Crossing Probabilities of Multiple Ising Interfaces

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Joint work with **Hao Wu**

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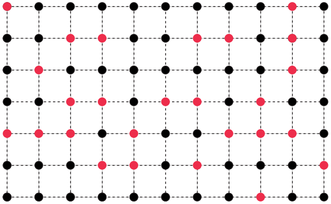
Analysis and Geometry of Random Shapes
@ IPAM, UCLA

Outline

1. Ising model
   - phase transition
   - conformal invariance at criticality?
2. crossing probabilities
   - critical Ising model
   - critical percolation
   - level lines of the Gaussian free field
   - (double-dimer pairings)
   - (multichordal loop-erased random walks / UST branches)
3. scaling limits of interfaces: SLE variants
   - chordal SLE_κ
   - multiple SLEs
   - partition functions

Ising model

ON THE PLANE



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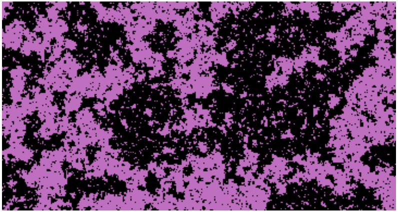


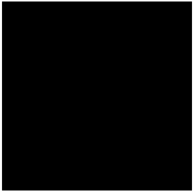
Ising model: ferromagnetic phase transition

[Lenz & Ising '20s, Peierls 30's, Kramers, Wannier, Onsager 40's →]

- random spins $\sigma_x = \pm 1$ at vertices x of a graph
- nearest neighbor interaction: $\mathbb{P}\,[\text{config.}] \propto \exp\left(\frac{1}{T} \sum_{x \sim y} \sigma_x \sigma_y\right)$
- **phase transition** at critical temperature $T = T_c$

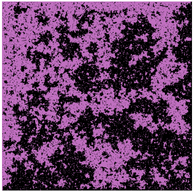
look at correlation of a pair of spins at x and y

$C(x,y) = \mathbb{E}\,[\sigma_x \sigma_y] - \mathbb{E}\,[\sigma_x]\,\mathbb{E}\,[\sigma_y]$ when $|x-y| >> 1$:



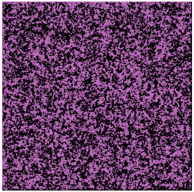
$T < T_c$

$C(x,y) \sim \text{const.}$



$T = T_c$

$C(x,y) \sim |x-y|^{-\beta}$



$T_c < T$

$C(x,y) \sim e^{-\frac{1}{\xi}|x-y|}$

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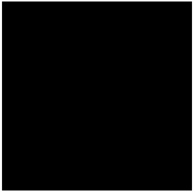

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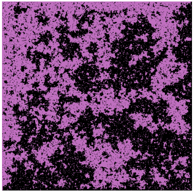

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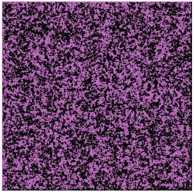

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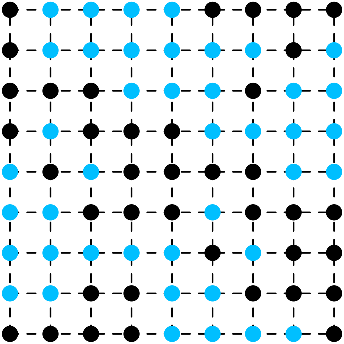
- scaling limit at *critical temperature* T_c: **conformal invariance**

Crossing probabilities



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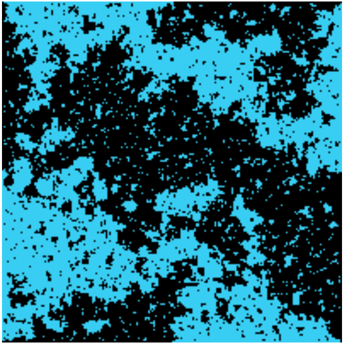
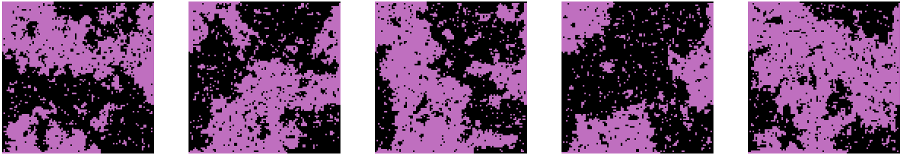


