Loewner energy via Brownian loop measure and action functional analogs of SLE/GFF couplings

Yilin Wang

ETH Zürich

January, 2019

IPAM UCLA

Contents

Introduction

- **Loewner's transform** [1923] consists of encoding the uniformizing conformal map of a simply connected domain D C into evolution of conformal distortions that flatten out the boundary iteratively,

  non self-intersecting curve ∂D ⇔ **real-valued driving function**.

  Main tool to solve Bieberbach's conjecture by De Branges in 1985.
- Random fractal non self-intersecting curves: the **Schramm-Loewner Evolution** introduced by Oded Schramm in 1999 which successfully describe interfaces in many statistical mechanics models.
- The **Loewner energy** is the action functional of SLE, also the large deviation rate function of SLE_κ as $\kappa \to 0$ [W. 2016].
- Loewner energy for Jordan curves (loops) on the Riemann's sphere, is non-negative, vanishing only on circles, and invariant under Möbius transformation [Rohde, W. 2017].
- **Weil-Petersson metric** is the unique homogeneous Kähler metric on the universal Teichmüller space. Loewner energy is Kähler potential of this metric.

Introduction

- **Loewner's transform** [1923] consists of encoding the uniformizing conformal map of a simply connected domain $D \subset \mathbb{C}$ into evolution of conformal distortions that flatten out the boundary iteratively,

  $$\text{non self-intersecting curve } \partial D \Leftrightarrow \textbf{real-valued driving function}.$$

  Main tool to solve Bieberbach's conjecture by De Branges in 1985.
- Random fractal non self-intersecting curves: the **Schramm-Loewner Evolution** introduced by Oded Schramm in 1999 which successfully describe interfaces in many statistical mechanics models.
- The **Loewner energy** is the action functional of SLE, also the large deviation rate function of SLE_κ as $\kappa \to 0$ [W. 2016].
- Loewner energy for Jordan curves (loops) on the Riemann's sphere, is non-negative, vanishing only on circles, and invariant under Möbius transformation [Rohde, W. 2017].
- **Weil-Petersson metric** is the unique homogeneous Kähler metric on the universal Teichmüller space. Loewner energy is Kähler potential of this metric.

Introduction

- **Loewner's transform** [1923] consists of encoding the uniformizing conformal map of a simply connected domain $D \subset \mathbb{C}$ into evolution of conformal distortions that flatten out the boundary iteratively,

  non self-intersecting curve $\partial D \Leftrightarrow$ **real-valued driving function**.

  Main tool to solve Bieberbach's conjecture by De Branges in 1985.
- Random fractal non self-intersecting curves: the **Schramm-Loewner Evolution** introduced by Oded Schramm in 1999 which successfully describe interfaces in many statistical mechanics models.
- The **Loewner energy** is the action functional of SLE, also the large deviation rate function of SLE_κ as $\kappa \to 0$ [W. 2016].
- Loewner energy for Jordan curves (loops) on the Riemann's sphere, is non-negative, vanishing only on circles, and invariant under Möbius transformation [Rohde, W. 2017].
- **Weil-Petersson metric** is the unique homogeneous Kähler metric on the universal Teichmüller space. Loewner energy is Kähler potential of this metric.

Introduction

- **Loewner's transform** [1923] consists of encoding the uniformizing conformal map of a simply connected domain $D \subset \mathbb{C}$ into evolution of conformal distortions that flatten out the boundary iteratively,

  $$\text{non self-intersecting curve } \partial D \Leftrightarrow \textbf{ real-valued driving function}.$$

  Main tool to solve Bieberbach's conjecture by De Branges in 1985.
- Random fractal non self-intersecting curves: the **Schramm-Loewner Evolution** introduced by Oded Schramm in 1999 which successfully describe interfaces in many statistical mechanics models.
- The **Loewner energy** is the action functional of SLE, also the large deviation rate function of SLE_κ as $\kappa \to 0$ [W. 2016].
- Loewner energy for Jordan curves (loops) on the Riemann's sphere, is non-negative, vanishing only on circles, and invariant under Möbius transformation [Rohde, W. 2017].
- **Weil-Petersson metric** is the unique homogeneous Kähler metric on the universal Teichmüller space. Loewner energy is Kähler potential of this metric.

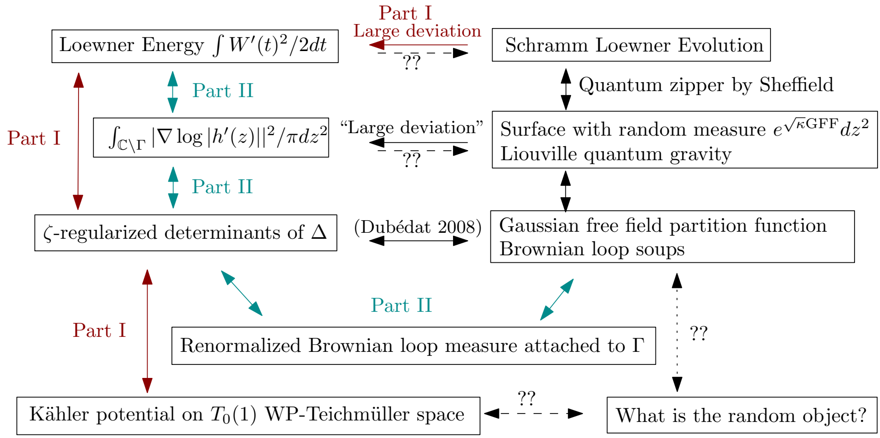
Introduction

- **Loewner's transform** [1923] consists of encoding the uniformizing conformal map of a simply connected domain D C into evolution of conformal distortions that flatten out the boundary iteratively,

  $$\text{non self-intersecting curve } \partial D \Leftrightarrow \textbf{ real-valued driving function}.$$

  Main tool to solve Bieberbach's conjecture by De Branges in 1985.
- Random fractal non self-intersecting curves: the **Schramm-Loewner Evolution** introduced by Oded Schramm in 1999 which successfully describe interfaces in many statistical mechanics models.
- The **Loewner energy** is the action functional of SLE, also the large deviation rate function of SLE_κ as $\kappa \to 0$ [W. 2016].
- Loewner energy for Jordan curves (loops) on the Riemann's sphere, is non-negative, vanishing only on circles, and invariant under Möbius transformation [Rohde, W. 2017].
- **Weil-Petersson metric** is the unique homogeneous Kähler metric on the universal Teichmüller space. Loewner energy is Kähler potential of this metric.

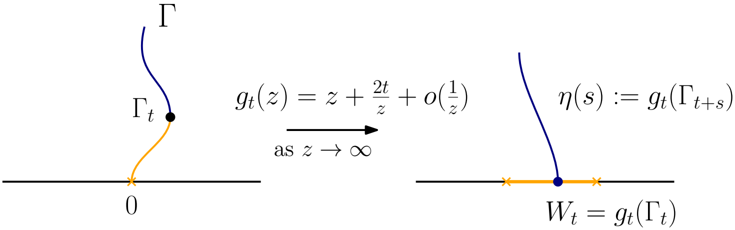
Action functionals vs. Random objects



Contents

Chordal Loewner chains

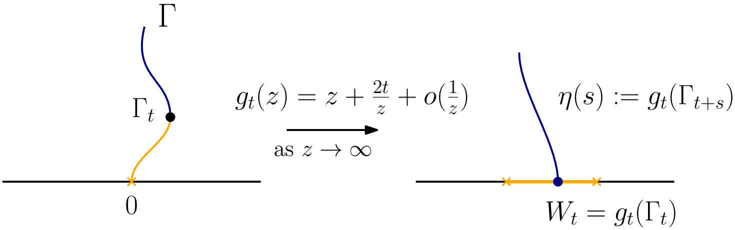
Let Γ be a simple chord in $\mathbb{H}$ from 0 to ∞.



- Γ is **capacity-parametrized** by $[0, \infty)$.
- $W : \mathbb{R}_+ \to \mathbb{R}$ is called the **driving function** of Γ.
- $W_0 = 0$.
- W is continuous.
- One can recover the curve Γ from W using Loewner's differential equation.
- We say that Γ is the **chordal Loewner chain** generated by W.
- The centered Loewner flow has the expansion $f_t(z) = g_t(z) - W_t = z - W_t + 2t/z + O(1/z)$.

Chordal Loewner chains

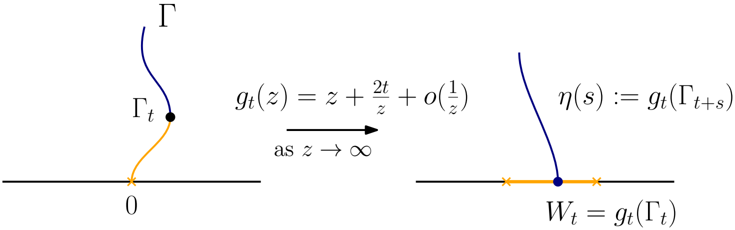
Let Γ be a simple chord in H from 0 to ∞.



- Γ is **capacity-parametrized** by $[0, \infty)$.
- $W : \mathbb{R}_+ \to \mathbb{R}$ is called the **driving function** of Γ.
- $W_0 = 0$.
- W is continuous.
- One can recover the curve Γ from W using Loewner's differential equation.
- We say that Γ is the **chordal Loewner chain** generated by W.
- The centered Loewner flow has the expansion $f_t(z) = g_t(z) - W_t = z - W_t + 2t/z + O(1/z)$.

Chordal Loewner chains

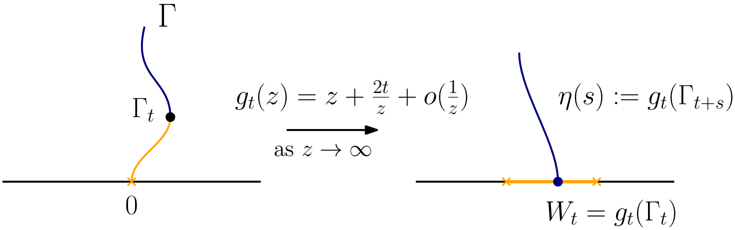
Let Γ be a simple chord in $\mathbb{H}$ from 0 to ∞.



- Γ is **capacity-parametrized** by $[0, \infty)$.
- $W : \mathbb{R}_+ \to \mathbb{R}$ is called the **driving function** of Γ.
- $W_0 = 0$.
- W is continuous.
- One can recover the curve Γ from W using Loewner's differential equation.
- We say that Γ is the **chordal Loewner chain** generated by W.
- The centered Loewner flow has the expansion $f_t(z) = g_t(z) - W_t = z - W_t + 2t/z + O(1/z)$.

Chordal Loewner chains

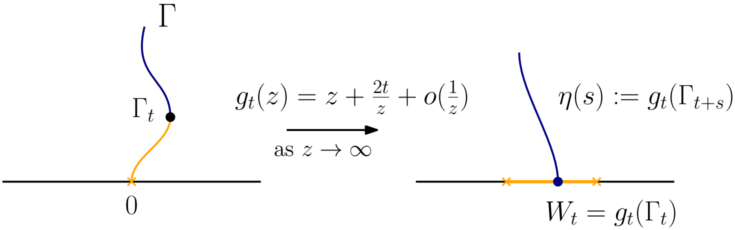
Let Γ be a simple chord in $\mathbb{H}$ from 0 to ∞.



- Γ is **capacity-parametrized** by $[0, \infty)$.
- $W : \mathbb{R}_+ \to \mathbb{R}$ is called the **driving function** of Γ.
- $W_0 = 0$.
- W is continuous.
- One can recover the curve Γ from W using Loewner's differential equation.
- We say that Γ is the **chordal Loewner chain** generated by W.
- The centered Loewner flow has the expansion $f_t(z) = g_t(z) - W_t = z - W_t + 2t/z + O(1/z)$.

Chordal Loewner chains

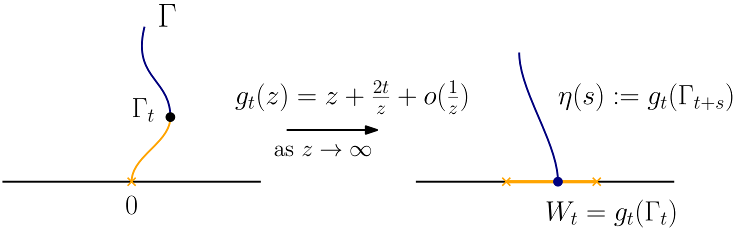
Let Γ be a simple chord in $\mathbb{H}$ from 0 to ∞.



- Γ is **capacity-parametrized** by $[0, \infty)$.
- $W : \mathbb{R}_+ \to \mathbb{R}$ is called the **driving function** of Γ.
- $W_0 = 0$.
- W is continuous.
- One can recover the curve Γ from W using Loewner's differential equation.
- We say that Γ is the **chordal Loewner chain** generated by W.
- The centered Loewner flow has the expansion $f_t(z) = g_t(z) - W_t = z - W_t + 2t/z + O(1/z)$.

