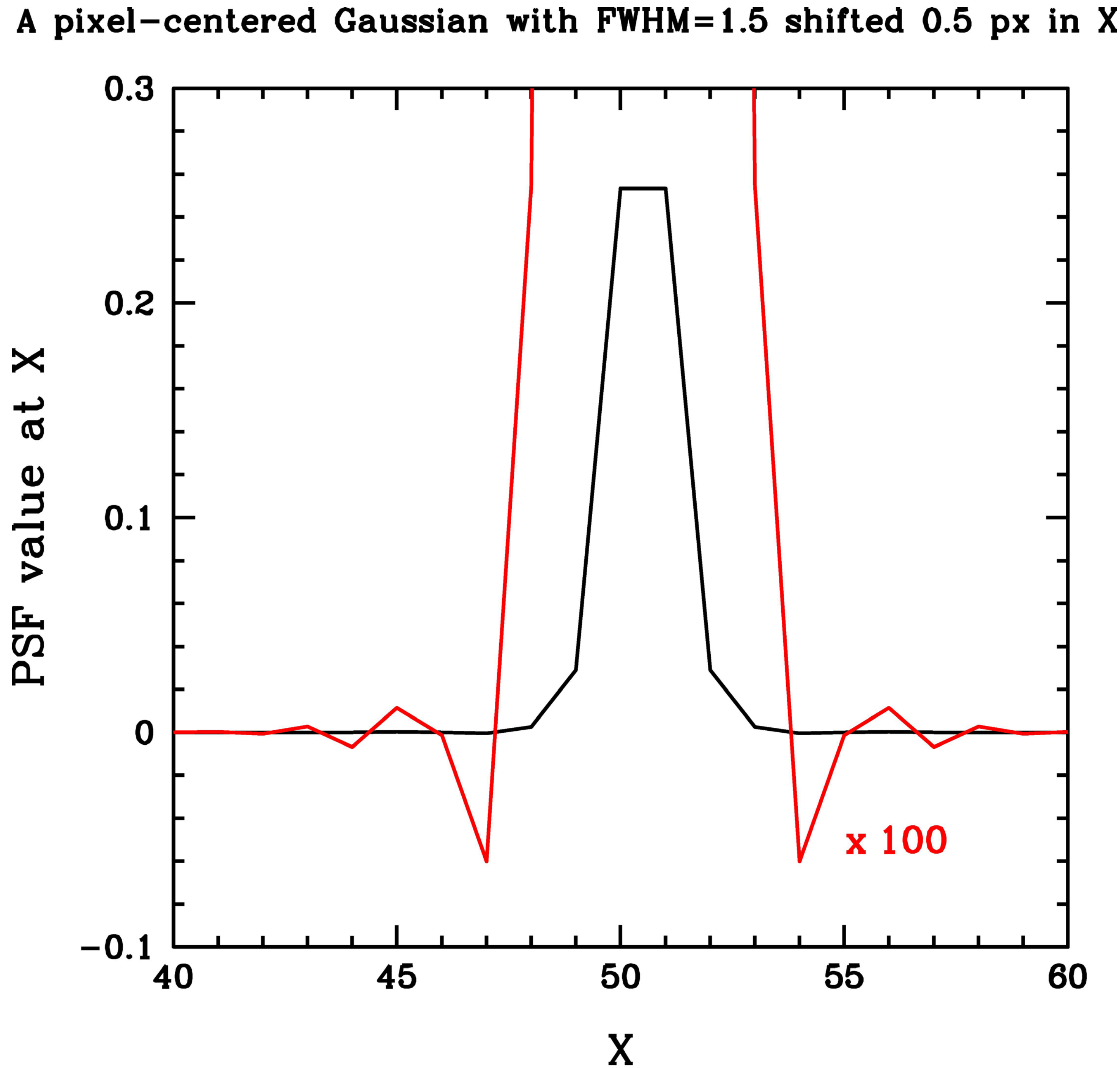


Why does this work so well?