Crossing probabilities in the critical Ising model

- consider critical Ising model on a graph (e.g. square grid)
- take marked points $x_1, \ldots, x_{2N}$ on the boundary
- impose **alternating $\oplus/\ominus$ boundary conditions**

$\Longrightarrow$ N **macroscopic interfaces** connect the marked points pairwise



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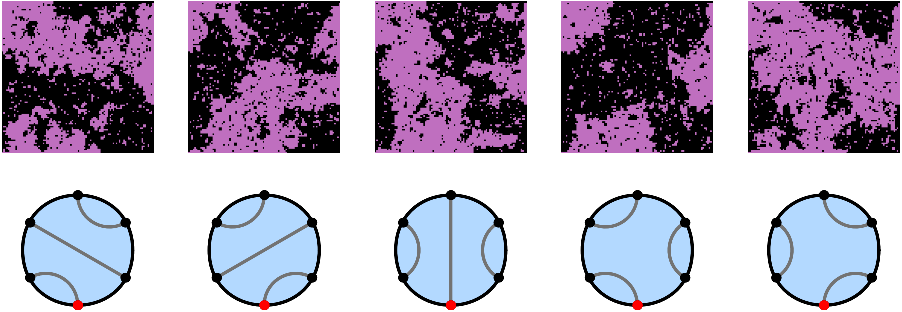

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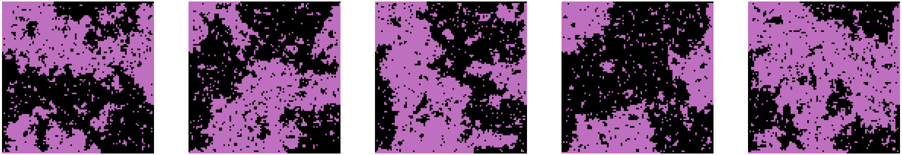
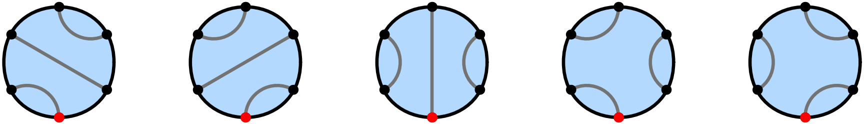




What are the probabilities of the various connectivities?

Proposition [Izyurov '11]

$$\lim_{\delta\to 0} \mathbb{P}[\text{ there exists a left-right } \oplus \text{ crossing }]$$

$$= \left(\int_0^1 \frac{s^{2/3}(1-s)^{2/3}}{1-s+s^2}\right)^{-1} \left(\int_0^\lambda \frac{s^{2/3}(1-s)^{2/3}}{1-s+s^2}\right)$$

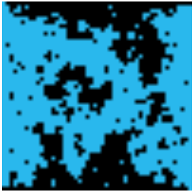
Proof: multi-point discrete holomorphic observable
+ FK-duality

- where $\varphi\colon \Omega \to \mathbb{H}$, $\varphi(x_4) = \infty$

$$\lambda = \frac{\varphi(x_1) - \varphi(x_2)}{\varphi(x_3) - \varphi(x_2)}$$

- conjectured in physics literature

[Cardy '80's; Bauer, Bernard, Kytölä '05]



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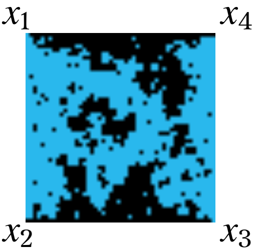

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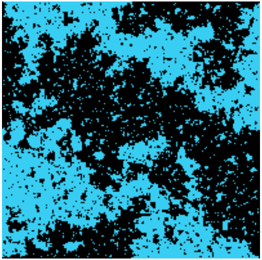
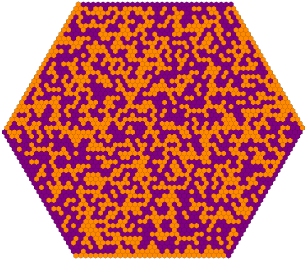
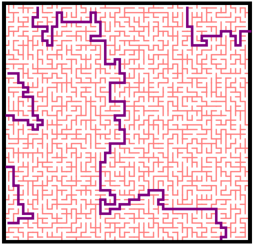
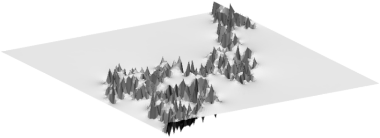
- [Izyurov '14]: convergence of 2^{N-1} connectivity events
- BUT there are $C_N := \frac{1}{N+1}\binom{2N}{N} \sim \frac{4^N}{N^{3/2}\sqrt{\pi}}$ events in total