Chordal Loewner chains

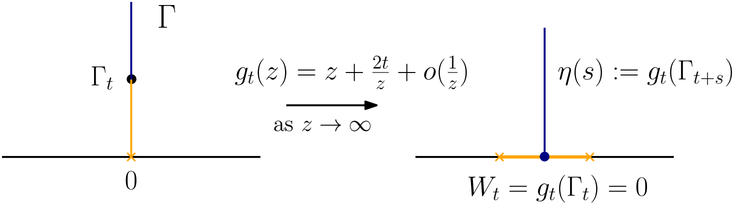
Let Γ be a simple chord in H from 0 to ∞.



- Γ is **capacity-parametrized** by $[0, \infty)$.
- $W : \mathbb{R}_+ \to \mathbb{R}$ is called the **driving function** of Γ.
- $W_0 = 0$.
- W is continuous.
- One can recover the curve Γ from W using Loewner's differential equation.
- We say that Γ is the **chordal Loewner chain** generated by W.
- The centered Loewner flow has the expansion $f_t(z) = g_t(z) - W_t = z - W_t + 2t/z + O(1/z)$.

Chordal Loewner chain

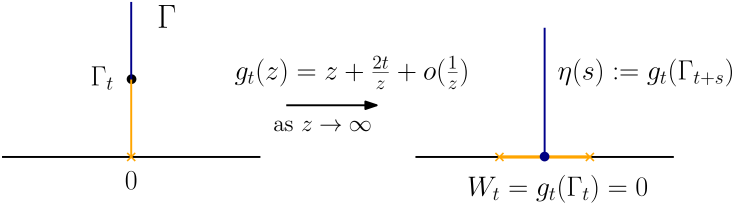
- If $W \equiv 0$, then $\Gamma = i\mathbb{R}_+$.



- When the curve is driven by $W = \sqrt{\kappa}B$ where B is 1-d Brownian motion, the curve generated is the **Schramm-Loewner Evolution of parameter** κ (SLE_κ).

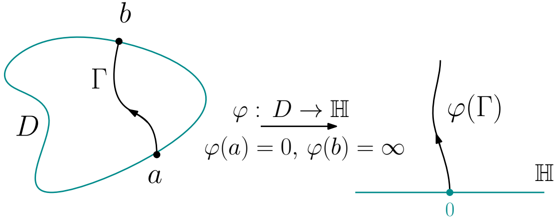
Chordal Loewner chain

- If $W \equiv 0$, then $\Gamma = i\mathbb{R}_+$.



- When the curve is driven by $W = \sqrt{\kappa}B$ where B is 1-d Brownian motion, the curve generated is the **Schramm-Loewner Evolution of parameter** κ (SLE_κ).

D C a simply connected domain, a, b are two boundary points of D.



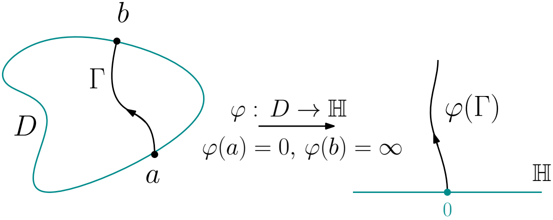
Definition: Loewner energy

We define the **Loewner energy of a simple chord Γ in** (D, a, b) to be

$$I_{D,a,b}(\Gamma) := I_{\mathbb{H},0,\infty}(\varphi(\Gamma)) := I(W) := \frac{1}{2}\int_0^\infty W\ (t)^2 dt$$

where W is the driving function of $\varphi(\Gamma)$.

The chordal Loewner energy

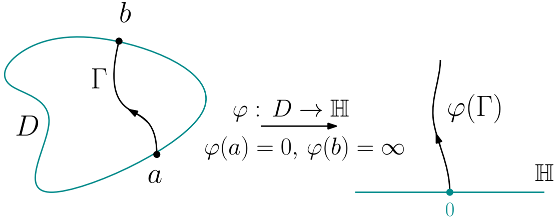


- The Loewner energy is well-defined in (D, a, b) since for $c > 0$,

$$I_{\mathbb{H},0,\infty}(\Gamma) = I_{\mathbb{H},0,\infty}(c\Gamma).$$

- $I_{D,a,b}(\Gamma) = 0$ *i* Γ is the hyperbolic geodesic connecting a and b.
- $I_{D,a,b}(\Gamma) < \infty$, then Γ is rectifiable *[Friz & Shekhar, 2015]*.

The chordal Loewner energy

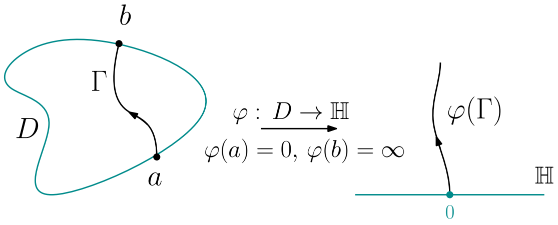


- The Loewner energy is well-defined in (D, a, b) since for $c > 0$,

$$I_{\mathbb{H},0,\infty}(\Gamma) = I_{\mathbb{H},0,\infty}(c\Gamma).$$

- $I_{D,a,b}(\Gamma) = 0$ *i* Γ is the hyperbolic geodesic connecting a and b.
- $I_{D,a,b}(\Gamma) < \infty$, then Γ is rectifiable *[Friz & Shekhar, 2015]*.

The chordal Loewner energy



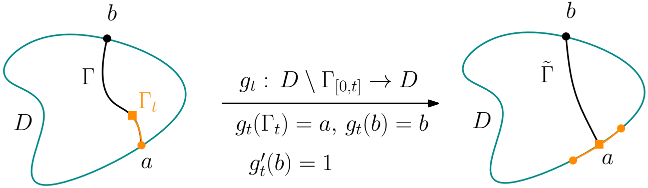
- The Loewner energy is well-defined in (D, a, b) since for $c > 0$,

$$I_{\mathbb{H},0,\infty}(\Gamma) = I_{\mathbb{H},0,\infty}(c\Gamma).$$

- $I_{D,a,b}(\Gamma) = 0$ *i* Γ is the hyperbolic geodesic connecting a and b.
- $I_{D,a,b}(\Gamma) < \infty$, then Γ is rectifiable *[Friz & Shekhar, 2015]*.

Upper half-plane vs. other domains

Assume that ∂D is smooth in a neighborhood of b, a continuously parametrized chord $\Gamma : [0, T] \to \overline{D}$ from a to b.



The **capacity parametrization** of Γ seen from b is chosen using the Schwarzian derivative of the mapping-out function:

$$\mathrm{cap}(\Gamma[0,t]) := -\frac{S(g_t)(b)}{12}.$$

The **driving function** is given by

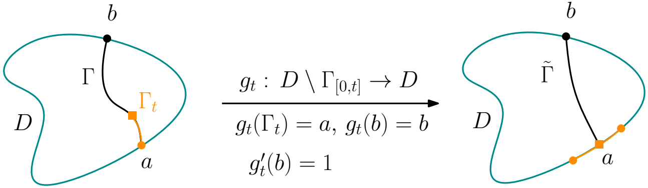
$$W_t = \frac{1}{2}\frac{g_t\ (b)}{g_t(b)}.$$

The **Loewner energy** is given by

$$I_{D,a,b}(\Gamma) = \sup_{0 \le T_0 < T_1 < \cdots < T_n = T} \sum_{i=0}^{n-1} \frac{(W_{T_{i+1}} - W_{T_i})^2}{\mathrm{cap}(\Gamma[0,T_{i+1}]) - \mathrm{cap}(\Gamma[0,T_i])}$$

Upper half-plane vs. other domains

Assume that ∂D is smooth in a neighborhood of b, a continuously parametrized chord $\Gamma : [0, T] \to \overline{D}$ from a to b.



The **capacity parametrization** of Γ seen from b is chosen using the Schwarzian derivative of the mapping-out function:

$$\mathrm{cap}(\Gamma[0,t]) := -\frac{S(g_t)(b)}{12}.$$

The **driving function** is given by

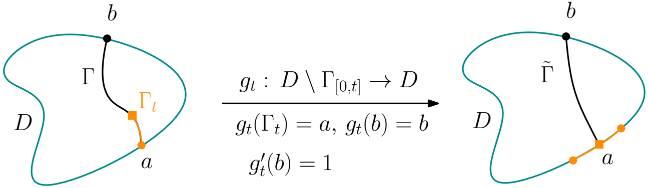
$$W_t = \frac{1}{2}\frac{g_t\ (b)}{g_t(b)}.$$

The **Loewner energy** is given by

$$I_{D,a,b}(\Gamma) = \sup_{0 \le T_0 < T_1 < \cdots < T_n = T} \sum_{i=0}^{n-1} \frac{(W_{T_{i+1}} - W_{T_i})^2}{\mathrm{cap}(\Gamma[0,T_{i+1}]) - \mathrm{cap}(\Gamma[0,T_i])}$$

Upper half-plane vs. other domains

Assume that ∂D is smooth in a neighborhood of b, a continuously parametrized chord $\Gamma : [0, T] \to \overline{D}$ from a to b.



The **capacity parametrization** of Γ seen from b is chosen using the Schwarzian derivative of the mapping-out function:

$$\mathrm{cap}(\Gamma[0,t]) := -\frac{S(g_t)(b)}{12}.$$

The **driving function** is given by

$$W_t = \frac{1}{2}\frac{g_t\ (b)}{g_t(b)}.$$

The **Loewner energy** is given by

$$I_{D,a,b}(\Gamma) = \sup_{0 \le T_0 < T_1 < \cdots < T_n = T} \sum_{i=0}^{n-1} \frac{(W_{T_{i+1}} - W_{T_i})^2}{\mathrm{cap}(\Gamma[0, T_{i+1}]) - \mathrm{cap}(\Gamma[0, T_i])}$$

SLE$_\kappa$ vs. Loewner energy

The Dirichlet energy $I(W)$ is the **action functional** of Brownian motion. Intuitively, the "Brownian path has the distribution on $C^0(\mathbb{R}_+, \mathbb{R})$ with density $\exp(-I(W))DW$."

However, $I(B) = \infty$ with probability 1.

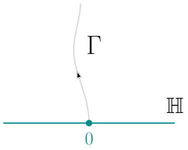
The Schilder's theorem states that $I(W)$ is also the **large deviation rate function** for Brownian motion $\sqrt{\kappa}B$ as $\kappa \to 0$. Loosely speaking,

$$\text{“}\mathbb{P}(\sqrt{\kappa}B \text{ stays close to } W) \approx \exp\left(-\frac{I(W)}{\kappa}\right).\text{”}$$

It should imply that the Loewner energy is the **large deviation rate function** of SLE$_\kappa$:

$$\text{“}\mathbb{P}(\text{SLE}_\kappa \text{ stays close to } \Gamma) \approx \exp\left(-\frac{I(\Gamma)}{\kappa}\right).\text{”} \tag{1}$$

The claim (1) is made precise in [W. 2016].



SLE$_\kappa$ vs. Loewner energy

The Dirichlet energy $I(W)$ is the **action functional** of Brownian motion. Intuitively, the "Brownian path has the distribution on $C^0(\mathbb{R}_+, \mathbb{R})$ with density $\exp(-I(W))DW$."

However, $I(B) = \infty$ with probability 1.

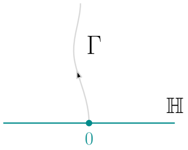
The Schilder's theorem states that $I(W)$ is also the **large deviation rate function** for Brownian motion $\sqrt{\kappa}B$ as $\kappa \to 0$. Loosely speaking,

$$\text{"}\mathbb{P}(\sqrt{\kappa}B \text{ stays close to } W) \approx \exp\left(-\frac{I(W)}{\kappa}\right).\text{"}$$

It should imply that the Loewner energy is the **large deviation rate function** of SLE$_\kappa$:

$$\text{"}\mathbb{P}(\text{SLE}_\kappa \text{ stays close to } \Gamma) \approx \exp\left(-\frac{I(\Gamma)}{\kappa}\right).\text{"} \quad (1)$$

The claim (1) is made precise in [W. 2016].



SLE_κ vs. Loewner energy

The Dirichlet energy $I(W)$ is the **action functional** of Brownian motion. Intuitively, the "Brownian path has the distribution on $C^0(\mathbb{R}_+, \mathbb{R})$ with density $\exp(-I(W))DW$."

$$\textit{However, } I(B) = \infty \textit{ with probability } 1.$$

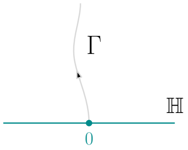
The Schilder's theorem states that $I(W)$ is also the **large deviation rate function** for Brownian motion $\sqrt{\kappa}B$ as $\kappa \to 0$. Loosely speaking,

$$\text{“}\mathrm{P}(\sqrt{\kappa}B \text{ stays close to } W) \approx \exp\left(-\frac{I(W)}{\kappa}\right).\text{”}$$

It should imply that the Loewner energy is the **large deviation rate function** of SLE_κ:

$$\text{“}\mathrm{P}(\mathrm{SLE}_\kappa \text{ stays close to } \Gamma) \approx \exp\left(-\frac{I(\Gamma)}{\kappa}\right).\text{”} \tag{1}$$

The claim (1) is made precise in [W. 2016].