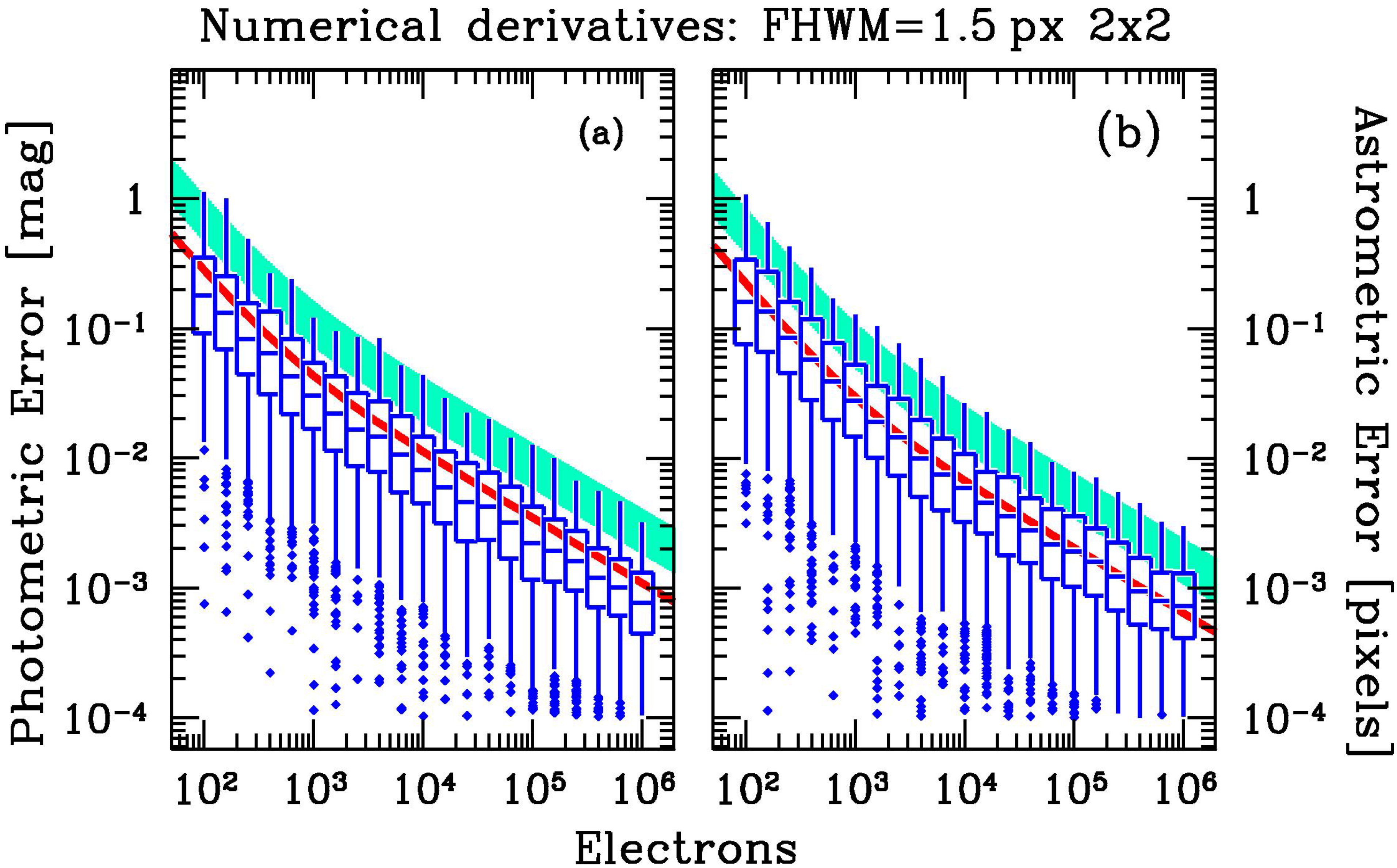
Photon Noise

▲**and That's A Good Thing®**


Numerical derivatives: FHWM=1.5 px 1x1
(a)
(b)
Photometric Error [mag]
Astrometric Error [pixels]
1
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^2 10^3 10^4 10^5 10^6
Electrons