Main theorem(s)









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- discrete polygons $(\Omega^\delta; x_1^\delta, \ldots, x_{2N}^\delta)$ $\quad \Omega^\delta \subset \delta\mathbb{Z}^2$
- $(\Omega^\delta; x_1^\delta, \ldots, x_{2N}^\delta) \xrightarrow{\delta \to 0} (\Omega; x_1, \ldots, x_{2N})$ in the Carathéodory sense

Theorem [P. & Wu '18]

For the critical **Ising model** on Ω^δ with alternating boundary conditions, for all connectivities $\alpha \in \mathrm{LP}_N$, we have

$$\lim_{\delta \to 0} \mathbb{P}[\text{ connectivity of interfaces} = \alpha] \quad = \quad \frac{\mathcal{Z}_\alpha^{(\kappa=3)}(\Omega; x_1, \ldots, x_{2N})}{\mathcal{Z}_{\mathrm{Ising}}^{(N)}(\Omega; x_1, \ldots, x_{2N})}$$

- $\mathcal{Z}_{\mathrm{Ising}}^{(N)} := \sum_{\alpha \in \mathrm{LP}_N} \mathcal{Z}_\alpha^{(\kappa=3)} = \mathrm{pf}\Big((x_j - x_i)^{-1}\Big)_{i,j}$
- $\{\mathcal{Z}_\alpha^{(\kappa=3)} : \alpha \in \mathrm{LP}_N\}$ "pure partition functions"

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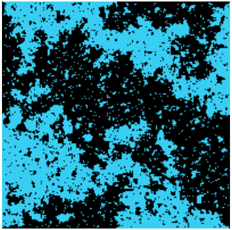
<u>Main inputs to the proof:</u>

- convergence of interfaces to SLE_3 variants
- good control of the martingale $\mathcal{Z}_\alpha / \mathcal{Z}_{\mathrm{Ising}}$



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Theorem [P. & Wu '18]

For the critical **Bernoulli percolation** on Ω^δ with alternating boundary conditions, for all connectivities $\alpha \in \mathrm{LP}_N$, we have

$$\lim_{\delta \to 0} \mathbb{P}[\text{ connectivity of interfaces} = \alpha] \quad = \quad \frac{\mathcal{Z}_\alpha^{(\kappa=6)}(\Omega; x_1, \ldots, x_{2N})}{\mathcal{Z}_{\mathrm{perco}}^{(N)}(\Omega; x_1, \ldots, x_{2N})}$$

- $\mathcal{Z}_{\mathrm{perco}}^{(N)} := \sum_{\alpha \in \mathrm{LP}_N} \mathcal{Z}_\alpha^{(\kappa=6)} = 1$
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Main inputs to the proof:

- convergence of interfaces to SLE_6 variants
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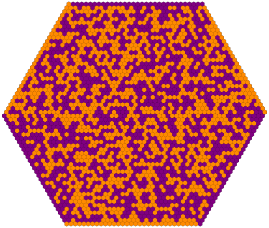
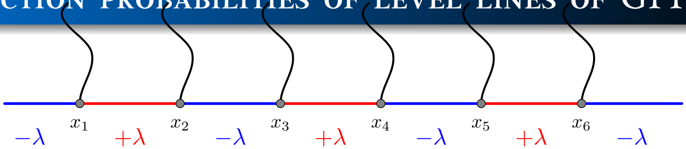


Connection probabilities of level lines of GFF



Theorem [Kenyon & Wilson '11, P. & Wu '17]

For the **Gaussian free field** with alternating boundary data, for all connectivities $\alpha \in \mathrm{LP}_N$, we have (KW: similar result for **double-dimers**)

$$\mathbb{P}[\text{ connectivity of level lines} = \alpha] = \frac{\mathcal{Z}_\alpha^{(\kappa=4)}(\Omega; x_1, \ldots, x_{2N})}{\mathcal{Z}_{\mathrm{GFF}}^{(N)}(\Omega; x_1, \ldots, x_{2N})}$$