SLE_κ vs. Loewner energy

The Dirichlet energy $I(W)$ is the **action functional** of Brownian motion. Intuitively, the "Brownian path has the distribution on $C^0(\mathbb{R}_+, \mathbb{R})$ with density $\exp(-I(W))DW$."

However, $I(B) = \infty$ with probability 1.

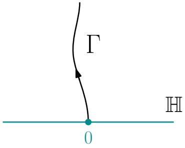
The Schilder's theorem states that $I(W)$ is also the **large deviation rate function** for Brownian motion $\sqrt{\kappa}B$ as $\kappa \to 0$. Loosely speaking,

$$\text{"}\mathbb{P}(\sqrt{\kappa}B \text{ stays close to } W) \approx \exp\left(-\frac{I(W)}{\kappa}\right).\text{"}$$

It should imply that the Loewner energy is the **large deviation rate function** of SLE_κ:

$$\text{"}\mathbb{P}(\text{SLE}_\kappa \text{ stays close to } \Gamma) \approx \exp\left(-\frac{I(\Gamma)}{\kappa}\right).\text{"} \qquad (1)$$

The claim (1) is made precise in [W. 2016].

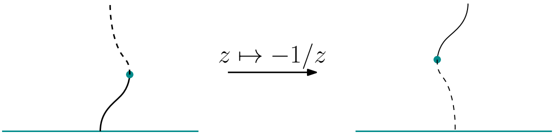


Reversibility of chordal Loewner energy

Theorem (W. 2016)

Let Γ be a simple chord in D connecting two boundary points a and b, we have

$$I_{D,a,b}(\Gamma) = I_{D,b,a}(\Gamma).$$



The deterministic result is based on

Theorem (Reversibility of SLE, Zhan 2008, Miller-Sheffield 2012)

For $\kappa \le 8$, the law of the trace of SLE_κ in (D, a, b), is the same as the law of SLE_κ in (D, b, a).

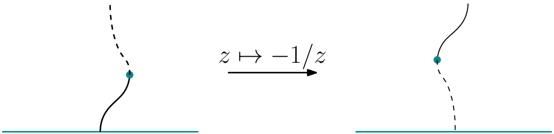
In fact, the decay rate as $\kappa \to 0$ of the probability of SLE_κ stays close to Γ is the same as the decay rate of being close to $-1/\Gamma$.

Reversibility of chordal Loewner energy

Theorem (W. 2016)

Let Γ be a simple chord in D connecting two boundary points a and b, we have

$$I_{D,a,b}(\Gamma) = I_{D,b,a}(\Gamma).$$



The deterministic result is based on

Theorem (Reversibility of SLE, Zhan 2008, Miller-Sheffield 2012)

For $\kappa \leq 8$, the law of the trace of SLE_κ in (D, a, b), is the same as the law of SLE_κ in (D, b, a).

In fact, the decay rate as $\kappa \to 0$ of the probability of SLE_κ stays close to Γ is the same as the decay rate of being close to $-1/\Gamma$.

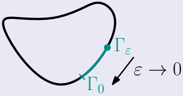
In fact, the Loewner energy has more symmetries.

Loewner loop energy

Definition (Rohde, W., 2017)

We define the **Loewner energy of a simple loop** $\Gamma : [0,1] \to \hat{\mathbb{C}}$ rooted at $\Gamma_0 = \Gamma_1$ to be

$$I^L(\Gamma, \Gamma_0) := \lim_{\varepsilon \to 0} I_{\hat{\mathbb{C}} \setminus \Gamma[0,\varepsilon], \Gamma_\varepsilon, \Gamma_0}(\Gamma[\varepsilon, 1]).$$



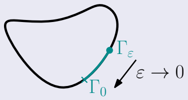
- $I^L(\Gamma, \Gamma_0) = 0$ if and only if Γ is a (round) circle.
- If $\Gamma[0, s]$ is a circular arc (including line segments), then the RHS is constant for $\varepsilon \leq s$, and $I^L(\Gamma, \Gamma_0)$ equals to the chordal energy $I_{\hat{\mathbb{C}} \setminus \Gamma[0,s], \Gamma_s, \Gamma_0}(\Gamma[s, 1])$.

Loewner loop energy

Definition (Rohde, W., 2017)

We define the **Loewner energy of a simple loop** $\Gamma : [0,1] \to \hat{\mathbb{C}}$ rooted at $\Gamma_0 = \Gamma_1$ to be

$$I^L(\Gamma, \Gamma_0) := \lim_{\varepsilon \to 0} I_{\hat{\mathbb{C}} \setminus \Gamma[0,\varepsilon], \Gamma_\varepsilon, \Gamma_0}(\Gamma[\varepsilon, 1]).$$



- $I^L(\Gamma, \Gamma_0) = 0$ if and only if Γ is a (round) circle.
- If $\Gamma[0, s]$ is a circular arc (including line segments), then the RHS is constant for $\varepsilon \le s$, and $I^L(\Gamma, \Gamma_0)$ equals to the chordal energy $I_{\hat{\mathbb{C}} \setminus \Gamma[0,s], \Gamma_s, \Gamma_0}(\Gamma[s, 1])$.

Root-invariance

Theorem (Rohde, W. 2017)

The Loewner loop energy is **independent** of the choice of root and orientation.

$\Longrightarrow$ I^L is invariant on the set of **free loops** under Möbius transformation;

$\Longrightarrow$ The loop setting is more natural than the chordal setting.

The proof is based on the reversibility of the chordal energy.

Moreover,

- $I^L(\Gamma) < \infty$, then Γ is a (rectifiable) quasicircle.
- If Γ is $C^{1.5+\varepsilon}$ for some $\varepsilon > 0$, then $I^L(\Gamma) < \infty$.

Root-invariance

Theorem (Rohde, W. 2017)

The Loewner loop energy is **independent** of the choice of root and orientation.

$=$ I^L is invariant on the set of **free loops** under Möbius transformation;

$=$ The loop setting is more natural than the chordal setting.

The proof is based on the reversibility of the chordal energy.

Moreover,

- $I^L(\Gamma) < \infty$, then Γ is a (rectifiable) quasicircle.
- If Γ is $C^{1.5+\varepsilon}$ for some $\varepsilon > 0$, then $I^L(\Gamma) < \infty$.

Root-invariance

Theorem (Rohde, W. 2017)

The Loewner loop energy is **independent** of the choice of root and orientation.

$\Longrightarrow$ I^L is invariant on the set of **free loops** under Möbius transformation;

$\Longrightarrow$ The loop setting is more natural than the chordal setting.

The proof is based on the reversibility of the chordal energy.

Moreover,

- $I^L(\Gamma) < \infty$, then Γ is a (rectifiable) quasicircle.
- If Γ is $C^{1.5+\varepsilon}$ for some $\varepsilon > 0$, then $I^L(\Gamma) < \infty$.

Root-invariance

Theorem (Rohde, W. 2017)

The Loewner loop energy is **independent** of the choice of root and orientation.

$\Longrightarrow$ I^L is invariant on the set of **free loops** under Möbius transformation;

$\Longrightarrow$ The loop setting is more natural than the chordal setting.

The proof is based on the reversibility of the chordal energy.

Moreover,

- $I^L(\Gamma) < \infty$, then Γ is a (rectifiable) quasicircle.
- If Γ is $C^{1.5+\varepsilon}$ for some $\varepsilon > 0$, then $I^L(\Gamma) < \infty$.

Zeta-regularizated determinants

- The Zeta-regularization of determinants is first introduced by *Ray & Singer (1976)*.
- *Hawking (1977)* has pointed out that it allows to regularize quadratic path integrals.
- *Osgood, Phillips & Sarnak (1988)* have shown that the results obtained by comparing two functional determinants of Laplacian in the QFT formalism agree with the results obtained by the zeta-regularized determinant (**Polyakov-Alvarez conformal anomaly formula**).

Zeta-regularizated determinants

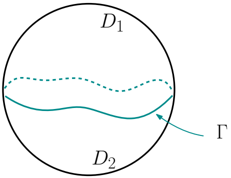
- The Zeta-regularization of determinants is first introduced by *Ray & Singer (1976)*.
- *Hawking (1977)* has pointed out that it allows to regularize quadratic path integrals.
- *Osgood, Phillips & Sarnak (1988)* have shown that the results obtained by comparing two functional determinants of Laplacian in the QFT formalism agree with the results obtained by the zeta-regularized determinant (**Polyakov-Alvarez conformal anomaly formula**).

Zeta-regularizated determinants

- The Zeta-regularization of determinants is first introduced by *Ray & Singer (1976)*.
- *Hawking (1977)* has pointed out that it allows to regularize quadratic path integrals.
- *Osgood, Phillips & Sarnak (1988)* have shown that the results obtained by comparing two functional determinants of Laplacian in the QFT formalism agree with the results obtained by the zeta-regularized determinant (**Polyakov-Alvarez conformal anomaly formula**).

The functional $\mathcal{H}$

- $g_0(z) = \frac{4}{(1+|z|^2)^2} \, dz^2$ denotes the spherical metric;
- $g = e^{2\varphi} g_0$ be a metric conformally equivalent to g_0;
- Γ a C^∞ **smooth** simple loop in C $\{\infty\}$ S^2;
- D_1 and D_2 two connected components $S^2 \setminus \Gamma$;
- $\Delta_g(D_i)$ the Laplace-Beltrami operator with Dirichlet boundary condition on D_i.



Definition

Let $\det_\zeta$ be the ζ-regularized determinant, we introduce

$$H(\Gamma, g) := \log \det_\zeta \Delta_g(S^2) - \log \mathrm{Area}_g(S^2) - \log \det_\zeta \Delta_g(D_1) - \log \det_\zeta \Delta_g(D_2).$$

Loewner Energy vs. Determinants

$$H(\Gamma, g) = \log \det{}_\zeta \Delta_g(S^2) - \log \mathrm{Area}_g(S^2) - \log \det{}_\zeta \Delta_g(D_1) - \log \det{}_\zeta \Delta_g(D_2).$$

Theorem (W., 2018)

If $g = e^{2\varphi} g_0$ is a metric conformally equivalent to the spherical metric g_0 on S^2, then:

1. $H(\cdot, g) = H(\cdot, g_0)$
2. Circles minimize $H(\cdot, g)$ among all C^∞ smooth Jordan curves.
3. Let Γ be a smooth Jordan curve on S^2. We have the identity

$$\begin{aligned} I^L(\Gamma, \Gamma(0)) &= 12H(\Gamma, g) - 12H(S^1, g) \\ &= 12 \log \frac{\det_\zeta(-\Delta_g(\mathrm{D}_1))\det_\zeta(-\Delta_g(\mathrm{D}_2))}{\det_\zeta(-\Delta_g(D_1))\det_\zeta(-\Delta_g(D_2))}, \end{aligned}$$

   where D_1 and D_2 are two connected components of the complement of S^1.

In particular, the above identity gives already the parametrization independence of the Loewner loop energy for smooth loops.

Loewner Energy vs. Determinants

$$H(\Gamma, g) = \log \det{}_\zeta \Delta_g(S^2) - \log \mathrm{Area}_g(S^2) - \log \det{}_\zeta \Delta_g(D_1) - \log \det{}_\zeta \Delta_g(D_2).$$

Theorem (W., 2018)

If $g = e^{2\varphi} g_0$ is a metric conformally equivalent to the spherical metric g_0 on S^2, then:

1. $H(\cdot, g) = H(\cdot, g_0)$
2. Circles minimize $H(\cdot, g)$ among all C^∞ smooth Jordan curves.
3. Let Γ be a smooth Jordan curve on S^2. We have the identity

$$\begin{aligned} I^L(\Gamma, \Gamma(0)) &= 12 H(\Gamma, g) - 12 H(S^1, g) \\ &= 12 \log \frac{\det_\zeta(-\Delta_g(\mathrm{D}_1))\det_\zeta(-\Delta_g(\mathrm{D}_2))}{\det_\zeta(-\Delta_g(D_1))\det_\zeta(-\Delta_g(D_2))}, \end{aligned}$$

where D_1 and D_2 are two connected components of the complement of S^1.

In particular, the above identity gives already the parametrization independence of the Loewner loop energy for smooth loops.

Remarks

- The regularity assumption on the curve is due to the constraint from the zeta-regularization and its variation formula [OPS].
- Picking different metrics g provide a wide range of identities with the Loewner energy that usually look different in their expression involving scalar curvatures, geodesic curvatures, conformal maps $D_1 \to \mathrm{D}_1$, etc., (but of course they are equal).
- One of the identities links to the Weil-Petersson class of the universal Teichmüller space.

