Quasisymmetric Uniformization and Heat Kernel Estimates

Analysis and Geometry of Random Shapes
Institute for Pure & Applied Mathematics

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Koch Snowflake

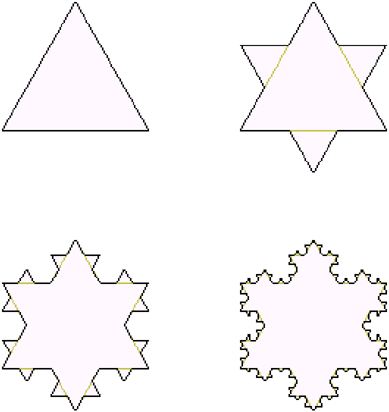


Image: Wikipedia

Snowball

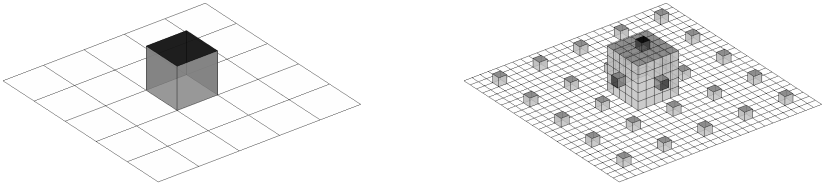
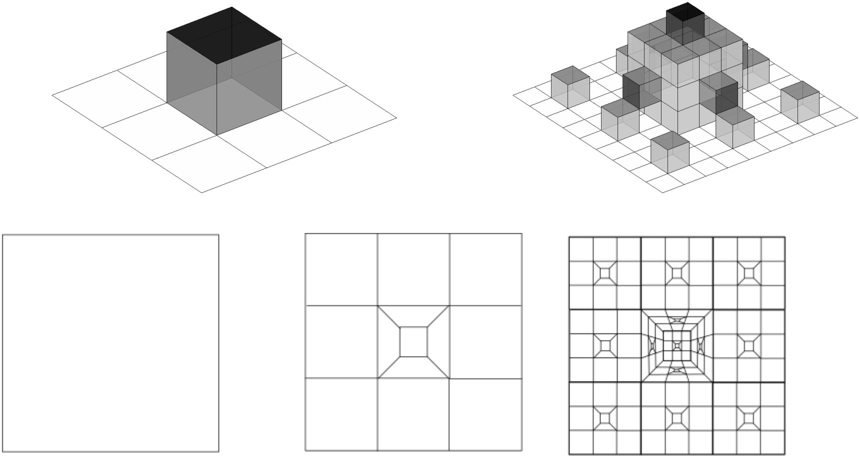


Image: Wikipedia

The intrinsic metric (shortest path metric) is bi-Lipschitz equivalent to the Euclidean metric in $\mathbb{R}^3$.

Snowball: abstract definition



Glue squares with same size.

Uniformization theorem

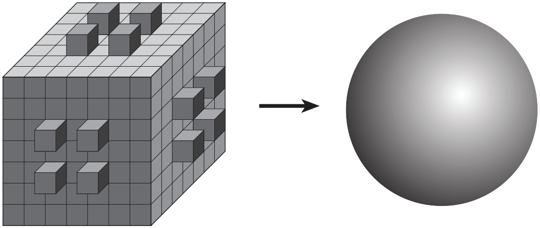


Image: D. Meyer

Theorem (Koebe 1907, Poincaré 1907)

Every simply connected Riemann surface $\mathcal{R}$ is conformally equivalent to $\mathbb{C}, \mathbb{U}$ or $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, i.e., there is an analytic bijection $f : \mathcal{R} \to \mathcal{M}$, where $\mathcal{M} = \mathbb{C}, \mathbb{U}$, or $\widehat{\mathbb{C}}$.

Corollary

The Brownian motion on any polyhedral surface homeomorphic to $\mathbb{S}^2$ can be viewed as a time change of the Brownian motion on $\mathbb{S}^2$.

Uniformization of a polyhedral approximation of Snowball

Can we view Brownian motion on snowball as a time change of the Brownian motion on $\mathbb{S}^2$?

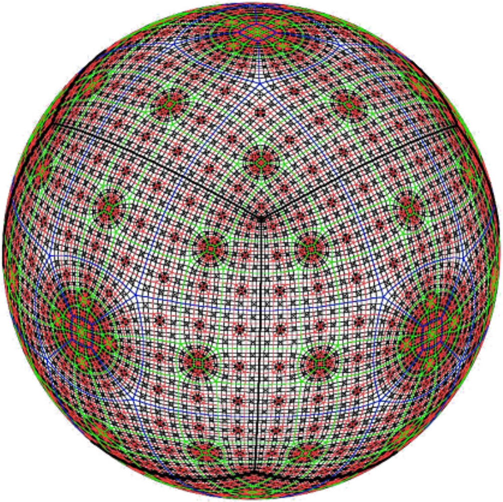


Image: Bowers and Stephenson

Can snowball be "conformally" mapped to $\widehat{\mathbb{C}}$?