- $\mathcal{Z}_{\mathrm{GFF}}^{(N)} := \sum_{\alpha \in \mathrm{LP}_N} \mathcal{Z}_\alpha^{(\kappa=4)} = \prod_{i<j} (x_j - x_i)^{\frac{1}{2}(-1)^{j-i}}$
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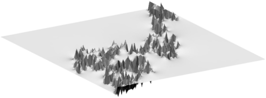
Main inputs to the proof:

- level lines: multiple SLE_4
- good control of the martingale $\mathcal{Z}_\alpha / \mathcal{Z}_{\mathrm{GFF}}$



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Theorem [Flores & Kleban '15, Kytölä & P. '15, P. & Wu '17, Wu '18]

(Proved so far for $\kappa \in (0,6]$.) **There exists a unique collection** $\{\mathcal{Z}_\alpha\}$ of functions with properties PDE, COV, ASY, and a growth bound.

Pure partition functions form a basis $\{\mathcal{Z}_\alpha\}_{\alpha \in \mathrm{LP}_N}$ for a space of **smooth positive** functions of $2N$ real variables $x_1 < \cdots < x_{2N}$.

(PDE): system of $2N$ **partial differential equations**

$$\left\{ \frac{\kappa}{2} \frac{\partial^2}{\partial x_j^2} + \sum_{i \neq j} \left(\frac{2}{x_i - x_j} \frac{\partial}{\partial x_i} - \frac{6/\kappa - 1}{(x_i - x_j)^2} \right) \right\} \mathcal{Z}(x_1, \ldots, x_{2N}) = 0 \qquad \forall\, 1 \leq j \leq 2N$$

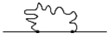
(COV): conformal **covariance**

$$\mathcal{Z}(f(x_1), \ldots, f(x_{2N})) = \prod_j \left| f'(x_j) \right|^{\frac{\kappa-6}{2\kappa}} \times \mathcal{Z}(x_1, \ldots, x_{2N})$$

(ASY): specific **asymptotics**

$$|x_{j+1} - x_j|^{\frac{6-\kappa}{\kappa}} \; \mathcal{Z}_\alpha(x_1, \ldots, x_{2N})$$

$$\xrightarrow{x_j, x_{j+1} \to \xi} \begin{cases} \mathcal{Z}_{\alpha \setminus \{j,j+1\}}(x_1, \ldots, x_{j-1}, x_{j+2}, \ldots, x_{2N}) & \text{if } \{j, j+1\} \in \alpha \\ 0 & \text{if } \{j, j+1\} \notin \alpha \end{cases}$$



- discrete polygons $(\Omega^\delta; x_1^\delta, \ldots, x_{2N}^\delta)$ $\quad \Omega^\delta \subset \delta\mathbb{Z}^2$
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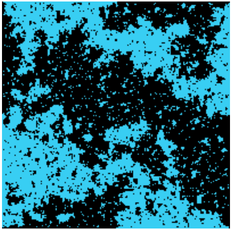
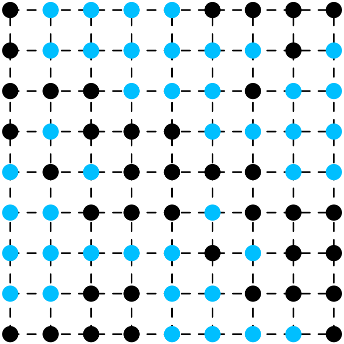


Proof ideas



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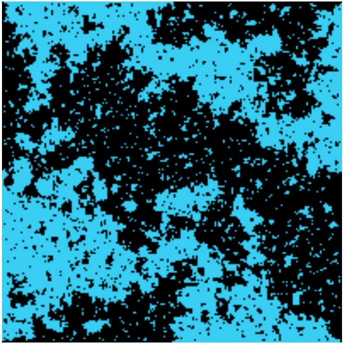

Proof idea: martingale argument

- WLOG: assume $\{1,2\} \in \alpha$.
- consider scaling limit η_{12} of Ising interface starting from x_1:

$$\mathrm{d}W_t = \sqrt{3}\,\mathrm{d}B_t + 3\,\partial_1 \log \mathcal{Z}_{\mathrm{Ising}}(W_t, V_t^2, V_t^3, \ldots, V_t^{2N})\mathrm{d}t$$

$$\mathrm{d}V_t^i = \frac{2\mathrm{d}t}{V_t^i - W_t}, \qquad V_0^i = x_i, \quad \text{for } i \neq 1, \quad W_0 = x_1$$