Remarks

- The regularity assumption on the curve is due to the constraint from the zeta-regularization and its variation formula [OPS].
- Picking different metrics g provide a wide range of identities with the Loewner energy that usually look different in their expression involving scalar curvatures, geodesic curvatures, conformal maps $D_1 \to \mathrm{D}_1$, etc., (but of course they are equal).
- One of the identities links to the Weil-Petersson class of the universal Teichmüller space.

Remarks

- The regularity assumption on the curve is due to the constraint from the zeta-regularization and its variation formula [OPS].
- Picking different metrics g provide a wide range of identities with the Loewner energy that usually look different in their expression involving scalar curvatures, geodesic curvatures, conformal maps $D_1 \to \mathrm{D}_1$, etc., (but of course they are equal).
- One of the identities links to the Weil-Petersson class of the universal Teichmüller space.

Universal Teichmüller space

- $QS(S^1)$ the group of quasisymmetric sense-preserving homeomorphism of S^1;

A sense-preserving homeomorphism $\varphi : S^1 \to S^1$ is **quasisymmetric** if there exists $M \geq 1$ such that for all $\theta \quad \mathbb{R}$ and $t \quad (0, \pi)$,

$$\frac{1}{M} \leq \left| \frac{\varphi(e^{i(\theta+t)}) - \varphi(e^{i\theta})}{\varphi(e^{i\theta}) - \varphi(e^{i(\theta-t)})} \right| \leq M.$$

- $\text{Möb}(S^1) \quad \text{PSL}(2, \mathbb{R})$ the subgroup of Möbius function of S^1.

The **universal Teichmüller space** is

$$T(1) := QS(S^1)/\text{Möb}(S^1) \quad \{\varphi \quad QS(S^1), \varphi \text{ fixes } -1, -i \text{ and } 1\}.$$

It can be modeled by **Beltrami coefficients** as well:

$$T(1) = L^\infty(\mathbb{D}, \mathbb{C})_1 / \quad ,$$

where

$$\mu \quad _\infty < 1, \; \nu \quad _\infty < 1, \quad \mu \quad \nu \Leftrightarrow w_\mu|_{S^1} = w_\nu|_{S^1}$$

w_μ is the normalized solution (fixes $-1, -i, 1$) $\mathbb{D} \to \mathbb{D}$ to the Beltrami equation

$$\bar{\partial} w_\mu(z) = \mu(z) \partial w_\mu(z).$$

Universal Teichmüller space

- $QS(S^1)$ the group of quasisymmetric sense-preserving homeomorphism of S^1;

A sense-preserving homeomorphism $\varphi : S^1 \to S^1$ is **quasisymmetric** if there exists $M \geq 1$ such that for all θ R and t $(0, \pi)$,

$$\frac{1}{M} \leq \left| \frac{\varphi(e^{i(\theta+t)}) - \varphi(e^{i\theta})}{\varphi(e^{i\theta}) - \varphi(e^{i(\theta-t)})} \right| \leq M.$$

- $\text{Möb}(S^1)$ $\quad$ $\text{PSL}(2, \text{R})$ the subgroup of Möbius function of S^1.

The **universal Teichmüller space** is

$$T(1) := QS(S^1)/\text{Möb}(S^1) \quad \{\varphi \quad QS(S^1), \varphi \text{ fixes } -1, -i \text{ and } 1\}.$$

It can be modeled by **Beltrami coefficients** as well:

$$T(1) = L^\infty(\text{D}, \text{C})_1 / \quad ,$$

where

$$\mu \quad _\infty < 1, \quad \nu \quad _\infty < 1, \quad \mu \quad \nu \Leftrightarrow w_\mu|_{S^1} = w_\nu|_{S^1}$$

w_μ is the normalized solution (fixes $-1, -i, 1$) D $\to$ D to the Beltrami equation

$$\bar{\partial} w_\mu(z) = \mu(z) \partial w_\mu(z).$$

Universal Teichmüller space

- $QS(S^1)$ the group of quasisymmetric sense-preserving homeomorphism of S^1;

A sense-preserving homeomorphism $\varphi : S^1 \to S^1$ is **quasisymmetric** if there exists $M \geq 1$ such that for all θ R and t $(0, \pi)$,

$$\frac{1}{M} \leq \left| \frac{\varphi(e^{i(\theta+t)}) - \varphi(e^{i\theta})}{\varphi(e^{i\theta}) - \varphi(e^{i(\theta-t)})} \right| \leq M.$$

- $\text{Möb}(S^1)$ $\text{PSL}(2, \mathbb{R})$ the subgroup of Möbius function of S^1.

The **universal Teichmüller space** is

$$T(1) := QS(S^1)/\text{Möb}(S^1) \quad \{\varphi \quad QS(S^1), \varphi \text{ fixes } -1, -i \text{ and } 1\}.$$

It can be modeled by **Beltrami coefficients** as well:

$$T(1) = L^\infty(\mathbb{D}, \mathbb{C})_1 / \quad ,$$

where

$$\mu \quad {}_\infty < 1, \quad \nu \quad {}_\infty < 1, \quad \mu \quad \nu \Leftrightarrow w_\mu|_{S^1} = w_\nu|_{S^1}$$

w_μ is the normalized solution (fixes $-1, -i, 1$) $\mathbb{D} \to \mathbb{D}$ to the Beltrami equation

$$\bar\partial w_\mu(z) = \mu(z) \partial w_\mu(z).$$

Universal Teichmüller space

- $QS(S^1)$ the group of quasisymmetric sense-preserving homeomorphism of S^1;

A sense-preserving homeomorphism $\varphi : S^1 \to S^1$ is **quasisymmetric** if there exists $M \geq 1$ such that for all $\theta \quad \mathbb{R}$ and $t \quad (0, \pi)$,

$$\frac{1}{M} \leq \left| \frac{\varphi(e^{i(\theta+t)}) - \varphi(e^{i\theta})}{\varphi(e^{i\theta}) - \varphi(e^{i(\theta-t)})} \right| \leq M.$$

- $\text{Möb}(S^1) \quad \text{PSL}(2, \mathbb{R})$ the subgroup of Möbius function of S^1.

The **universal Teichmüller space** is

$$T(1) := QS(S^1)/\text{Möb}(S^1) \quad \{\varphi \quad QS(S^1), \varphi \text{ fixes } -1, -i \text{ and } 1\}.$$

It can be modeled by **Beltrami coefficients** as well:

$$T(1) = L^{\infty}(\mathbb{D}, \mathbb{C})_1 / \quad ,$$

where

$$\mu \quad _\infty < 1, \quad \nu \quad _\infty < 1, \quad \mu \quad \nu \Leftrightarrow w_\mu|_{S^1} = w_\nu|_{S^1}$$

w_μ is the normalized solution (fixes $-1, -i, 1$) $\mathbb{D} \to \mathbb{D}$ to the Beltrami equation

$$\bar{\partial} w_\mu(z) = \mu(z) \partial w_\mu(z).$$

Universal Teichmüller space

- $QS(S^1)$ the group of quasisymmetric sense-preserving homeomorphism of S^1;

A sense-preserving homeomorphism $\varphi : S^1 \to S^1$ is **quasisymmetric** if there exists $M \geq 1$ such that for all $\theta \quad$ R and $t \quad (0, \pi)$,

$$\frac{1}{M} \leq \left| \frac{\varphi(e^{i(\theta+t)}) - \varphi(e^{i\theta})}{\varphi(e^{i\theta}) - \varphi(e^{i(\theta-t)})} \right| \leq M.$$

- Möb(S^1) $\quad$ PSL(2, R) the subgroup of Möbius function of S^1.

The **universal Teichmüller space** is

$$T(1) := QS(S^1)/\text{Möb}(S^1) \quad \{\varphi \quad QS(S^1), \varphi \text{ fixes } -1, -i \text{ and } 1\}.$$

It can be modeled by **Beltrami coefficients** as well:

$$T(1) = L^{\infty}(\mathrm{D}, \mathrm{C})_1 / \quad ,$$

where

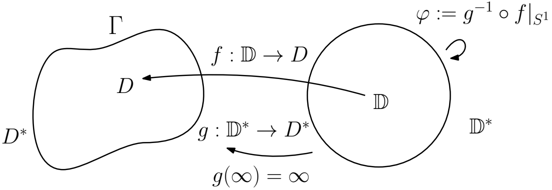
$$\mu \quad _{\infty} < 1, \;\; \nu \quad _{\infty} < 1, \quad \mu \quad \nu \Leftrightarrow w_\mu|_{S^1} = w_\nu|_{S^1}$$

w_μ is the normalized solution (fixes $-1, -i, 1$) D $\to$ D to the Beltrami equation

$$\bar{\partial} w_\mu(z) = \mu(z) \partial w_\mu(z).$$

Welding function

- Associate Γ with its **welding function** φ:



[Rohde, W. 2017]: $I^L(\Gamma) < \infty$ Γ is a quasicircle ⇔ φ $QS(S^1)$.

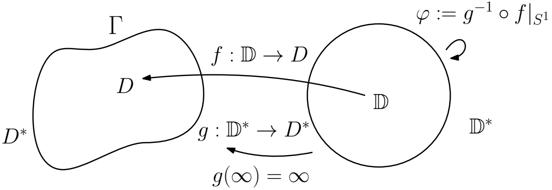
- “ ” is not true, there are quasicircles with ∞ Loewner energy.

Question

What is the class of finite energy loops in $T(1)$?

Welding function

- Associate Γ with its **welding function** φ:



[Rohde, W. 2017]: $I^L(\Gamma) < \infty$ Γ is a quasicircle $\Leftrightarrow \varphi$ $QS(S^1)$.

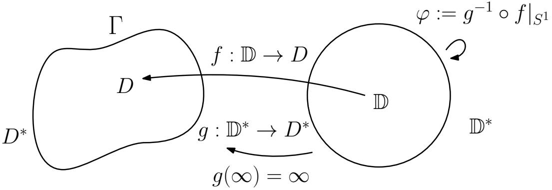
- " " is not true, there are quasicircles with ∞ Loewner energy.

Question

What is the class of finite energy loops in $T(1)$?

Welding function

- Associate Γ with its **welding function** φ:



[Rohde, W. 2017]: $I^L(\Gamma) < \infty$ Γ is a quasicircle ⇔ φ $QS(S^1)$.

- “ ” is not true, there are quasicircles with ∞ Loewner energy.

Question

What is the class of finite energy loops in $T(1)$?

- The homogeneous space of C^∞-smooth diffeomorphisms

$$M := \mathrm{Diff}(S^1)/\mathrm{M\ddot{o}b}(S^1) \qquad T(1)$$

has a Kähler structure ***[Witten, Bowick, Rajeev, etc.]***.

- There is a **unique** homogeneous Kähler metric (up to constant factor): the **Weil-Petersson metric**.

Weil-Petersson Class

- The homogeneous space of C^∞-smooth diffeomorphisms

$$M := \mathrm{Diff}(S^1)/\mathrm{Möb}(S^1) \qquad T(1)$$

  has a Kähler structure *[Witten, Bowick, Rajeev, etc.]*.
- There is a **unique** homogeneous Kähler metric (up to constant factor): the **Weil-Petersson metric**.

Weil-Petersson metric

The tangent space at id of M consists of C^∞ vector fields on S^1:

$$v = v(\theta)\frac{\partial}{\partial\theta} = \sum_{m \in \mathbb{Z}\setminus\{-1,0,1\}} v_m e^{im\theta}\frac{\partial}{\partial\theta}, \text{ where } v_{-m} = \overline{v_m}.$$

The almost complex structure $J^2 = -Id$ is given by the Hilbert transform:

$$J(v)_m = -i\,\mathrm{sgn}(m) v_m, \text{ for } m \in \mathbb{Z}\setminus\{-1,0,1\}.$$

In particular,

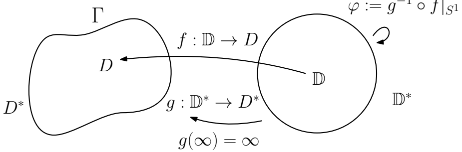
$$J\left(\cos(m\theta)\frac{\partial}{\partial\theta}\right) = \sin(m\theta)\frac{\partial}{\partial\theta}; \quad J\left(\sin(m\theta)\frac{\partial}{\partial\theta}\right) = -\cos(m\theta)\frac{\partial}{\partial\theta}.$$

The Weil-Petersson symplectic form $\omega(\cdot,\cdot)$ and the Riemannian metric $\langle\cdot,\cdot\rangle_{WP}$ is given at the origin by

$$\omega(v,w) = i\sum_{m \in \mathbb{Z}\setminus\{-1,0,1\}} (m^3 - m) v_m w_{-m},$$

$$\langle v, w\rangle_{WP} = \omega(v, J(w)) = \sum_{m=2}^{\infty}(m^3 - m)\,\mathrm{Re}(v_m w_{-m}).$$

- **Weil-Petersson Teichmüller space** $T_0(1)$ is the closure of $\mathrm{Diff}(S^1)/\mathrm{M\ddot{o}b}(S^1) \quad T(1)$ under the WP-metric. **Weil-Petersson class** $\mathrm{WP}(S^1) \quad QS(S^1)$ are homeomorphisms representing points in $T_0(1)$.
- The above description and many other characterizations are provided by *[Nag, Verjovski, Sullivan, Cui, Takhtajan, Teo, Shen, etc]*.