Problem: Negative values caused by shifting an undersampled PSF

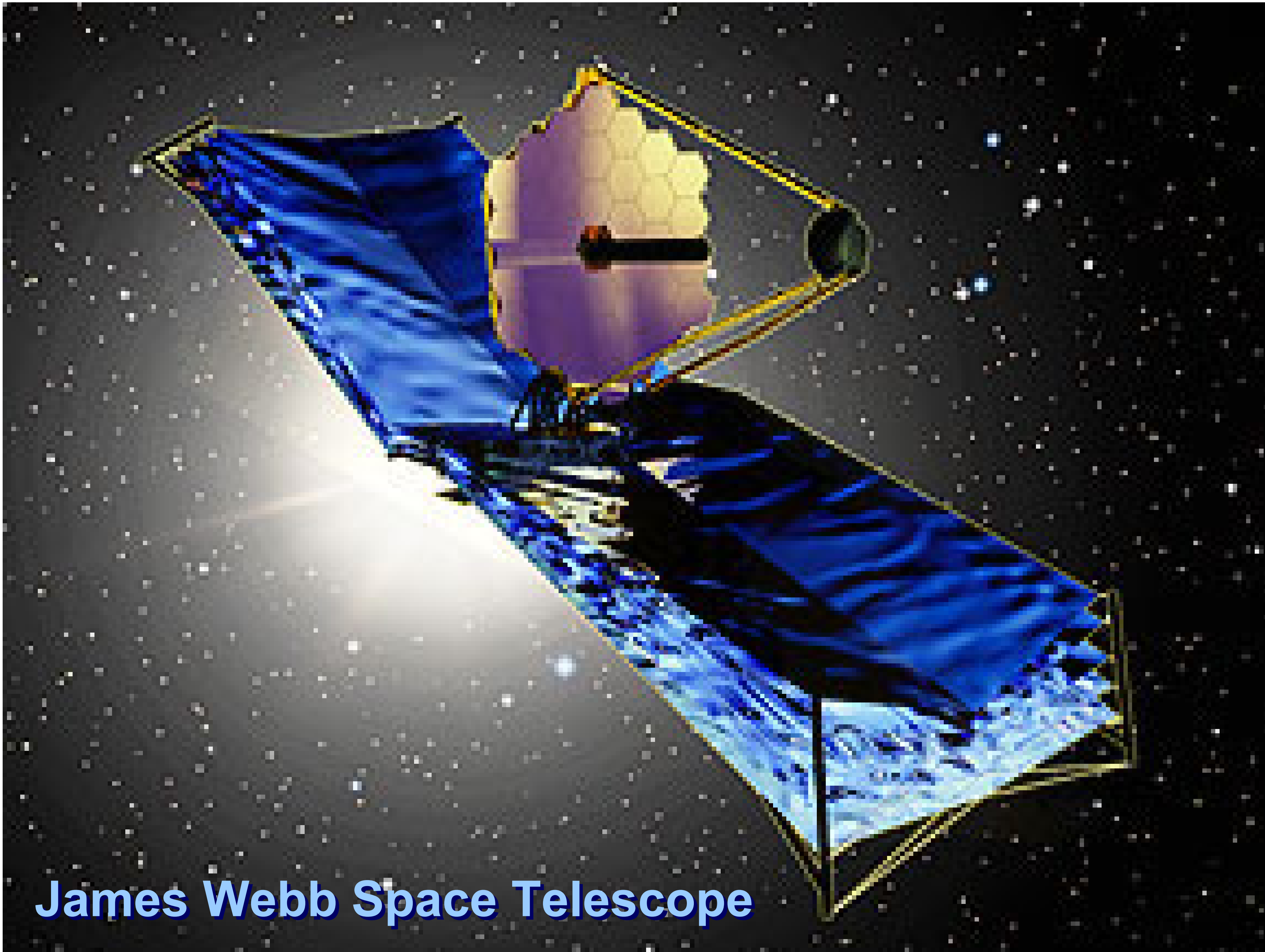


Solution: Use *supersampled* Point Spread Functions

Why did it work with faint stars?