(**convergence of discrete interface: [Izyurov '15; P. & Wu '18]**)

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- pure partition function $\mathcal{Z}_\alpha$ gives **bounded martingale**

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- **Main proposition**: [P. & Wu '18] at continuation threshold T,

$$M_t \quad \xrightarrow{t \to T} \quad 1\{\eta_{12}(T) = x_2\} \frac{\mathcal{Z}_{\hat{\alpha}}(V_T^3, \ldots, V_T^{2N})}{\mathcal{Z}_{\mathrm{Ising}}^{(N-1)}(V_T^3, \ldots, V_T^{2N})} =: M_T$$

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$$M_t := \frac{\mathcal{Z}_\alpha(W_t, V_t^2, V_t^3, \ldots, V_t^{2N})}{\mathcal{Z}_{\mathrm{Ising}}(W_t, V_t^2, V_t^3, \ldots, V_t^{2N})}$$

- **Main proposition:** [P. & Wu '18] at continuation threshold T,

$$M_t \quad \xrightarrow{t \to T} \quad \mathbb{1}\{\eta_{12}(T) = x_2\} \frac{\mathcal{Z}_{\hat{\alpha}}(V_T^3, \ldots, V_T^{2N})}{\mathcal{Z}_{\mathrm{Ising}}^{(N-1)}(V_T^3, \ldots, V_T^{2N})} =: M_T$$

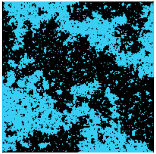
- therefore, by the optional stopping theorem,

$$\frac{\mathcal{Z}_\alpha(x_1, x_2, x_3, \ldots, x_{2N})}{\mathcal{Z}_{\mathrm{Ising}}(x_1, x_2, x_3, \ldots, x_{2N})} = M_0 = \mathbb{E}[M_T]$$

Proof idea: induction on N

Induction hypothesis: for Ising model with $N-1$ interfaces,

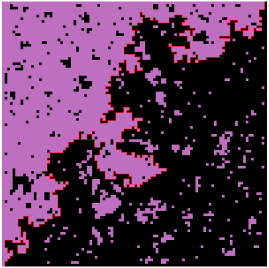
$$\lim_{\delta\to 0} \mathbb{P}[\hat{\alpha}] = \frac{\mathcal{Z}_{\hat{\alpha}}(x_3, \ldots, x_{2N})}{\mathcal{Z}^{(N-1)}_{\text{Ising}}(x_3, \ldots, x_{2N})}$$

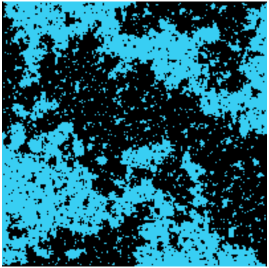


Then: optional stopping argument + technical details give

$$\begin{aligned}
\frac{\mathcal{Z}_{\alpha}(x_1, x_2, x_3, \ldots, x_{2N})}{\mathcal{Z}_{\text{Ising}}(x_1, x_2, x_3, \ldots, x_{2N})} &= \mathbb{E}[M_T] \\
&= \mathbb{E}\left[1\{\eta_{12}(T) = x_2\} \frac{\mathcal{Z}_{\hat{\alpha}}(V_T^3, \ldots, V_T^{2N})}{\mathcal{Z}^{(N-1)}_{\text{Ising}}(V_T^3, \ldots, V_T^{2N})} \right] \\
&= \mathbb{E}\left[1\{\eta_{12}(T) = x_2\} \lim_{\delta\to 0} \mathbb{P}[\hat{\alpha}] \right] \\
&= \lim_{\delta\to 0} \mathbb{E}\left[1\{\eta_{12}^{\delta} \text{ terminates at } x_2^{\delta}\} \, \mathbb{E}[\alpha \mid \eta_{12}^{\delta}] \right] \\
&= \lim_{\delta\to 0} \mathbb{P}[\alpha]
\end{aligned}$$