Theorem (Takhtajan & Teo, 2006)

The **universal Liouville action $\mathbf{S}_1$** : $T_0(1) \to \mathrm{R}$,

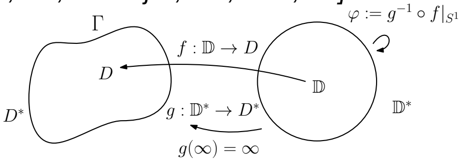
$$\mathbf{S}_1([\varphi]) := \int_{\mathrm{D}} \left| \frac{f}{f}(z) \right|^2 \mathrm{d}z^2 + \int_{\mathrm{D}} \left| \frac{g}{g}(z) \right|^2 \mathrm{d}z^2 + 4\pi \log \left| \frac{f\ (0)}{g\ (\infty)} \right|$$

is a Kähler potential of the Weil-Petersson metric, where

$$g\ (\infty) = \lim_{z \to \infty} g\ (z) = \tilde{g}\ (0)^{-1} \text{ and } \tilde{g}(z) = 1/g(1/z).$$

Weil-Petersson Class

- **Weil-Petersson Teichmüller space** $T_0(1)$ is the closure of $\mathrm{Diff}(S^1)/\mathrm{Möb}(S^1)$ $\quad T(1)$ under the WP-metric. **Weil-Petersson class** $\mathrm{WP}(S^1)$ $\quad QS(S^1)$ are homeomorphisms representing points in $T_0(1)$.
- The above description and many other characterizations are provided by *[Nag, Verjovski, Sullivan, Cui, Takhtajan, Teo, Shen, etc]*.



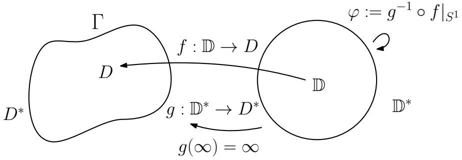
Theorem (Takhtajan & Teo, 2006)

The **universal Liouville action $\mathbf{S}_1$** : $T_0(1) \to \mathrm{R}$,

$$\mathbf{S}_1([\varphi]) := \int_{\mathrm{D}} \left|\frac{f}{f}(z)\right|^2 \mathrm{d}z^2 + \int_{\mathrm{D}} \left|\frac{g}{g}(z)\right|^2 \mathrm{d}z^2 + 4\pi \log \left|\frac{f\ (0)}{g\ (\infty)}\right|$$

is a Kähler potential of the Weil-Petersson metric, where

$$g\ (\infty) = \lim_{z \to \infty} g\ (z) = \tilde{g}\ (0)^{-1} \text{ and } \tilde{g}(z) = 1/g(1/z).$$



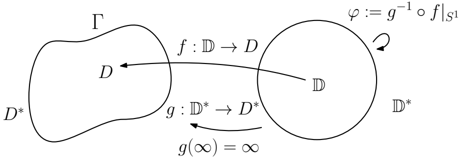
Theorem (W. 2018)

A bounded simple loop Γ in $\hat{\mathbb{C}}$ has finite Loewner energy *if and only if* $[\varphi] \quad T_0(1)$. Moreover,

$$I^L(\Gamma) = \mathbf{S}_1([\varphi])/\pi.$$

- There is no regularity assumption on the loop for the identity to hold.
- This gives a new characterization of the WP-Class, and a new viewpoint on the Kähler potential on $T_0(1)$ (or alternatively a way to look at the Loewner energy).

Loewner Energy vs. Weil-Petersson Class



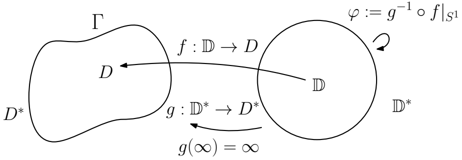
Theorem (W. 2018)

A bounded simple loop Γ in $\hat{\mathbb{C}}$ has finite Loewner energy *if and only if* $[\varphi] \quad T_0(1)$. Moreover,

$$I^L(\Gamma) = \mathbf{S_1}([\varphi])/\pi.$$

- There is no regularity assumption on the loop for the identity to hold.
- This gives a new characterization of the WP-Class, and a new viewpoint on the Kähler potential on $T_0(1)$ (or alternatively a way to look at the Loewner energy).

Loewner Energy vs. Weil-Petersson Class



Theorem (W. 2018)

A bounded simple loop Γ in $\hat{\mathbb{C}}$ has finite Loewner energy *if and only if* $[\varphi]$ $T_0(1)$. Moreover,

$$I^L(\Gamma) = \mathbf{S}_1([\varphi])/\pi.$$

- There is no regularity assumption on the loop for the identity to hold.
- This gives a new characterization of the WP-Class, and a new viewpoint on the Kähler potential on $T_0(1)$ (or alternatively a way to look at the Loewner energy).

Characterizations of the WP-Class (an incomplete list)

[Nag, Verjovsky, Sullivan, Cui, Taktajan, Teo, Shen, etc.] The following are equivalent:

- The welding function φ is in Weil-Petersson class;
- $\int_{D} |\ \ \log|f\ \ (z)||^2\, dz^2 = \int_{D} |f\ \ (z)/f\ \ (z)|^2\, dz^2 < \infty$;
- $\int_{D}\ \ |g\ \ (z)/g\ (z)|^2\, dz^2 < \infty$;
- $\int_{D} |S(f)|^2 \rho^{-1}(z)\, dz^2 < \infty$;
- $\int_{D}\ \ |S(g)|^2 \rho^{-1}(z)\, dz^2 < \infty$;
- φ has quasiconformal extension to D, whose complex dilation $\mu = \partial_{\bar{z}}\varphi / \partial_z \varphi$ satisfies

$$\int_{D} |\mu(z)|^2 \rho(z)\, dz^2 < \infty;$$

- φ is absolutely continuous with respect to arc-length measure, such that $\log|\varphi\ |$ belongs to the Sobolev space $H^{1/2}(S^1)$;
- Grunsky operator associated to f or g is Hilbert-Schmidt,

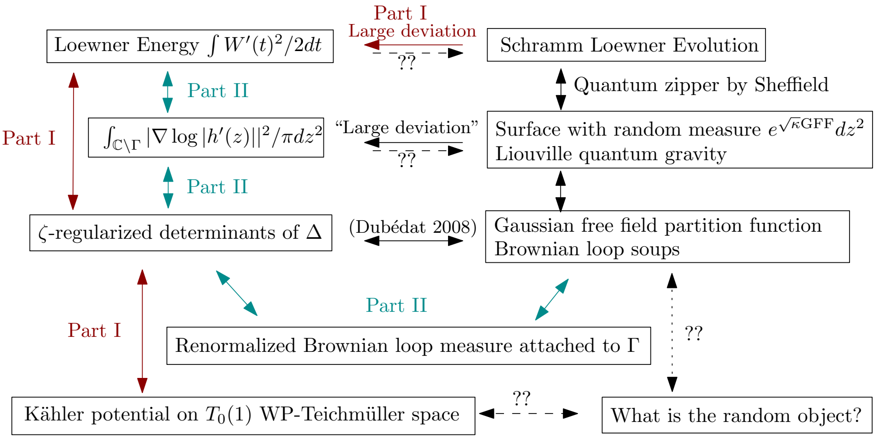
where $\rho(z)\, dz^2 = 1/(1 - |z|^2)^2\, dz^2$ is the hyperbolic metric on D or D and

$$S(f) = \frac{f}{f} - \frac{3}{2}\left(\frac{f}{f}\right)^2$$

is the Schwarzian derivative of f.

Contents

Action functionals vs. Random objects



Loewner Energy vs. Determinants

Recall $H(\Gamma, g) = \log \det_\zeta \Delta_g(S^2) - \log \mathrm{Area}_g(S^2) - \log \det_\zeta \Delta_g(D_1) - \log \det_\zeta \Delta_g(D_2)$.

Theorem (W., 2018)

If $g = e^{2\varphi} g_0$ is a metric conformally equivalent to the spherical metric g_0 on S^2, then:

1. $H(\cdot, g) = H(\cdot, g_0)$
2. Circles minimize $H(\cdot, g)$ among all C^∞ smooth Jordan curves.
3. Let Γ be a smooth Jordan curve on S^2. We have the identity

$$\begin{aligned} I^L(\Gamma, \Gamma(0)) &= 12 H(\Gamma, g) - 12 H(S^1, g) \\ &= 12 \log \frac{\det_\zeta(-\Delta_g(\mathrm{D}_1))\det_\zeta(-\Delta_g(\mathrm{D}_2))}{\det_\zeta(-\Delta_g(D_1))\det_\zeta(-\Delta_g(D_2))}, \end{aligned}$$

where D_1 and D_2 are two connected components of the complement of S^1.

Zeta-regularizated determinants

- $\Delta_g(S^2)$ is non-negative, essentially self-adjoint for the L^2 product.
- The spectrum is

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \cdots$$

- Define the Zeta-function

$$\zeta_\Delta(s) := \sum_{i \geq 1} \lambda_i^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty \mathrm{Tr}(e^{-t\Delta}) t^{s-1} dt,$$

it can be analytically continued to a neighborhood of 0.
- Define (following Ray & Singer 1976)

$$\log \det{}_\zeta(\Delta_g(S^2)) := -\zeta_\Delta(0)$$
$$\text{``} = \sum_{i \geq 1} \log(\lambda_i)\lambda_i^{-s}|_{s=0} = \log(\prod_{i \geq 1} \lambda_i).\text{''}$$

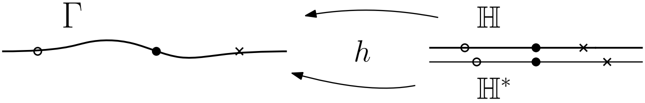
Proof of the identity (sketch)

$$I^L(\Gamma, \Gamma(0)) = 12 \log \frac{\det_\zeta(-\Delta_{\mathrm{D}_1, g_0})\det_\zeta(-\Delta_{\mathrm{D}_2, g_0})}{\det_\zeta(-\Delta_{D_1, g_0})\det_\zeta(-\Delta_{D_2, g_0})}$$

- When Γ passes through ∞, we show

$$I^L(\Gamma, \infty) = D_{\mathrm{H} \quad \mathrm{H}}(\log |h \;|) := \frac{1}{\pi}\left(\int_{\mathrm{H} \quad \mathrm{H}} | \quad \log |h \;(z)|\Big|^2 \, \mathrm{d}z^2\right),$$

where h maps conformally H H to the complement of Γ and fixes ∞.



The right-hand side does not involve Loewner iteration of conformal maps.

- Use the Polyakov-Alvarez conformal anomaly formula to compare determinants of Laplacians.

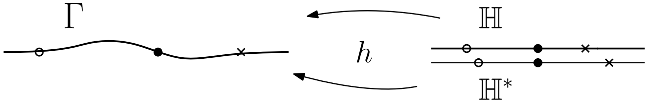
Proof of the identity (sketch)

$$I^L(\Gamma, \Gamma(0)) = 12 \log \frac{\det_\zeta(-\Delta_{D_1,g_0})\det_\zeta(-\Delta_{D_2,g_0})}{\det_\zeta(-\Delta_{D_1,g_0})\det_\zeta(-\Delta_{D_2,g_0})}$$

- When Γ passes through ∞, we show

$$I^L(\Gamma, \infty) = D_{H \quad H}(\log |h \ |) := \frac{1}{\pi}\left(\int_{H \quad H} | \quad \log |h \ (z)||^2 \, dz^2\right),$$

where h maps conformally H H to the complement of Γ and fixes ∞.



The right-hand side does not involve Loewner iteration of conformal maps.

- Use the Polyakov-Alvarez conformal anomaly formula to compare determinants of Laplacians.

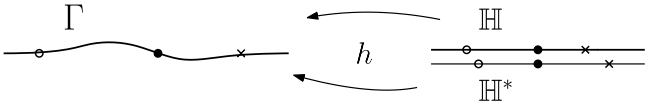
Proof of the identity (sketch)

$$I^L(\Gamma,\Gamma(0)) = 12 \log \frac{\det_\zeta(-\Delta_{\mathrm{D}_1,g_0})\det_\zeta(-\Delta_{\mathrm{D}_2,g_0})}{\det_\zeta(-\Delta_{D_1,g_0})\det_\zeta(-\Delta_{D_2,g_0})}$$

- When Γ passes through ∞ , we show

$$I^L(\Gamma,\infty) = D_{\mathrm{H}\quad\mathrm{H}}\ (\log \left|h\ \right|) := \frac{1}{\pi}\left(\int_{\mathrm{H}\quad\mathrm{H}} \left|\quad \log|h\ (z)|\right|^2 \mathrm{d}z^2\right),$$

where h maps conformally H H to the complement of Γ and fixes ∞ .



The right-hand side does not involve Loewner iteration of conformal maps.