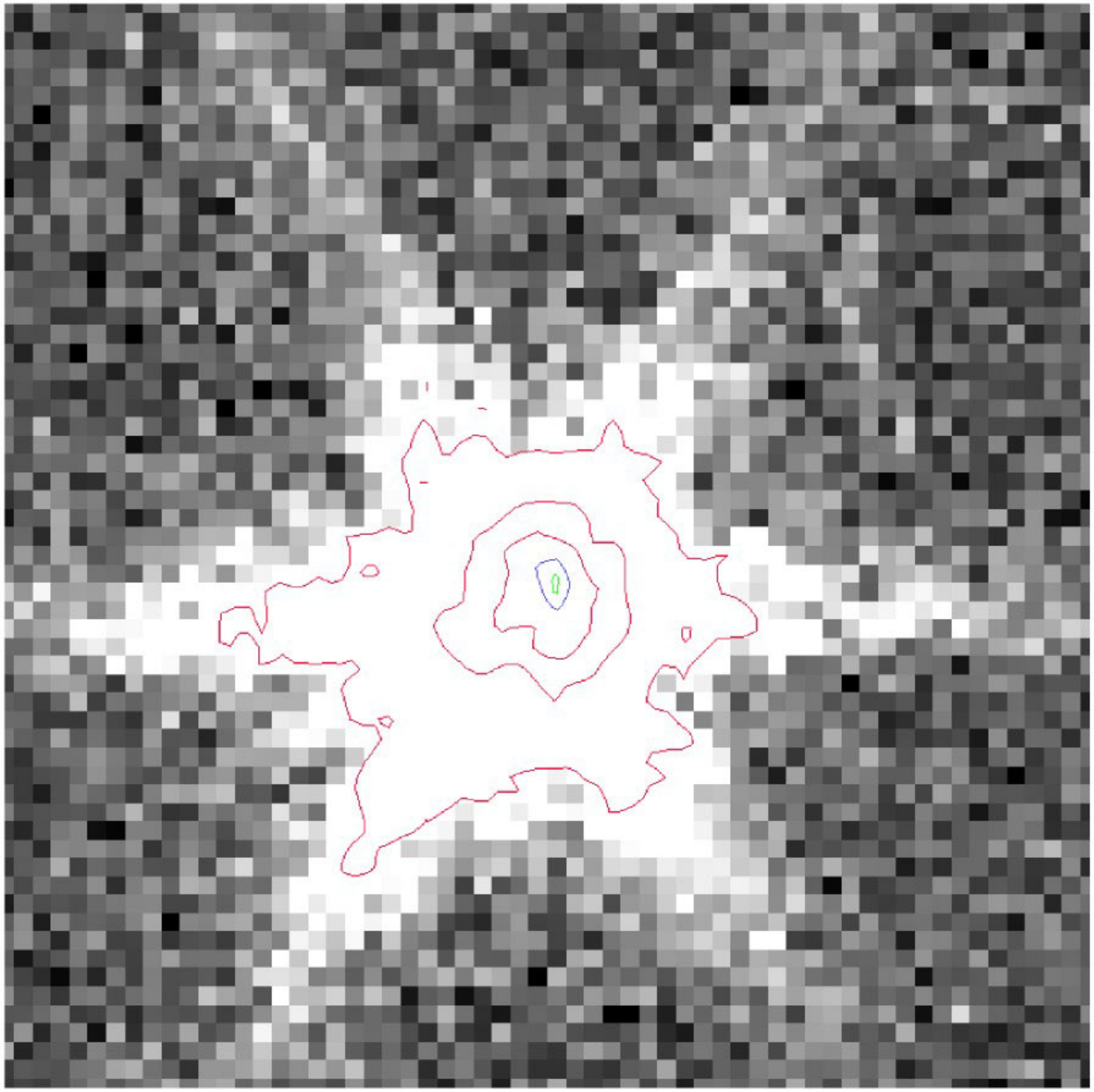
One need lots of photons to properly sample the higher spatial frequencies of a PSF!

Ugly (real) PSFs?