SCALING LIMITS OF INTERFACES

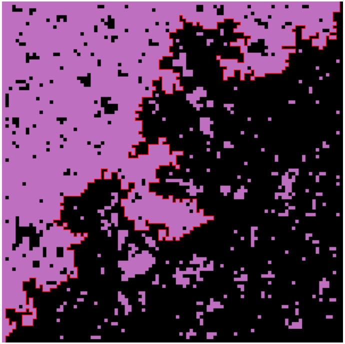


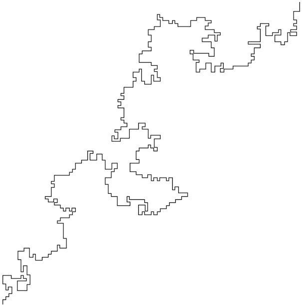


SLE$_\kappa$ VARIANTS
& PARTITION FUNCTIONS

$$\mathrm{d}W_t = \sqrt{\kappa}\,\mathrm{d}B_t + \kappa\,\partial_1 \log \mathcal{Z}\left(W_t, V_t^2, V_t^3, \ldots, V_t^{2N}\right)\mathrm{d}t$$

Dobrushin boundary conditions: $\partial\Omega^{\delta} = \{\oplus \text{ segment}\} \bigcup \{\ominus \text{ segment}\}$





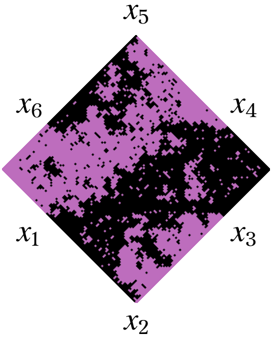
interface of Ising model $\xrightarrow{\delta\to 0}$ Schramm-Loewner evolution, SLE_3

NB: Convergence in the topology of curves (and driving functions)

[Chelkak, Duminil-Copin, Hongler, Kemppainen, Smirnov '14]

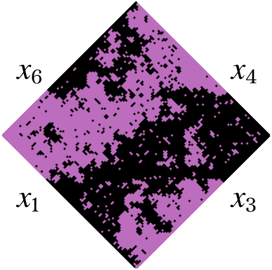
Proof: tightness (RSW type estimates) + discrete holomorphic observable

Scaling limits of multiple Ising interfaces (I)



- fix discrete domain data $(\Omega^\delta; x_1^\delta, \ldots, x_{2N}^\delta)$
- consider critical **Ising model** in $\Omega^\delta \subset \delta\mathbb{Z}^2$ with alternating $\oplus/\ominus$ b.c.
- let $(\Omega^\delta; x_1^\delta, \ldots, x_{2N}^\delta) \xrightarrow{\delta \to 0} (\Omega; x_1, \ldots, x_{2N})$ in the Carathéodory sense

Scaling limits of multiple Ising interfaces (I)



- fix discrete domain data $(\Omega^\delta; x_1^\delta, \ldots, x_{2N}^\delta)$
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- let $(\Omega^\delta; x_1^\delta, \ldots, x_{2N}^\delta) \xrightarrow{\delta\to 0} (\Omega; x_1, \ldots, x_{2N})$ in the Carathéodory sense
- *condition* on the event that the interfaces connect the boundary points according to a given connectivity α

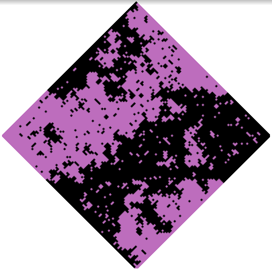
Theorem [Beffara, P. & Wu '18]

The law of the N macroscopic interfaces of the critical Ising model **converges in the scaling limit $\delta \to 0$ to the N-SLE$_\kappa$ with $\kappa = 3$.**

Wu **[arXiv:1703.02022]**
Beffara, P. & Wu **[arXiv:1801.07699]**

Proof: convergence for $N = 1$ and classification of multiple SLE$_3$

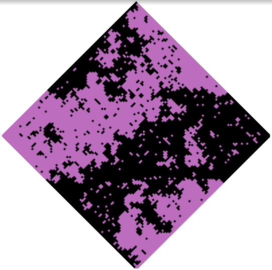
Scaling limits of multiple Ising interfaces (I)



- consider critical **Ising model** in $\Omega^\delta \subset \delta\mathbb{Z}^2$ with alternating $\oplus/\ominus$ b.c.
- *condition* on having given connectivity α

Theorem [Beffara, P. & Wu '18]

Ising interfaces $\xrightarrow{\delta \to 0}$ N-SLE_3 associated to α



- consider critical **Ising model** in $\Omega^\delta \subset \delta\mathbb{Z}^2$ with alternating $\oplus/\ominus$ b.c.
- *condition* on having given connectivity α

