Removability, rigidity of circle domains and Koebe's conjecture.

(Joint work with D. Ntalampekos)

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January 8, 2019

Introduction

A classical theorem

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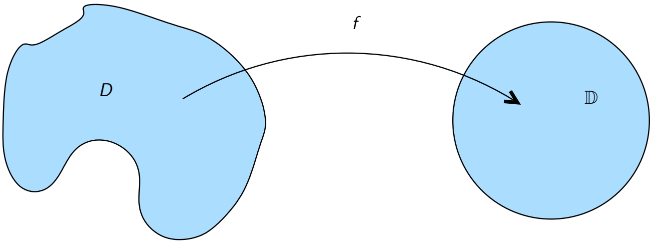
Theorem (Riemann mapping theorem)

Every simply connected domain $D \subsetneq \mathbb{C}$ is conformally equivalent to the open unit disk $\mathbb{D}$.



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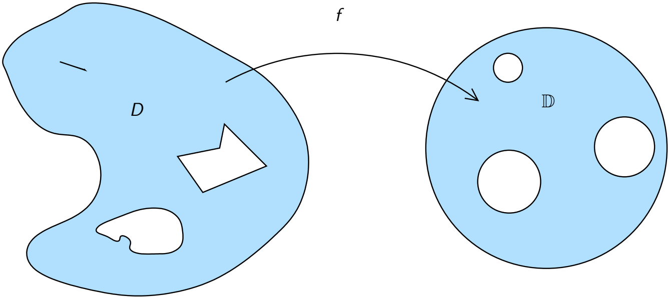

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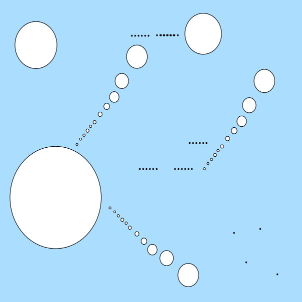

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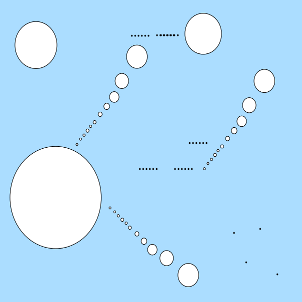

- The boundary of any circle domain contains at most countably many circles.

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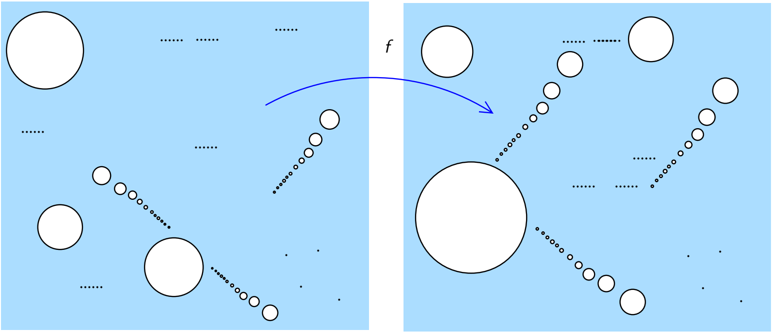
Uniqueness of the Koebe map



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Non-rigid circle domains?

Conformal removability

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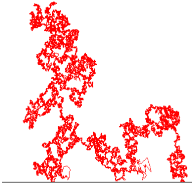

Figure: SLE_6

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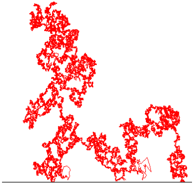

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Is SLE_κ *conformally removable, for* $\kappa \in [4, 8)$*?*

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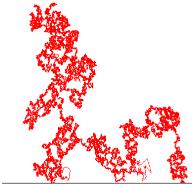

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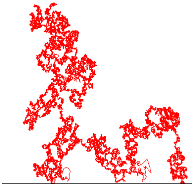

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- Uniqueness of conformal welding of random surfaces (Sheffield (2016), Duplantier–Miller–Sheffield (2014), Mcenteggart–Miller–Qian (2018)).
- The Sierpinski Gasket is not conformally removable (Ntalampekos (2018)).

An important observation

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The complement of a non-removable Cantor set is a non-rigid circle domain.

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- $\mu_f := \partial_{\overline{z}} f / \partial_z f$ is the **Beltrami coefficient** of f (measure of non-conformality : Weyl's lemma).
- Given any μ measurable on U with $\|\mu\|_\infty < 1$, there exists a quasiconformal mapping f on U with $\mu_f = \mu$ a.e. on U, unique up to postcomposition with a conformal map (Measurable Riemann mapping theorem).