James Webb Space Telescope

Next Generation Space Telescope

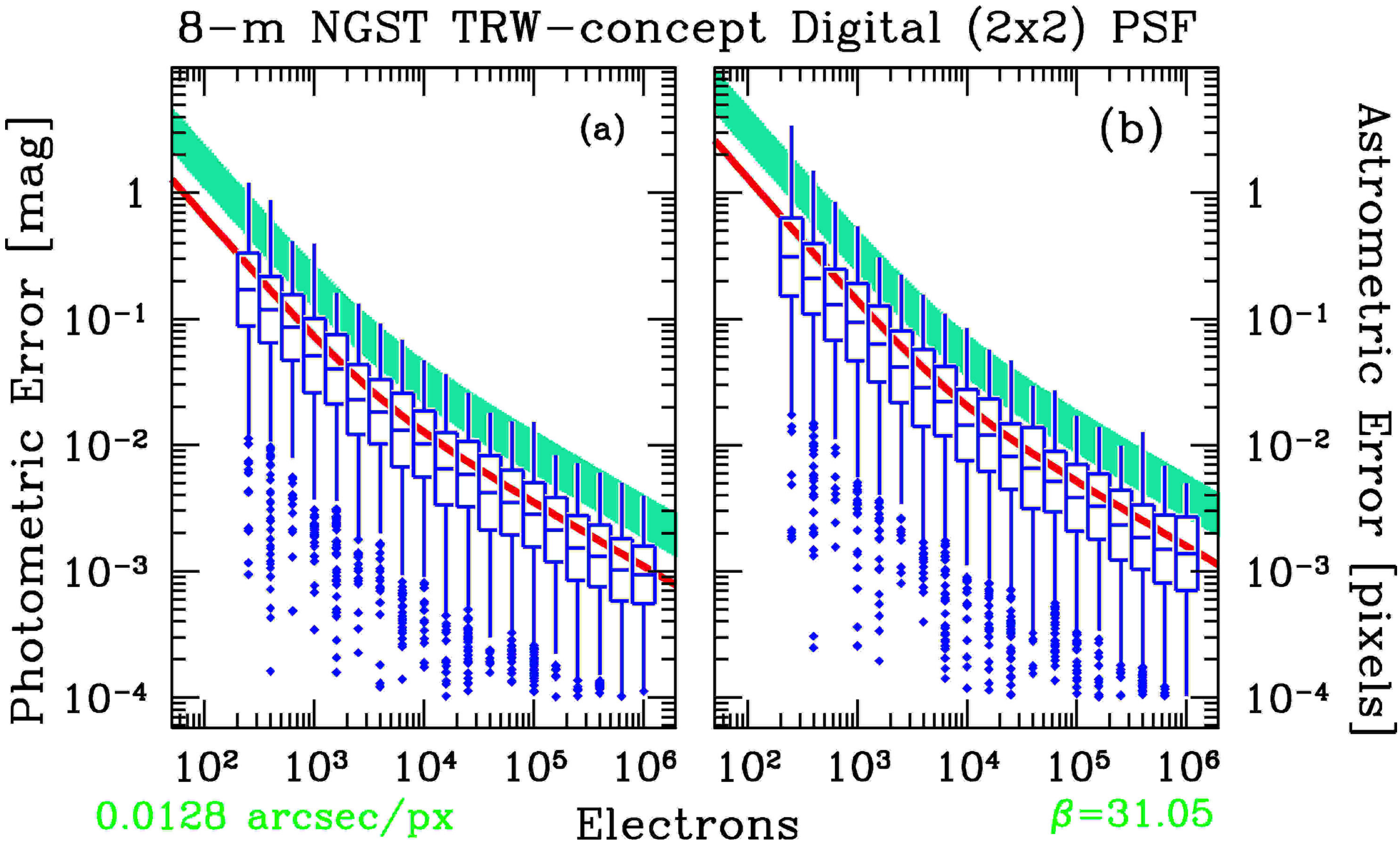
8-m TRW-concept



PSF by John Krist

0.0128 arcsec/pixel β=31.05

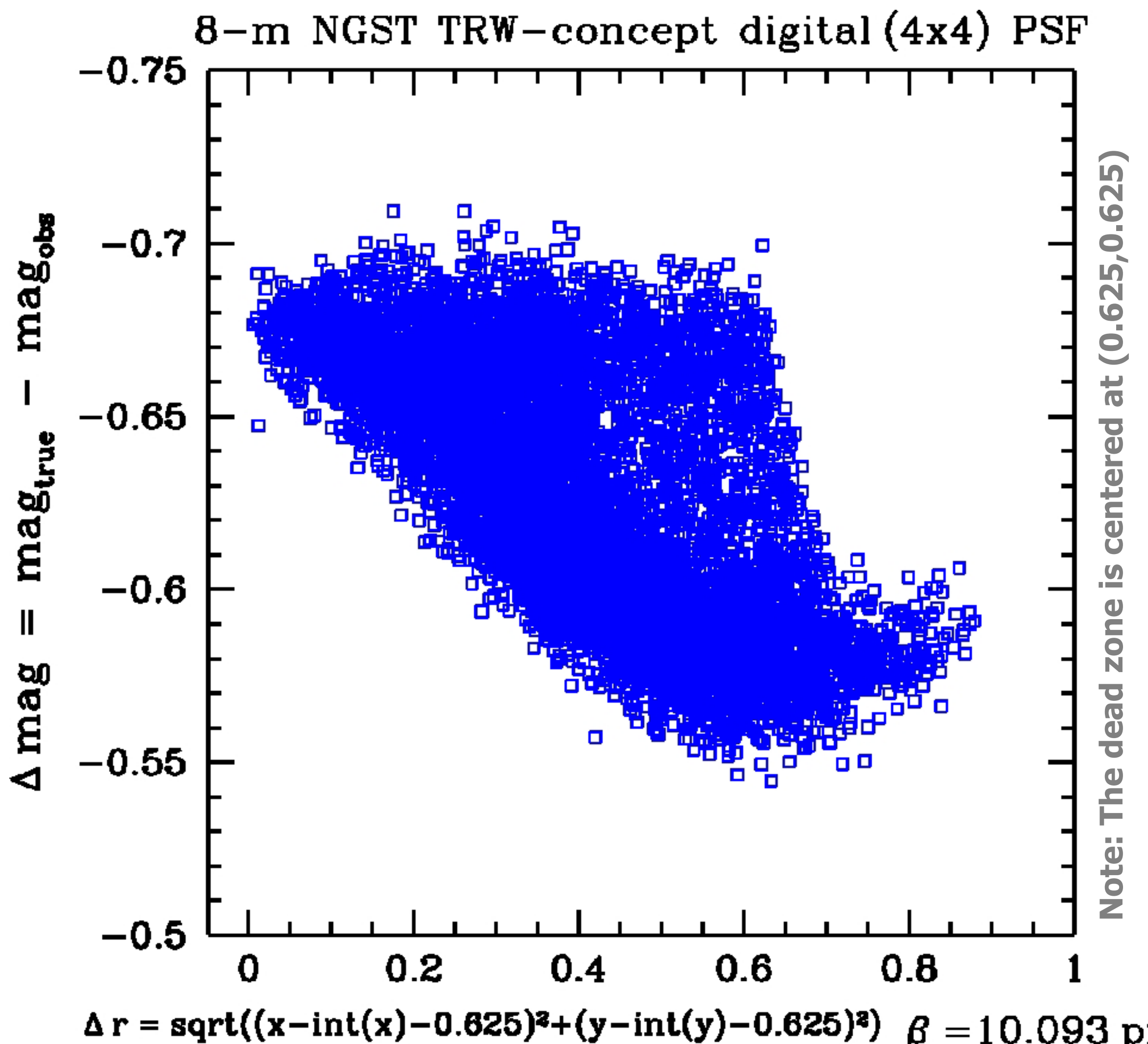
contours: 90%, 50%, 10%, 1%, 0.1% of peak



8-m NGST TRW-concept Digital (2x2) PSF
(a)
(b)
Photometric Error [mag]
Astrometric Error [pixels]
1
10−1
10−2
10−3
10−4
10²
10³
10⁴
10⁵
10⁶
0.0128 arcsec/px
Electrons
β=31.05