- **Definition** (Grötzsch 1928): A homeo $f : (X, d_X) \to (Y, d_Y)$ is said to be K-quasiconformal (QC) if the dilitation [picture]

$$H_f(x,r) := \frac{\sup\{d_Y(f(x), f(z) : d_X(x,z) \le r\}}{\inf\{d_Y(f(x), f(z) : d_X(x,z) \ge r\}},$$

satisfies

$$\limsup_{r \downarrow 0} H_f(x,r) \le K \quad \text{for all } x \in X.$$

- **Fact**: An orientation preserving homeo $f : U \to V$ between domains $U, V \subset \mathbb{C}$ is 1-QC if and only if it is conformal (Menshov 1937).
- **Defintion** (Ahlfors, Beurling 1956): A homeomorphism $f : (X, d_X) \to (Y, d_Y)$ is said to be K-quasisymmetric (QS) if

$$H_f(x,r) \le K \quad \text{for all } x \in X, \text{ and for all } r > 0.$$

- We say f is QS (resp. QC) if it is K-QS (resp. K-QC) for some $K \ge 1$.

Snowballs are quasisymmetric to $\mathbb{S}^2$ (D. Meyer 2002, 2010)

Idea behind the proof: Take the limit of uniformizing maps.

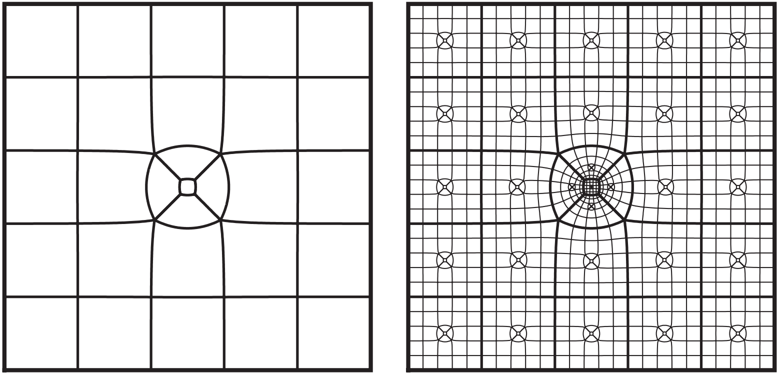


Image: Snowballs are Quasiballs by Meyer.

Two exponents associated with a graph $G = (V, E)$.

- Let $B(x, r) = \{y \in V \ : \ d(x, y) \le r\}$ denote the closed ball and let $V(x, r) = |B(x, r)|$ denote its volume.
- We say that d_f is the volume growth exponent or fractal dimension if

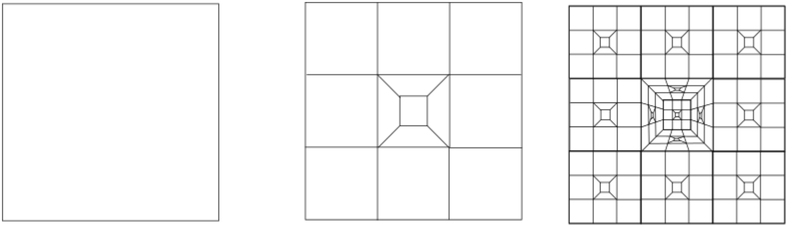
$$V(x, r) \asymp r^{d_f} \quad \forall x \in V, \forall r \ge 1.$$

- We say that d_w is the escape time exponent or walk dimension if the exit time on balls for the simple random walk satisfies

$$\mathbb{E}_x \tau_{B(x,r)} \asymp r^{d_w} \quad \forall x \in V, \forall r \ge 1.$$

- Example: $\mathbb{Z}^d$ has $d_f = d$ and $d_w = 2$.

Graphical snowball



These graphs viewed from the central square converges to an infinite quadrangulation of the plane with volume growth exponent $d_f = \log_3(13)$.

Question: How fast does the random walk travel in this quadrangulation?

Sub-Gaussian estimate on graphs

Definition

We say a graph $\mathbb{G} = (V_{\mathbb{G}}, E_{\mathbb{G}})$ satisfies the sub-Gaussian estimate with volume growth exponent d_f and escape time exponent d_w if there exists constants $C_1, C_2, c_1, c_2 > 0$ such that the transition probability for the simple random walk satisfies the estimates

$$P_n(x,y) \le \frac{C_1}{n^{d_f/d_w}} \exp\left[-\left(\frac{d(x,y)^{d_w}}{C_2 n}\right)^{\frac{1}{d_w-1}}\right], \forall n \ge 1, \forall x, y \in V_{\mathbb{G}}$$

and

$$P_n(x,y) + P_{n+1}(x,y) \ge \frac{c_1}{n^{d_f/d_w}} \exp\left[-\left(\frac{d(x,y)^{d_w}}{c_2 n}\right)^{\frac{1}{d_w-1}}\right], \forall x, y \in V_{\mathbb{G}}, \forall n \ge 1 \vee d(x,y).$$

Sub-Gaussian estimate implies $\mathbb{E}(d(X_0, X_n)) \asymp n^{1/d_w}$.