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- There exist non-removable sets of Hausdorff dimension one (Bishop (1994)) and removable sets of Hausdorff dimension two.

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In particular, if E is a Cantor set with $m(E) > 0$, then $\Omega := \widehat{\mathbb{C}} \setminus E$ is non-rigid.

The rigidity conjecture

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Conjecture (He–Schramm, 1994)

Let Ω be a circle domain. The following are equivalent :

- (A) *Ω is conformally rigid*
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Let Ω be a circle domain. The following are equivalent :

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- If there are no circles in $\partial\Omega$, then **(A)** $\Rightarrow$ **(B)**.

Known cases

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	$\partial\Omega$ **removable?**	Ω **rigid?**
finite	y	y (Koebe 1918)
countable	y	y (He–Schramm 1993)
σ-**finite**	y (Besicovitch 1931)	y (He–Schramm 1994)
John	y (Jones–Smirnov 2000)	y (Ntalampekos-Y. 2018)
Hölder	y (Jones–Smirnov 2000)	y (Ntalampekos-Y. 2018)
Quasi	y (Jones–Smirnov 2000)	y (Ntalampekos-Y. 2018)
Area > 0	NO	NO (Sibner 1968)

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$$k_D(x_1, x_2) = \inf_{\gamma} \int_{\gamma} \frac{1}{\delta_D(x)} \, ds,$$

over all rectifiable paths $\gamma \subset D$ *that connect* x_1 *and* x_2*.*

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- John domains (and more generally Hölder domains) satisfy the quasihyperbolic condition.

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- If $\|\mu_{\widetilde{f}}\|_\infty > 0$, construct a conformal map g of Ω onto another circle domain that satisfies $\|\mu_{\widetilde{g}}\|_\infty > c$, contradiction.

Another approach to the rigidity conjecture

Properties of conformally removable sets

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- Conformally removable sets are quasiconformally invariant : If $f : \mathbb{C} \to \mathbb{C}$ is quasiconformal and E is conformally removable, then $f(E)$ also is.

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Corollary (Y., 2016)

Let Ω be a circle domain and let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a quasiconformal mapping which maps Ω onto another circle domain $f(\Omega)$.
If Ω is conformally rigid, then $f(\Omega)$ also is.

Quasiconformally rigid implies conformally rigid : An important lemma

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- Proof is based on *trans-quasiconformal deformation of Schottky groups*, a generalization of a technique introduced by Sibner (1968) to prove that a domain is conformally equivalent to a circle domain if and only if it is quasiconformally equivalent to a circle domain.
- Uses David homeomorphisms.

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- Every measurable function $\mu : \overline{\Omega} \to \mathbb{D}$ can be extended to a measurable function $\widetilde{\mu} : \widehat{\mathbb{C}} \to \mathbb{D}$ invariant with respect to $\Gamma(\Omega)$.

When is the qc image of a circle domain also a circle domain?

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Proposition

Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a quasiconformal mapping whose Beltrami coefficient μ_f is invariant with respect to the Schottky group $\Gamma(\Omega)$. Then $f(\Omega)$ is a circle domain.

Important lemma, revisited

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Lemma

Let Ω be a circle domain in $\widehat{\mathbb{C}}$. If Ω is quasiconformally rigid, then $\partial\Omega$ has zero area.

Further remarks on the rigidity conjecture

Question

If $E \subset \mathbb{C}$ *is a conformally removable Cantor set, is* $\Omega := \widehat{\mathbb{C}} \setminus E$ *a conformally rigid circle domain?*

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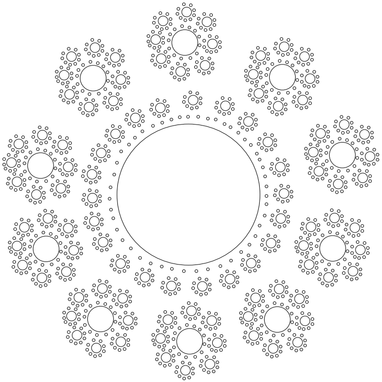
Proposition (Ntalampekos–Y. (2018))

Every $w \in \partial\Omega^*$ that is not a point boundary component is the accumulation point of an infinite sequence of distinct circles in $\partial\Omega^*$.

A Sierpinski-type circle domain



THANK YOU!