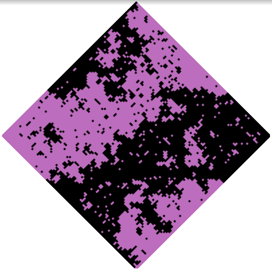
Theorem [Beffara, P. & Wu '18]

Ising interfaces $\xrightarrow{\delta\to 0}$ N-SLE$_3$ associated to α

- On $(\mathbb{H}; x_1, \ldots, x_{2N})$ the marginal law of the curve starting from x_1 is given by the **Loewner chain with driving process**

$$\mathrm{d}W_t = \sqrt{3}\,\mathrm{d}B_t + 3\,\partial_1 \log \mathcal{Z}_\alpha\left(W_t, V_t^2, V_t^3, \ldots, V_t^{2N}\right)\mathrm{d}t, \qquad W_0 = x_1$$

$$\mathrm{d}V_t^i = \frac{2\mathrm{d}t}{V_t^i - W_t}, \qquad V_0^i = x_i, \quad \text{for } i \neq 1$$



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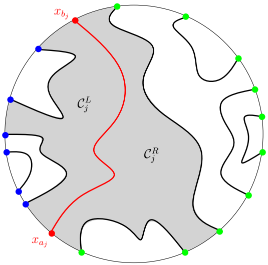
$$\mathrm{d}V_t^i = \frac{2\mathrm{d}t}{V_t^i - W_t}, \qquad V_0^i = x_i, \quad \text{for } i \neq 1$$

- Almost surely generated by a **continuous transient curve**, which hits the boundary only at its endpoint, determined by α.

P. & Wu **[arXiv:1703.00898]** Proof: control drift + compare with chordal SLE

Global multiple SLE$_\kappa$, aka N-SLE$_\kappa$

- family of **random curves** in $(\Omega; x_1, \ldots, x_{2N})$
- various connectivities encoded in **planar pair partitions** $\alpha \in \mathrm{LP}_N$
- **unique "pure" measure for each** α:

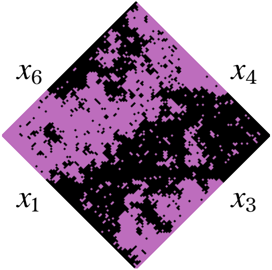


Theorem [Beffara, P. & Wu '18]

Let $\kappa \in (0, 4] \cup \{16/3, 6\}$. For any fixed connectivity α of $2N$ points, **there exists a unique probability measure on N curves** such that *conditionally on $N-1$ of the curves, the remaining one is the chordal* SLE$_\kappa$ *in the random domain where it can live.*

Dubédat (2006); Kozdron & Lawler (2007–2009); Miller & Sheffield (2016); Miller, Sheffield & Werner (2018); P. & Wu (2017); Beffara, P. & Wu (2018)

Proof: Markov chain on space of curve families



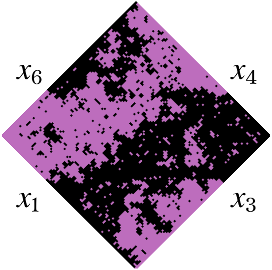
- consider critical **Ising model** in $\Omega^\delta \subset \delta\mathbb{Z}^2$ with alternating $\oplus/\ominus$ b.c.
- let $(\Omega^\delta; x_1^\delta, \ldots, x_{2N}^\delta) \xrightarrow{\delta \to 0} (\Omega; x_1, \ldots, x_{2N})$
- *allow any connectivity* of the interfaces

Theorem [Izyurov '15]

Ising interfaces $\xrightarrow{\delta \to 0}$ (local) multiple SLE_3

Proof: multi-point holomorphic observable

Scaling limits of multiple Ising interfaces (II)



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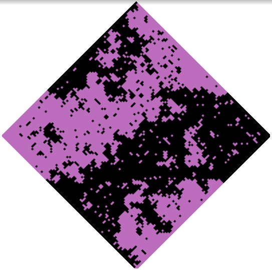
Scaling limit of the interface starting from x_1 is given by the **Loewner chain with driving process**

$$\mathrm{d}W_t = \sqrt{3}\,\mathrm{d}B_t + 3\,\partial_1 \log \mathcal{Z}_{\mathrm{Ising}}\left(W_t, V_t^2, V_t^3, \ldots, V_t^{2N}\right)\mathrm{d}t, \qquad W_0 = x_1$$

$$\mathrm{d}V_t^i = \frac{2\mathrm{d}t}{V_t^i - W_t}, \qquad V_0^i = x_i, \quad \text{for } i \neq 1$$