- Use the Polyakov-Alvarez conformal anomaly formula to compare determinants of Laplacians.

Polyakov-Alvarez conformal anomaly formula

Take $g = e^{2\sigma} g_0$ a metric conformally equivalent to g_0. (Here think $\sigma = \log |h\ |$.)

Theorem ([Polyakov 1981], [Alvarez 1983], [Osgood, et al. 1988])

For a compact surface M without boundary,

$$\left(\log \det_\zeta(-\Delta_g) - \log \mathrm{vol}_g(M)\right) - \left(\log \det_\zeta(-\Delta_0) - \log \mathrm{vol}_0(M)\right)$$
$$= -\frac{1}{6\pi}\left[\frac{1}{2}\int_M |\ \ _0\sigma|^2 \,\mathrm{dvol}_0 + \int_M K_0 \sigma \,\mathrm{dvol}_0.\right]$$

The analogue for a compact surface D with smooth boundary is:

$$\log \det_\zeta(-\Delta_g) - \log \det_\zeta(-\Delta_0)$$
$$= -\frac{1}{6\pi}\left[\frac{1}{2}\int_D |\ \ _0\sigma|^2 \,\mathrm{dvol}_0 + \int_D K_0\sigma \,\mathrm{dvol}_0 + \int_{\partial D} k_0 \sigma \,\mathrm{dl}_0\right] - \frac{1}{4\pi}\int_{\partial D} \partial_n \sigma \,\mathrm{dl}_0.$$

"Taking $g_0 = dz^2$", we have $K_0 \equiv 0$ and $k_0 \equiv 0$. We get:

$$I^L(\Gamma, \Gamma(0)) = \frac{1}{\pi}\left(\int_{\mathrm{H}\ \ \mathrm{H}} \left|\ \ \log|h\ (z)|\right|^2 \,dz^2\right) = 12H(\Gamma, g_0) - 12H(S^1, g_0). \quad \square$$

Polyakov-Alvarez conformal anomaly formula

Take $g = e^{2\sigma} g_0$ a metric conformally equivalent to g_0. (Here think $\sigma = \log |h\ |$.)

Theorem ([Polyakov 1981], [Alvarez 1983], [Osgood, et al. 1988])

For a compact surface M without boundary,

$$\Big(\log \det_\zeta(-\Delta_g) - \log \mathrm{vol}_g(M)\Big) - \Big(\log \det_\zeta(-\Delta_0) - \log \mathrm{vol}_0(M)\Big)$$
$$= -\frac{1}{6\pi}\left[\frac{1}{2}\int_M |\ \ _0\sigma|^2 \,\mathrm{dvol}_0 + \int_M K_0 \sigma \,\mathrm{dvol}_0.\right]$$

The analogue for a compact surface D with smooth boundary is:

$$\log \det_\zeta(-\Delta_g) - \log \det_\zeta(-\Delta_0)$$
$$= -\frac{1}{6\pi}\left[\frac{1}{2}\int_D |\ \ _0\sigma|^2 \,\mathrm{dvol}_0 + \int_D K_0 \sigma \,\mathrm{dvol}_0 + \int_{\partial D} k_0 \sigma \,\mathrm{dl}_0\right] - \frac{1}{4\pi}\int_{\partial D} \partial_n \sigma \,\mathrm{dl}_0.$$

"Taking $g_0 = dz^2$", we have $K_0 \equiv 0$ and $k_0 \equiv 0$. We get:

$$I^L(\Gamma, \Gamma(0)) = \frac{1}{\pi}\left(\int_{\mathrm{H} \quad \mathrm{H}} \left|\quad \log |h\ (z)|\right|^2 \mathrm{d}z^2\right) = 12H(\Gamma, g_0) - 12H(S^1, g_0). \quad \square$$

Brownian loop measure

Introduced by Greg Lawler and Wendelin Werner.

[Following J. Dubédat] Let $x \quad M$, $t > 0$, consider the sub-probability measure W_x^t on the path of Brownian motion (diffusion generated by $-\Delta_M$) on M started from x on the time interval $[0, t]$, killed if it hits the boundary of M.
The measures $\mathrm{W}_{x \to y}^t$ on paths from x to y are obtained from the disintegration of W_x^t according to its endpoint y:

$$\mathrm{W}_x^t = \int_M \mathrm{W}_{x \to y}^t \,\mathrm{dvol}(y).$$

Define the **Brownian loop measure** on M:

$$\mu_M^{loop} := \int_0^\infty \frac{\mathrm{d}t}{t} \int_M \mathrm{W}_{x \to x}^t \,\mathrm{dvol}(x).$$

In particular,

$$\left|\mathrm{W}_{x \to x}^t\right| = p_t(x, x).$$

We consider μ_M^{loop} as measure on **unrooted** Brownian loops by forgetting the starting point.

Brownian loop measure

Introduced by Greg Lawler and Wendelin Werner.

[Following J. Dubédat] Let $x \quad M$, $t > 0$, consider the sub-probability measure W_x^t on the path of Brownian motion (diffusion generated by $-\Delta_M$) on M started from x on the time interval $[0, t]$, killed if it hits the boundary of M.
The measures $\mathrm{W}_{x \to y}^t$ on paths from x to y are obtained from the disintegration of W_x^t according to its endpoint y:

$$\mathrm{W}_x^t = \int_M \mathrm{W}_{x \to y}^t \, \mathrm{dvol}(y).$$

Define the **Brownian loop measure** on M:

$$\mu_M^{loop} := \int_0^\infty \frac{\mathrm{d}t}{t} \int_M \mathrm{W}_{x \to x}^t \, \mathrm{dvol}(x).$$

In particular,

$$\left| \mathrm{W}_{x \to x}^t \right| = p_t(x, x).$$

We consider μ_M^{loop} as measure on **unrooted** Brownian loops by forgetting the starting point.

Brownian loop measure

Introduced by Greg Lawler and Wendelin Werner.

[Following J. Dubédat] Let x M, $t > 0$, consider the sub-probability measure W_x^t on the path of Brownian motion (diffusion generated by $-\Delta_M$) on M started from x on the time interval $[0, t]$, killed if it hits the boundary of M.
The measures $\mathrm{W}_{x \to y}^t$ on paths from x to y are obtained from the disintegration of W_x^t according to its endpoint y:

$$\mathrm{W}_x^t = \int_M \mathrm{W}_{x \to y}^t \, \mathrm{dvol}(y).$$

Define the **Brownian loop measure** on M:

$$\mu_M^{loop} := \int_0^\infty \frac{\mathrm{d}t}{t} \int_M \mathrm{W}_{x \to x}^t \, \mathrm{dvol}(x).$$

In particular,

$$\left| \mathrm{W}_{x \to x}^t \right| = p_t(x, x).$$

We consider μ_M^{loop} as measure on **unrooted** Brownian loops by forgetting the starting point.

Brownian loop measure

Introduced by Greg Lawler and Wendelin Werner.

[Following J. Dubédat] Let $x \quad M$, $t > 0$, consider the sub-probability measure W_x^t on the path of Brownian motion (diffusion generated by $-\Delta_M$) on M started from x on the time interval $[0, t]$, killed if it hits the boundary of M.
The measures $\mathrm{W}_{x \to y}^t$ on paths from x to y are obtained from the disintegration of W_x^t according to its endpoint y:

$$\mathrm{W}_x^t = \int_M \mathrm{W}_{x \to y}^t \,\mathrm{dvol}(y).$$

Define the **Brownian loop measure** on M:

$$\mu_M^{loop} := \int_0^\infty \frac{\mathrm{d}t}{t} \int_M \mathrm{W}_{x \to x}^t \,\mathrm{dvol}(x).$$

In particular,

$$\left| \mathrm{W}_{x \to x}^t \right| = p_t(x, x).$$

We consider μ_M^{loop} as measure on **unrooted** Brownian loops by forgetting the starting point.

The Brownian loop measure satisfies the following two remarkable properties

- *(Restriction property)* If $M \quad M$, then

$$d\mu_M^{loop}(\delta) = 1_{\delta \quad M} \, d\mu_M^{loop}(\delta).$$

- *(Conformal invariance)* On the surfaces $M_1 = (M, g)$ and $M_2 = (M, e^{2\sigma} g)$ be two conformally equivalent Riemann surface, where $\sigma \quad C^{\infty}(M, \mathrm{R})$, then

$$\mu_{M_1}^{loop} = \mu_{M_2}^{loop}.$$

Loop measure vs. determinant of Laplacian

$$\text{“}\left|\mu_M^{loop}\right| = -\log \det{}_\zeta(\Delta).\text{”}$$

If we compute formally, the total mass of μ_M^{loop} is given by

$$\text{“}\left|\mu_M^{loop}\right| = \int_0^\infty \frac{\mathrm{d}t}{t}\int_M p_t(x,x)\,\mathrm{dvol}(x) = \int_0^\infty t^{-1}\mathrm{Tr}\left(e^{-t\Delta}\right)\mathrm{d}t.\text{”}$$

On the other hand, $1/\Gamma(s)$ is analytic and has the expansion near 0 as

$$1/\Gamma(s) = s + O(s^2).$$

Therefore for any analytic function f in a neighborhood of 0,

$$\left(\frac{f(s)}{\Gamma(s)}\right)\bigg|_{s=0} = f(0).$$

Take formally $f(s) = \int_0^\infty t^{s-1}\mathrm{Tr}(e^{-t\Delta})\,\mathrm{d}t$, we have

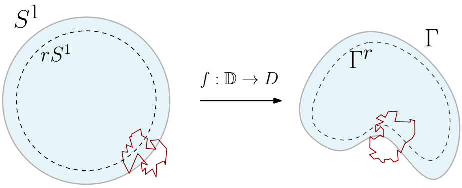
$$\text{“}-\log\det{}_\zeta(\Delta) = \zeta_\Delta(0) = \left(\frac{f(s)}{\Gamma(s)}\right)\bigg|_{s=0} = \int_0^\infty t^{-1}\mathrm{Tr}(e^{-t\Delta})\,\mathrm{d}t = \left|\mu_M^{loop}\right|\text{”}. \qquad (2)$$

$$\text{“}\left|\mu_M^{loop}\right| = -\log \det{}_\zeta(\Delta).\text{”}$$

The determinant expression of Loewner energy suggests that we have formally

$$\begin{aligned}
\text{“}\frac{1}{12} I^L(\Gamma) &= \log \frac{\det_\zeta(\Delta_{\mathrm{D}_1,g})\det_\zeta(\Delta_{\mathrm{D}_2,g})}{\det_\zeta(\Delta_{D_1,g})\det_\zeta(\Delta_{D_2,g})} \\
&= \left|\mu_{D_1}^{loop}\right| + \left|\mu_{D_2}^{loop}\right| - \left|\mu_{\mathrm{D}_1}^{loop}\right| - \left|\mu_{\mathrm{D}_2}^{loop}\right| + \left|\mu_{S^2}^{loop}\right| - \left|\mu_{S^2}^{loop}\right| \\
&= \mu_{S^2}^{loop}(\{\delta; \delta \cap S^1 = \quad\}) - \mu_{S^2}^{loop}(\{\delta; \delta \cap \Gamma = \quad\}).\text{”}
\end{aligned}$$

However, both terms diverge due to the small and large Brownian loops (from the conformal invariance).



For a Brownian loop δ D, where D D is simply connected, we denote δ^{out} its outer boundary (therefore of $\mathrm{SLE}_{8/3}$ type).
Let A, B C be disjoint compact sets,

$$W(A, B; D) := \left|\mu^{loop}\{\delta \quad D; \delta^{out} \text{ intersects both } A \text{ and } B\}\right| < \infty.$$

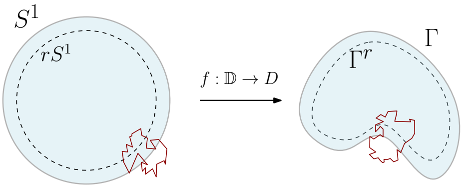
Introduced by W. Werner.

Theorem (W., 2018)

For all Jordan curve Γ (no regularity assumption),

$$\frac{1}{12} I^L(\Gamma) = \lim_{r \to 1} W(S^1, rS^1; \mathrm{C}) - W(\Gamma, \Gamma^r; \mathrm{C}).$$

Loop measure vs. Loewner energy



For a Brownian loop δ D, where D D is simply connected, we denote δ^{out} its outer boundary (therefore of $\mathrm{SLE}_{8/3}$ type).
Let A, B C be disjoint compact sets,

$$W(A, B; D) := \left|\mu^{loop}\{\delta \quad D; \delta^{out} \text{ intersects both } A \text{ and } B\}\right| < \infty.$$

Introduced by W. Werner.