Now consider a realistic IR detector …

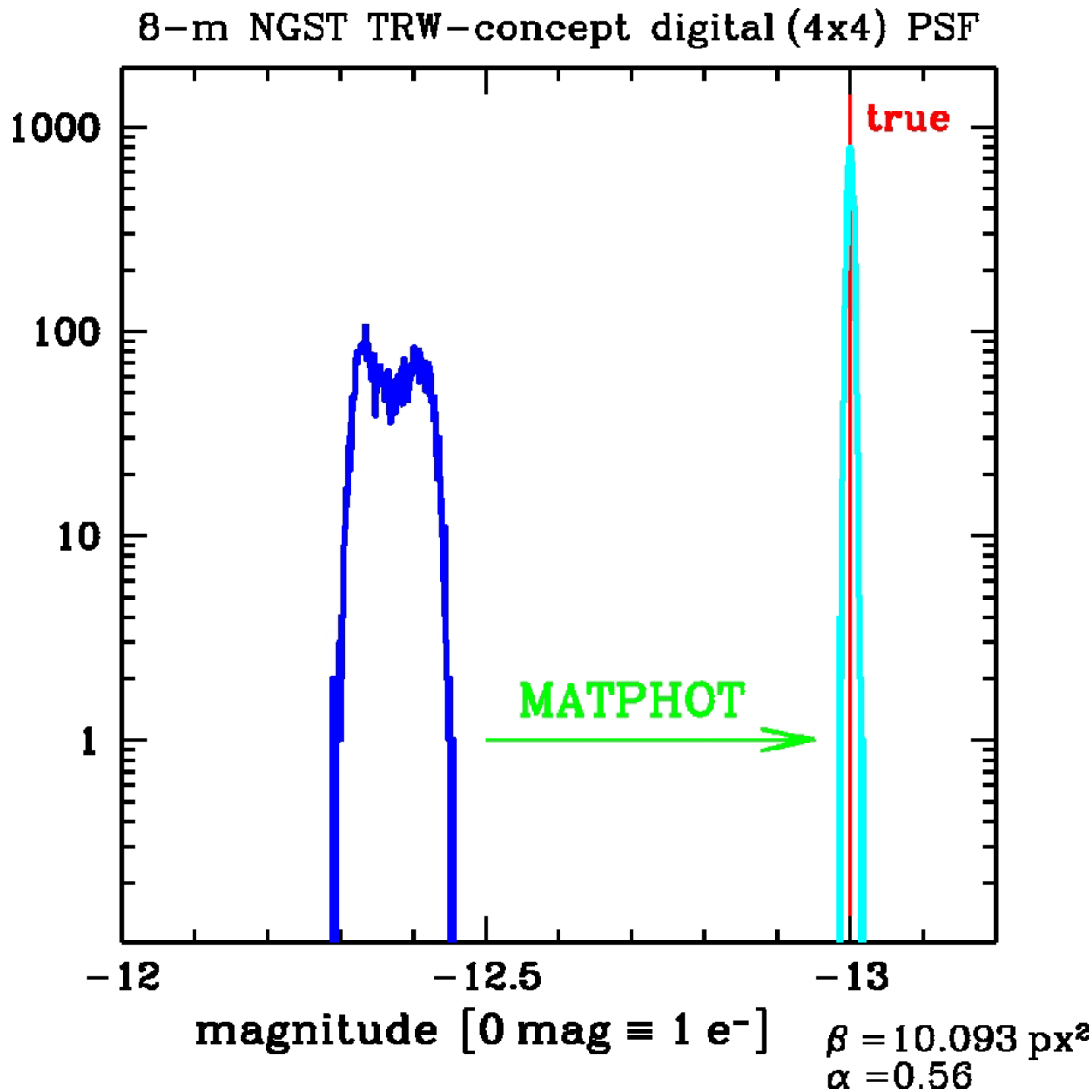
Non-Uniform Pixel Response Functions



$\beta = 10.093\ \mathrm{px}^2$
$\alpha = 0.56$

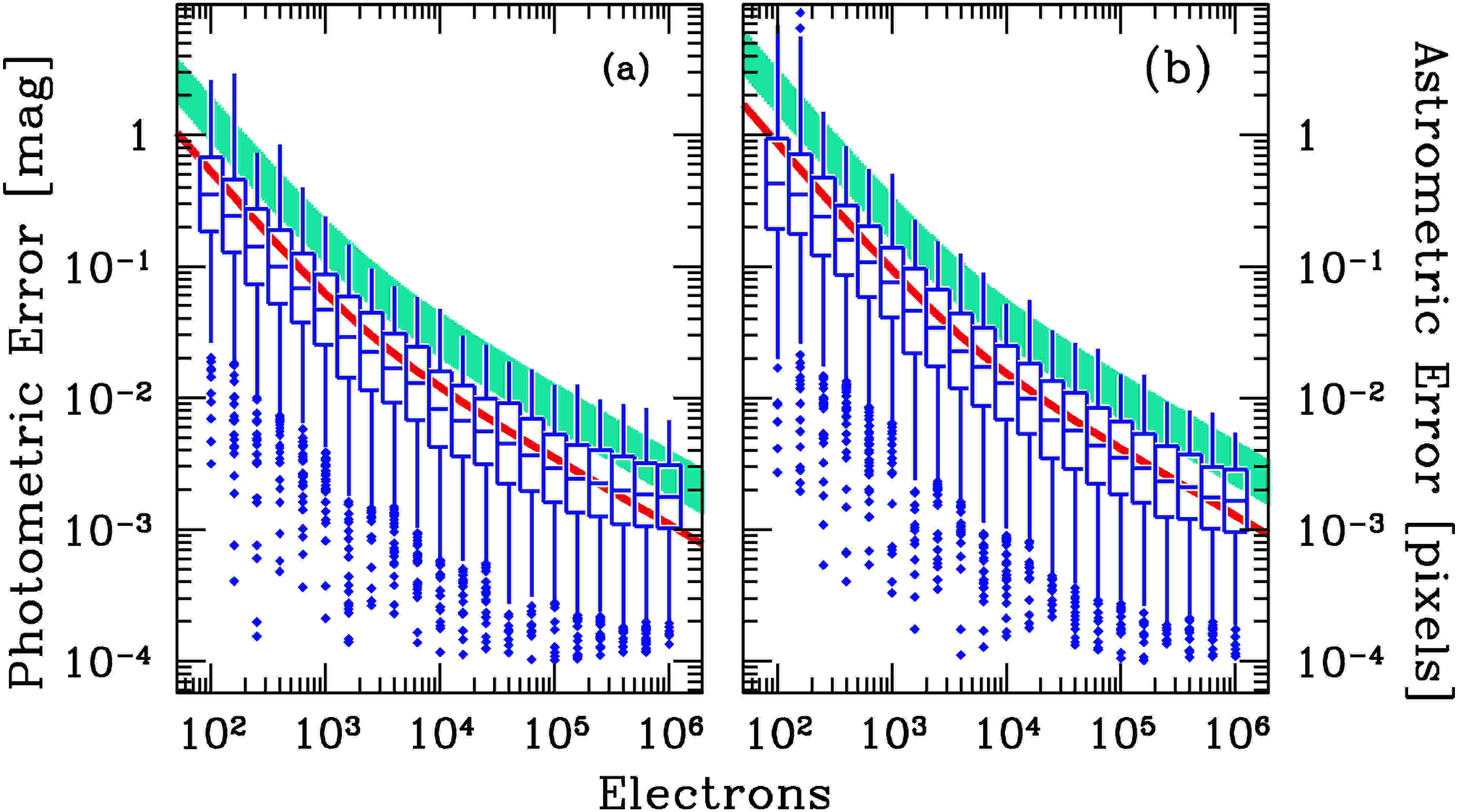
^ 44% of the pixel is dead!

Excellent IR stellar photometry should be possible with accurate knowledge of the Pixel Response Function



Calibration Errors

1% flat-field error: FHWM=3.0 px 3x3



Challenges of PSF Extraction

- highly variable PSF within the field of view
- too few stars
- which are not bright enough
- from undersampled observations
- that are poorly dithered
- with
  - significant Charge Transfer Efficiency variations
  - variable diffusion
  - loss of photons due to charge leakage
- and may possibly be nonlinear at <1% level

This work is supported by a grant from the **Applied Information Systems Research Program (AISRP)** of the Office of Space Science of the National Aeronautics and Space Administration (NASA) [NRA 01-OSS-01: Interagency Order No. S-13811-G] .