Example: Graphical Sierpinski gasket

Graphical Sierpinski gasket has $d_f = \log_2 3$, $d_w = \log_2 5$ and satisfies the sub-Gaussian estimate (Barlow and Perkins 1988).

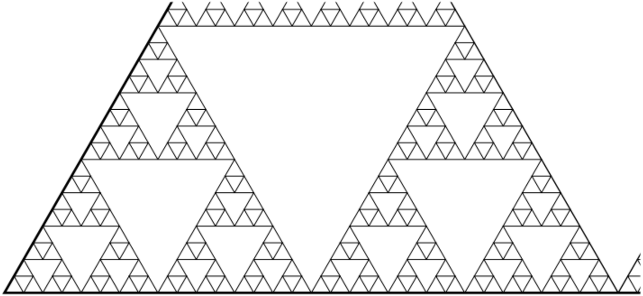


Image: Barlow.

How to uniformize a planar graph?

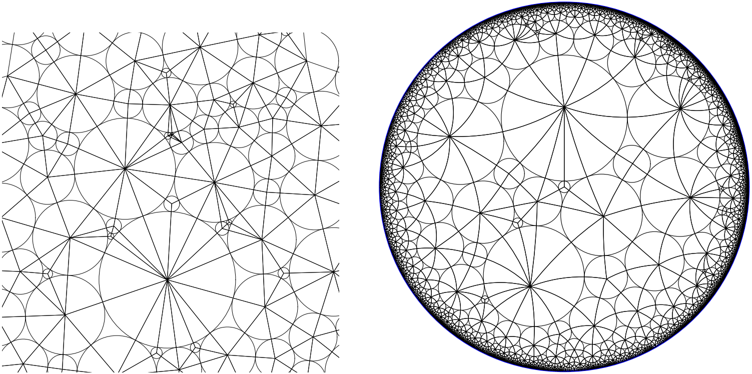


Image: Angel, Hutchcroft, Nachmias, Ray.

Good embedding

An embedding with straight lines of a planar graph $\mathbb{G} = (V_{\mathbb{G}}, E_{\mathbb{G}})$ is a map sending the vertices to points in the plane and edges to straight lines connecting the corresponding vertices such that no two edges cross.

The carrier of the embedding, denoted by $\mathrm{carr}(\mathbb{G})$ is the union of closed faces of the embedding.

Definition (ABGN=Angel, Barlow, Gurel-Gurevich, Nachmias 2016)

Let $D, \eta \in (0, \infty)$. We say than an embedding with straight lines of a planar graph $\mathbb{G} = (V_{\mathbb{G}}, E_{\mathbb{G}})$ is (D, η)-good if

(a) No flat angles. For any face, all the inner angles are at most $\pi - \eta$. In particular, all faces are convex, there is no outer face, and the number of edges in a face is at most $2\pi/\eta$.

(b) Adjacent edges have comparable lengths. For any two adjacent edges $e_1 = \{u, v\}$ and $e_2 = \{u, w\}$, we have $|u - v| / |u - w| \in [D^{-1}, D]$.

Two metrics on the graph

- The cable system $\mathcal{X} = \mathcal{X}(\mathbb{G})$ corresponding to the graph $\mathbb{G}$ is the topological space obtained by replacing each edge $e \in E_{\mathbb{G}}$ by a copy of the unit interval $[0, 1]$, glued together in the obvious way, with the endpoints corresponding to the vertices.
- A length function $\ell : E_{\mathbb{G}} \to (0, \infty)$ induces a metric on the cable system as follows. We define the length of each unit interval $[0, 1]$ corresponding to an edge e to be $\ell(e)$ and consider the induced intrinsic metric (embedding metric).
- Any good embedding induces a length function $\ell(e)$ defined as the length of the straight line joining the vertices.
- Fact (ABGN 2016): For a good embedding, the Euclidean metric is bi-Lipschitz equivalent to the embedding metric on the cable system.
- The metric on the cable system defined by $\ell \equiv 1$ is same as the graph metric when restricted to vertices.