Theorem (W., 2018)

For all Jordan curve Γ (no regularity assumption),

$$\frac{1}{12} I^L(\Gamma) = \lim_{r \to 1} W(S^1, rS^1; \mathrm{C}) - W(\Gamma, \Gamma^r; \mathrm{C}).$$

Proof: Chordal Conformal restriction

Lemma 1: Chordal Conformal restriction

Let (D, a, b) and $(D\ , a, b)$ be two simply connected domains in C coinciding in a neighborhood of a and b, and Γ a simple curve in both (D, a, b) and $(D\ , a, b)$. Then we have

$$\begin{aligned} I_{D\ ,a,b}(\Gamma) - I_{D,a,b}(\Gamma) =& I_{D,a,b}(\psi(\Gamma)) - I_{D,a,b}(\Gamma) \\ =& 3 \log \left|\psi\ (a)\psi\ (b)\right| + 12W(\Gamma, D \setminus D\ ; D) - 12W(\Gamma, D\ \setminus D; D\ ), \end{aligned}$$

where $\psi : D\ \ \to D$ is a conformal map fixing a and b.

Deterministic proof, similar computation as in SLE conformal restriction.
Intuition: The SLE partition function is

$$Z^{\mathrm{SLE}_\kappa}_{(D,a,b)} = H_D(a,b)^{\beta} \det{}_{\zeta}(\Delta)^{-c/2},$$

where as $\kappa \to 0$,

$$\beta = \frac{6-\kappa}{2\kappa} \quad \frac{3}{\kappa}, \quad c = \frac{(3\kappa-8)(6-\kappa)}{2\kappa} \quad -\frac{24}{\kappa}.$$

The Energy = "$-\kappa \log(\cdot)$"

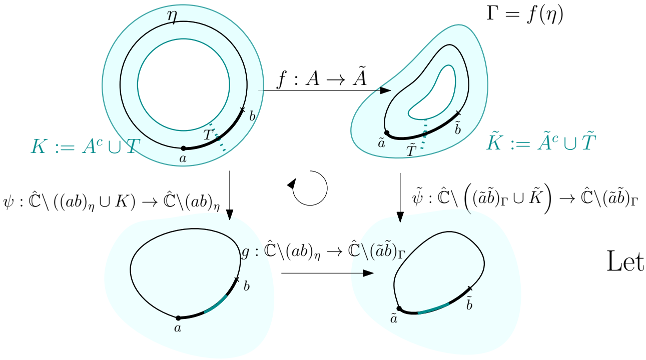
Proof: Loop Conformal restriction

Lemma 2: Loop conformal restriction

If η is a Jordan curve with finite energy and $\Gamma = f(\eta)$, where $f : A \to \tilde{A}$ is conformal on a neighborhood A of η, then

$$I^L(\Gamma) - I^L(\eta) = 12W(\eta, A^c; \mathbb{C}) - 12W(\Gamma, \tilde{A}^c; \mathbb{C}).$$

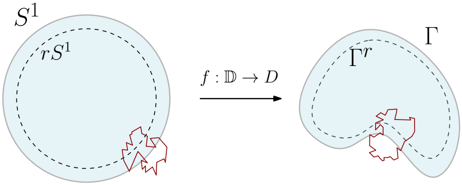
Proof of Lemma 2:



Let $b \to a$.

Proof: Equipotentials

When $\eta = rS^1$, $\Gamma^r = f(rS^1)$ is the equipotential, and $A = D$.



We deduce

$$I^L(\Gamma^r) = 12W(rS^1, S^1; \mathbb{C}) - 12W(\Gamma^r, \Gamma; \mathbb{C}).$$

Lemma 3

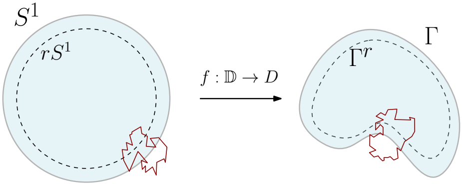
We have: $I^L(\Gamma^r) \xrightarrow{r \to 1} I^L(\Gamma)$.

□

In fact, $r \to I^L(\Gamma^r)$ is increasing if $I^L(\Gamma) > 0$, namely when Γ is not a circle. It will follow from the flow-line coupling for finite energy curve [Viklund, W. 2019+].

Proof: Equipotentials

When $\eta = rS^1$, $\Gamma^r = f(rS^1)$ is the equipotential, and $A = \mathrm{D}$.



We deduce

$$I^L(\Gamma^r) = 12W(rS^1, S^1; \mathrm{C}) - 12W(\Gamma^r, \Gamma; \mathrm{C}).$$

Lemma 3

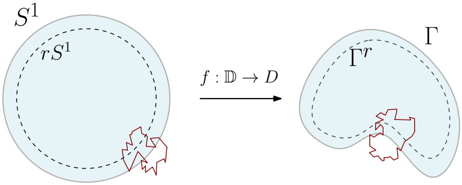
We have: $I^L(\Gamma^r) \xrightarrow{r \to 1} I^L(\Gamma)$.

□

In fact, $r \to I^L(\Gamma^r)$ is increasing if $I^L(\Gamma) > 0$, namely when Γ is not a circle. It will follow from the flow-line coupling for finite energy curve [Viklund, W. 2019+].

Proof: Equipotentials

When $\eta = rS^1$, $\Gamma^r = f(rS^1)$ is the equipotential, and $A = D$.



We deduce

$$I^L(\Gamma^r) = 12W(rS^1, S^1; \mathbb{C}) - 12W(\Gamma^r, \Gamma; \mathbb{C}).$$

Lemma 3

We have: $I^L(\Gamma^r) \xrightarrow{r \to 1} I^L(\Gamma)$.

□

In fact, $r \to I^L(\Gamma^r)$ is increasing if $I^L(\Gamma) > 0$, namely when Γ is not a circle. It will follow from the flow-line coupling for finite energy curve [Viklund, W. 2019+].

SLE/GFF coupling analogs: A Dictionary

Work in progress with F. Viklund. With $\gamma = \sqrt{\kappa}$, $\chi = \gamma/2 - 2/\gamma$:

Random Conformal Geometry		**Action Functional Analogs**
Neumann GFF on $\mathbb{H}$	$\longleftrightarrow$	$2u_1 : \mathbb{H} \to \mathbb{R}$ with finite Dirichlet energy;
Neumann GFF on $\mathbb{H}$	$\longleftrightarrow$	$2u_2 : \mathbb{H} \to \mathbb{R}$ with finite Dirichlet energy;
γ-LQG measure on $\mathbb{H}, e^{\gamma GFF} dz^2$	$\longleftrightarrow$	$e^{2u_1(z)} dz^2$;
γ-LQG boundary measure on $\mathbb{R} = \partial\mathbb{H}$	$\longleftrightarrow$	$e^{u_1(z)} \lvert dz \rvert$, $\quad u_1\lvert_{\mathbb{R}} \quad H^{1/2}(\mathbb{R})$;
"SLE_κ loop"	$\longleftrightarrow$	finite energy loop Γ;
γ-LQG on $\mathbb{C}$	$\longleftrightarrow$	$e^{2\varphi(z)} dz^2$;
γ-quantum chaos wrt. natural parametrization on SLE loop	$\longleftrightarrow$	trace of φ on $\Gamma \quad H^{1/2}(\Gamma)$;
independent couple	$\longleftrightarrow$	sum up their rate functions;
$e^{iGFF/\chi}$	$\longleftrightarrow$	$e^{i\varphi(z)}$ unit vector field;

Isometric conformal welding

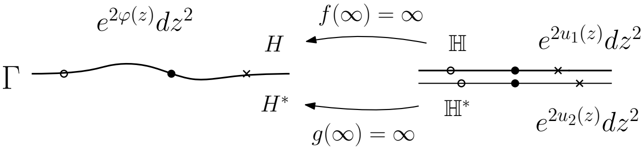
Let D_1, D_2 $\mathbb{C}$ be Jordan domains bounded respectively by rectifiable curves Γ_1 and Γ_2 of same total length. Let $\psi : \Gamma_1 \to \Gamma_2$ be an isometry (preserves the arc-length).

- *[Huber 1976]* The solution does not always exist.
- *[Bishop 1990]* Even if the solution exists, Γ can be a curve of positive area $=$ non-uniqueness of solution.
- *[David 1982, Zinsmeister 1982...]* If D_1 and D_2 are chord-arc, then the solution exists and is unique, which is an quasi-circle. *[Bishop 1990]* The Hausdorff dimension of Γ can take any value in $1 < d < 2$.
- *[David 1982]* If the chord-arc constant of domains are close enough to 1, Γ is also chord-arc.
- *[Viklund, W. 2019+]* We will see that isometric welding of two finite energy domains has also finite energy (solution exists and is unique).

Isometric conformal welding

Let D_1, D_2 C be Jordan domains bounded respectively by rectifiable curves Γ_1 and Γ_2 of same total length. Let $\psi : \Gamma_1 \to \Gamma_2$ be an isometry (preserves the arc-length).

- *[Huber 1976]* The solution does not always exist.
- *[Bishop 1990]* Even if the solution exists, Γ can be a curve of positive area $=$ non-uniqueness of solution.
- *[David 1982, Zinsmeister 1982...]* If D_1 and D_2 are chord-arc, then the solution exists and is unique, which is an quasi-circle. *[Bishop 1990]* The Hausdorff dimension of Γ can take any value in $1 < d < 2$.
- *[David 1982]* If the chord-arc constant of domains are close enough to 1, Γ is also chord-arc.
- *[Viklund, W. 2019+]* We will see that isometric welding of two finite energy domains has also finite energy (solution exists and is unique).



Let φ $W^{1,2}_{loc}(C)$ with finite Dirichlet energy:

$$D_C(\varphi) := \frac{1}{\pi}\int_C | \quad \varphi(z)|^2 \, dz^2 < \infty,$$

Γ an infinite Jordan curve, f, g the conformal maps from H, H onto H, H , respectively.

Theorem (Welding coupling 2019+)

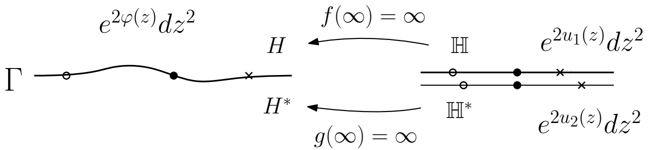
We have $e^{2\varphi}$ $L^1_{loc}(C)$, so the measure $e^{2\varphi}dz^2$ is well-defined and locally finite. The pull-back measures e^{2u_1} by f on H (resp. e^{2u_2} by g on H) satisfy

$$u_1(z) = \varphi \circ f(z) + \log\left|f \ (z)\right|, \quad u_2(z) = \varphi \circ g(z) + \log\left|g \ (z)\right|.$$

We have the identity

$$D_H(u_1) + D_H \ (u_2) = I^L(\Gamma) + D_C(\varphi).$$

Welding-coupling uniqueness



Theorem (Welding-coupling uniqueness, 2019+)

Suppose u_1 and u_2 with finite Dirichlet energy are given. Then there exist unique Γ, φ, f, and g such that the following holds:

1. Γ is an infinite Jordan curve passing through 0 and 1;
2. If H and H^* are the connected components of $\mathbb{C}\setminus\Gamma$, then $f : \mathbb{H} \to H$ is the conformal map fixing $0, 1$ and ∞ and $g : \mathbb{H}^* \to H^*$ is the conformal map fixing $0, \infty$;
3. $\varphi \in W^{1,2}_{loc}(\mathbb{C})$ and $D_{\mathbb{C}}(\varphi) < \infty$;
4. $u_1(z) = \varphi \circ f(z) + \log|f'(z)|$, $z \in \mathbb{H}$;
5. $u_2(z) = \varphi \circ g(z) + \log|g'(z)|$, $z \in \mathbb{H}^*$.

In fact, Γ is obtained from the isometric conformal welding of $\mathbb{H}$ and $\mathbb{H}^*$ according to the boundary lengths $e^{u_1}|dz|$ and $e^{u_2}|dz|$. Moreover, $I^L(\Gamma) < \infty$.

Isometric welding of finite energy domains

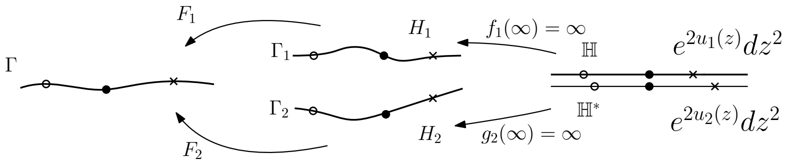
Assume $I^L(\Gamma_1) < \infty, I^L(\Gamma_2) < \infty$, both curves pass through ∞.

Corollary

The isometric conformal welding of Euclidean domain H_1 bounded by Γ_1 and H_2 bounded by Γ_2 has a unique solution Γ up to Möbius transformation. Moreover,

$$I^L(\Gamma) < I^L(\Gamma_1) + I^L(\Gamma_2)$$

if $I^L(\Gamma_1) + I^L(\Gamma_2) = 0$.



In fact, let $u_1 = \log|f_1'|$, $u_2 = \log|g_2'|$,

$$D(u_1) \leq I^L(\Gamma_1), \quad I^L(\Gamma) \leq D(u_1) + D(u_2) \leq I^L(\Gamma_1) + I^L(\Gamma_2).$$

The first equality holds only when $I^L(\Gamma_1) = 0$.