<u>**LOCAL:**</u> A priori, holds only **before blow-up**

Scaling limits of multiple Ising interfaces (II)



- consider critical **Ising model** in $\Omega^\delta \subset \delta\mathbb{Z}^2$ with alternating $\oplus/\ominus$ b.c.
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Theorem [Izyurov '15]

Ising interfaces $\xrightarrow{\delta \to 0}$ (local) multiple SLE_3

$$\mathrm{d}W_t = \sqrt{3}\,\mathrm{d}B_t + 3\,\partial_1 \log \mathcal{Z}_{\text{Ising}}\left(W_t, V_t^2, V_t^3, \ldots, V_t^{2N}\right)\mathrm{d}t$$

Proposition: "Globality of the scaling limit" [P. & Wu '18]

1. Convergence holds also in the space of curves.
2. Scaling limit is a.s. a **continuous transient curve**, that hits the boundary only at its endpoint = one of the marked points.

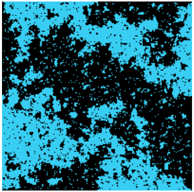
Proof: 1. RSW bounds by [Chelkak, Duminil-Copin & Hongler '16]
+ results of [Aizenman & Burchard '99; Kemppainen & Smirnov '17]
2. control drift + compare with chordal SLE **[arXiv:1808.09438]**

(I): *conditionally* on given connectivity α:

interfaces $\xrightarrow{\delta \to 0}$ N-SLE$_3$ associated to α,

which has (pure) partition function $\mathcal{Z}_\alpha$

$$\mathrm{d}W_t = \sqrt{3}\,\mathrm{d}B_t + 3\,\partial_1 \log \mathcal{Z}\left(W_t, V_t^2, V_t^3, \ldots, V_t^{2N}\right)\mathrm{d}t$$



[arXiv:1703.00898]

Relation of partition functions in (I) and (II)

(I): *conditionally* on given connectivity α:

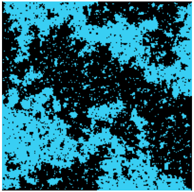
interfaces $\xrightarrow{\delta \to 0}$ N-SLE$_3$ associated to α,

which has (pure) partition function $\mathcal{Z}_\alpha$

(II): *allow any connectivity* of the interfaces:

interfaces $\xrightarrow{\delta \to 0}$ multiple SLE$_3$ with partition function $\mathcal{Z}_{\text{Ising}}$

$$\mathrm{d}W_t = \sqrt{3}\,\mathrm{d}B_t + 3\,\partial_1 \log \mathcal{Z}\left(W_t, V_t^2, V_t^3, \ldots, V_t^{2N}\right)\mathrm{d}t$$



[arXiv:1703.00898]

(I): *conditionally* on given connectivity α:

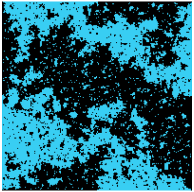
interfaces $\xrightarrow{\delta \to 0}$ N-SLE$_3$ associated to α,

which has (pure) partition function $\mathcal{Z}_\alpha$

(II): *allow any connectivity* of the interfaces:

interfaces $\xrightarrow{\delta \to 0}$ multiple SLE$_3$ with partition function $\mathcal{Z}_{\mathrm{Ising}}$

$$\mathrm{d}W_t = \sqrt{3}\,\mathrm{d}B_t + 3\,\partial_1 \log \mathcal{Z}\left(W_t, V_t^2, V_t^3, \ldots, V_t^{2N}\right)\mathrm{d}t$$



Lemma [Kytölä & P. '16, P. & Wu '17]

$$\mathcal{Z}_{\mathrm{Ising}} := \mathrm{pf}\left(\frac{1}{x_j - x_i}\right)_{i,j} = \sum_{\alpha} \mathcal{Z}_\alpha$$

[arXiv:1703.00898]

Proof: uniqueness of partition functions using
ideas of [Dubédat '06; Kozdron & Lawler '07] + PDE results [Flores & Kleban '15]