Circle packing, quasisymmetry, and Sub-Gaussian estimate

Theorem (M. 2018+)

Let $\mathbb{G} = (V_{\mathbb{G}}, E_{\mathbb{G}})$ be planar graph with volume growth exponent d_f such that it admits a good embedding with carrier $\mathbb{R}^2$ or $\mathbb{U}$. Then the following are equivalent

(a) *The embedding metric d_E (or equivalently the Euclidean metric) and the graph metric $d_{\mathbb{G}}$ are quasisymmetric.*

(b) *The simple random on $\mathbb{G}$ satisfies sub-Gaussian estimate with walk dimension $d_w = d_f$.*

Corollary (M. 2018+)

Let $\mathbb{G} = (V_{\mathbb{G}}, E_{\mathbb{G}})$ be one-ended, planar triangulation with volume growth exponent d_f. Then the circle packing metric is quasisymmetric to the graph metric if and only if the simple random on $\mathbb{G}$ satisfies sub-Gaussian estimate with walk dimension $d_w = d_f$.

Proof sketch: from QS to heat kernel estimate

Key Ingredients:

- Theorem (ABGN 2016): The cable process (after a time change) satisfies Gaussian heat kernel estimate with respect to the (good) embedding metric ($d_w = 2$).
- Comparability of annuli: Let d_1, d_2 be two quasisymmetric metrics. Then every annulus $B_1(x, A_1 r) \setminus B_1(x, r)$ is contained in an annulus $B_2(x, A_2 s) \setminus B_2(x, s)$ where A_2 does not depend on x, r.
- Given $A_2 > 1$, there exists $A_1 > 1$, such that the annulus $B_1(x, A_1 r) \setminus B_1(x, r)$ contains $B_2(x, A_2 s) \setminus B_2(x, s)$ for all x, r.
- Elliptic Harnack inequality is preserved under quasisymmetry. [picture]

Proof sketch: from QS to heat kernel estimate

- Theorem (Grigor'yan–Telcs 2002): To obtain sub-Gaussian heat kernel estimate with $d_f = d_w$, it suffices to verify elliptic Harnack inequality and the following resistance estimate on annuli

$$\exists A > 1: \ \mathrm{Cap}(B_{\mathbb{G}}(x, r), B_{\mathbb{G}}(x, Ar)^c) \asymp 1 \ \text{ for all } r \geq 1, x.$$

- Using the above proposition and comparability of annuli, we obtain the following two sided bounds on capacity: [picture]

$$\exists A > 1: \ \mathrm{Cap}(B_{\mathbb{G}}(x, r), B_{\mathbb{G}}(x, Ar)^c) \asymp 1 \ \text{ for all } r \geq 1, x.$$

- The heat kernel bounds now follow from EHI, capacity bounds and Grigor'yan–Telcs theorem.

From heat kernel estimates to QS: QC vs QS

- Every QS map is QC but the converse is false; for example, $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x + e^x$.
- **Theorem** (Local to global principle; Gehring 1960): If $f : \mathbb{R}^n \to \mathbb{R}^n$, $n \geq 2$ is quasiconformal, then f is quasisymmetric.
- We follow the approach of Heinonen and Koskela (1998) who significantly generalized Gehring's local to global principle.
- The key ingredients to carry out this approach are Loewner property of the circle packing embedding and capacity upper bounds across annuli on the graph.

Concluding remarks

- The proof of heat kernel estimates using QS also applies to diffusion on snowballs (fractal).
- A large family of random maps (Brownian map, UIPT, Liouville Quantum Gravity, mated CRT map) are satisfy a weaker variant of sub-Gaussian heat kernel estimate with $d_f = d_w$. (Gwynne–Miller 2017, Gwynne–Hutchcroft 2018).

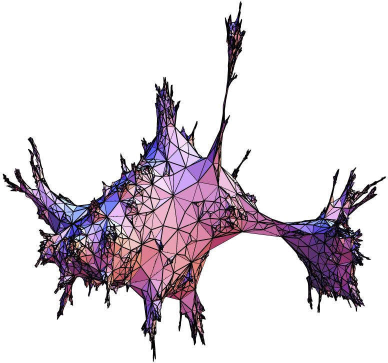


Image: Uniform triangulation of $\mathbb{S}^2$ (due to Curien)

Conjecture: Show that the uniform infinite planar triangulation (resp. Brownian map) is "almost quasisymmetric" to its circle packing embedding (resp. $\hat{\mathbb{C}}$).

Thank you

References:

1. Angel, Barlow, Gurel-Gurevich, Nachmias, Boundaries of planar graphs, via circle packings, Ann. Probab (2016).
2. Heinonen, Koskela, Quasiconformal maps in metric spaces with controlled geometry, Acta Math (1998).
3. Gwynne, Miller, Random walk on random planar maps: spectral dimension, resistance, and displacement (preprint)
4. Gwynee, Hutchcroft, Anomalous diffusion of random walk on random planar maps (preprint)
5. M., Quasisymmetric Uniformization and heat kernel estimates, Trans. AMS (to appear).