Isometric welding of finite energy domains

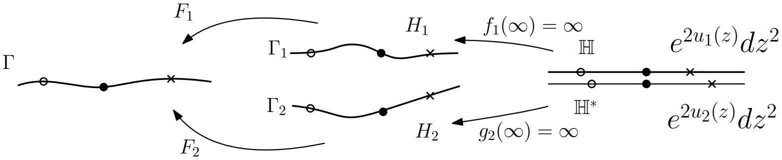
Assume $I^L(\Gamma_1) < \infty, I^L(\Gamma_2) < \infty$, both curves pass through ∞.

Corollary

The isometric conformal welding of Euclidean domain H_1 bounded by Γ_1 and H_2 bounded by Γ_2 has a unique solution Γ up to Möbius transformation. Moreover,

$$I^L(\Gamma) < I^L(\Gamma_1) + I^L(\Gamma_2)$$

if $I^L(\Gamma_1) + I^L(\Gamma_2) = 0$.



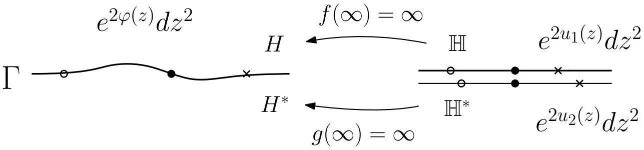
In fact, let $u_1 = \log|f_1|$, $u_2 = \log|g_2|$,

$$D(u_1) \le I^L(\Gamma_1), \quad I^L(\Gamma) \le D(u_1) + D(u_2) \le I^L(\Gamma_1) + I^L(\Gamma_2).$$

The first equality holds only when $I^L(\Gamma_1) = 0$.

Welding coupling identity

$$D_{\mathrm{H}}(u_1) + D_{\mathrm{H}}\ (u_2) = I^L(\Gamma) + D_{\mathrm{C}}(\varphi).$$



- Recall that $u_1(z) = \varphi \circ f(z) + \log |f\ (z)|$, $\quad u_2(z) = \varphi \circ g(z) + \log |g\ (z)|$.
- Use the identity $I^L(\Gamma) = D_{\mathrm{H}}(\log |f\ |) + D_{\mathrm{H}}\ (\log |g\ |)$.
- Prove that the cross-terms cancel out. □

Notice that since the harmonic conjugate $\arg(f\ )$ has the same Dirichlet energy as $\log|f\ |$. We have the identity

$$I^L(\Gamma) = D_{\mathrm{H}}(\arg f\ ) + D_{\mathrm{H}\ }(\arg g\ ).$$

the analog to the forward SLE/GFF coupling (flow-line coupling).

Notice that since the harmonic conjugate $\arg(f\ )$ has the same Dirichlet energy as $\log|f\ |$. We have the identity

$$I^L(\Gamma) = D_{\mathrm{H}}(\arg f\ ) + D_{\mathrm{H}}\ (\arg g\ ).$$

the analog to the forward SLE/GFF coupling (flow-line coupling).

Analog to flow-line coupling

Let η be a bounded C^1 Jordan curve and $\Gamma := \mu(\eta)$, where μ is a Möbius function mapping one point of η to ∞.
For $z = \Gamma(s)$, define the function $\tau : \Gamma \rightarrow$ R such that τ is continuous and

$$\tau(z) := \arg(\Gamma\ (s)) = -\arg(f^{-1})\ (z).$$

We denote by $P[\tau](z) = -\arg(f^{-1})\ (z)$ the Poisson integral of τ in C (defined from both sides of Γ).

Theorem (Flowline coupling analog 2019+)

We have the identity

$$I^L(\Gamma) = D_{\mathrm{C}}(P[\tau]) = \min_{\varphi, \varphi|_\Gamma = \tau} D_{\mathrm{C}}(\varphi).$$

Conversely, under regularity condition of φ and $D_{\mathrm{C}}(\varphi) < \infty$, then for all z_0 C, the solution to the differential equation

$$\Gamma\ (t) = \exp\left(i\varphi(\Gamma(t))\right), \quad t \quad \mathrm{R} \quad \text{and} \quad \Gamma(0) = z_0$$

is an infinite arclength parametrized simple curve and

$$I^L(\Gamma) \leq D_{\mathrm{C}}(\varphi).$$

Analog to flow-line coupling

Let η be a bounded C^1 Jordan curve and $\Gamma := \mu(\eta)$, where μ is a Möbius function mapping one point of η to ∞.
For $z = \Gamma(s)$, define the function $\tau : \Gamma \to \mathbb{R}$ such that τ is continuous and

$$\tau(z) := \arg(\Gamma\ (s)) = -\arg(f^{-1})\ (z).$$

We denote by $P[\tau](z) = -\arg(f^{-1})\ (z)$ the Poisson integral of τ in $\mathbb{C}$ (defined from both sides of Γ).

Theorem (Flowline coupling analog 2019+)

We have the identity

$$I^L(\Gamma) = D_{\mathbb{C}}(P[\tau]) = \min_{\varphi, \varphi|_\Gamma = \tau} D_{\mathbb{C}}(\varphi).$$

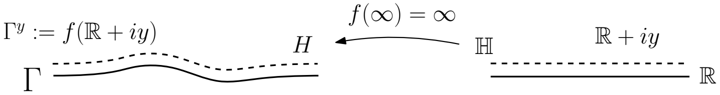
Conversely, under regularity condition of φ and $D_{\mathbb{C}}(\varphi) < \infty$, then for all z_0 $\mathbb{C}$, the solution to the differential equation

$$\Gamma\ (t) = \exp\left(i\varphi(\Gamma(t))\right), \quad t \quad \mathbb{R} \quad \text{and} \quad \Gamma(0) = z_0$$

is an infinite arclength parametrized simple curve and

$$I^L(\Gamma) \le D_{\mathbb{C}}(\varphi).$$

Equipotential energy decrease

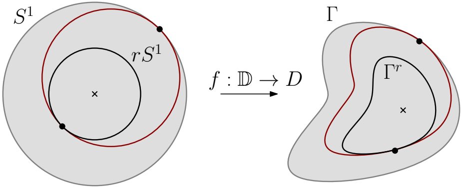


Corollary

We have $I^L(\Gamma^y) \le I^L(\Gamma)$. The equality holds if and only if $I^L(\Gamma) = 0$.

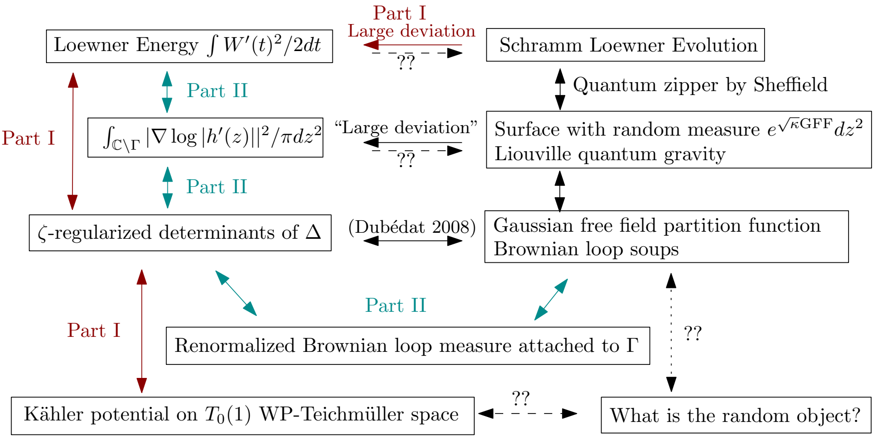
Proof: Since on Γ^y, $\tau^y = P[\tau]$. We have

$$I^L(\Gamma^y) = D_{\mathbb{C}}(P[\tau^y]) \le D_{\mathbb{C}}(P[\tau]) = I^L(\Gamma). \quad \square$$



Contents

Action functionals vs. Random objects



What random model?

- What is the random model naturally associated to the WP-Teichmüller space? Malliavin's measure on diffeomorphisms of the circle?
- In which space does the random welding belong to? (What analytic framework beyond quasiconformal geometry?)
- What is the gradient flow of the Loewner energy and what meaning in Loewner's framework? Other natural dynamics? Stochastic gradient flow?
- Random model = an intrinsic description of SLE loop ($\kappa \leq 4$)? = Reversibility?

What random model?

- What is the random model naturally associated to the WP-Teichmüller space? Malliavin's measure on diffeomorphisms of the circle?
- In which space does the random welding belong to? (What analytic framework beyond quasiconformal geometry?)
- What is the gradient flow of the Loewner energy and what meaning in Loewner's framework? Other natural dynamics? Stochastic gradient flow?
- Random model = an intrinsic description of SLE loop ($\kappa \le 4$)? = Reversibility?

What random model?

- What is the random model naturally associated to the WP-Teichmüller space? Malliavin's measure on diffeomorphisms of the circle?
- In which space does the random welding belong to? (What analytic framework beyond quasiconformal geometry?)
- What is the gradient flow of the Loewner energy and what meaning in Loewner's framework? Other natural dynamics? Stochastic gradient flow?
- Random model = an intrinsic description of SLE loop ($\kappa \le 4$)? = Reversibility?

What random model?

- What is the random model naturally associated to the WP-Teichmüller space? Malliavin's measure on diffeomorphisms of the circle?
- In which space does the random welding belong to? (What analytic framework beyond quasiconformal geometry?)
- What is the gradient flow of the Loewner energy and what meaning in Loewner's framework? Other natural dynamics? Stochastic gradient flow?
- Random model $=$ an intrinsic description of SLE loop ($\kappa \le 4$)? $=$ Reversibility?

Exploring the connection

- How is the Kähler structure on the WP-Teichmüller space encoded in the Loewner's driving function? Why there is such a coincidence?
- Topological group structure on WP-Teichmüller space = what meaning in the Loewner setting?
- Use driving function to find purely geometric characterization of WP-quasicircles? (Jones' Conjecture)
- *[TT06]* WP-quasicircle ⇔ associated Grunsky operator G is Hilbert-Schmidt. Moreover,

$$I^L(\Gamma) \quad \log \det{}_F(I - G \;\; G),$$

where $\det_F$ is the Fredholm determinant (only well-defined when G is HS).
Is it a better object to look at than zeta-regularized determinant of Laplacian?
Interpretation of Grunsky operator?

Exploring the connection

- How is the Kähler structure on the WP-Teichmüller space encoded in the Loewner's driving function? Why there is such a coincidence?
- Topological group structure on WP-Teichmüller space $=$ what meaning in the Loewner setting?
- Use driving function to find purely geometric characterization of WP-quasicircles? (Jones' Conjecture)
- *[TT06]* WP-quasicircle $\Leftrightarrow$ associated Grunsky operator G is Hilbert-Schmidt. Moreover,

$$I^L(\Gamma) \quad \log \det{}_F(I - G \ G),$$

where $\det_F$ is the Fredholm determinant (only well-defined when G is HS). Is it a better object to look at than zeta-regularized determinant of Laplacian? Interpretation of Grunsky operator?

Exploring the connection

- How is the Kähler structure on the WP-Teichmüller space encoded in the Loewner's driving function? Why there is such a coincidence?
- Topological group structure on WP-Teichmüller space $=$ what meaning in the Loewner setting?
- Use driving function to find purely geometric characterization of WP-quasicircles? (Jones' Conjecture)
- *[TT06]* WP-quasicircle $\Leftrightarrow$ associated Grunsky operator G is Hilbert-Schmidt. Moreover,

$$I^L(\Gamma) \quad \log \det{}_F(I - G \; G),$$

where $\det_F$ is the Fredholm determinant (only well-defined when G is HS).
Is it a better object to look at than zeta-regularized determinant of Laplacian?
Interpretation of Grunsky operator?

Exploring the connection

- How is the Kähler structure on the WP-Teichmüller space encoded in the Loewner's driving function? Why there is such a coincidence?
- Topological group structure on WP-Teichmüller space $=$ what meaning in the Loewner setting?
- Use driving function to find purely geometric characterization of WP-quasicircles? (Jones' Conjecture)
- *[TT06]* WP-quasicircle $\Leftrightarrow$ associated Grunsky operator G is Hilbert-Schmidt. Moreover,

$$I^L(\Gamma) \quad \log\det{}_F(I - G \;\; G),$$

where $\det_F$ is the Fredholm determinant (only well-defined when G is HS).
Is it a better object to look at than zeta-regularized determinant of Laplacian?
Interpretation of Grunsky operator?

Other topology?

- Multiple-chord Loewner energy, large deviation of multiple SLE (work in progress with E. Peltola).
- Energy of (multiple) loops in higher genus surface?
- Probabilistic interpretation of Weil-Petersson metric on Teichmüller space of compact surfaces (genus ≥ 2)? Natural measure on Teichmüller/moduli space?
- Conformal field theory (SLE, statistical mechanics models) $=$ String theory (Kähler geometry on universal Teichmüller space)???

Thanks for your attention!