Weakly exponential measures and time changes of the Brownian motion

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Our space: $K = [0,1]^2$

μ: a Borel regular probability measure on K.

μ **has weak exponential decay** $\underset{\text{def}}{\Leftrightarrow}$

$\exists c_1, c_2, \alpha_1, \alpha_2 > 0$, $\forall x \in [0,1]^2, \forall r \in (0,1]$

$$c_1 r^{\alpha_1} \leq \mu(B_*(x,r)) \leq c_2 r^{\alpha_2}, \tag{0.1}$$

where $B_*(x,r) = \{y | y \in [0,1]^2, |x-y| < r\}$.

Examples

(1) Liouville measure (or Liouville quantum gravity) associated with Gaussian free field

(2) Random self-similar measures

Gaussian Free Field: a very short introduction based on [9, 2]

First Ingredient: $(\Omega, \mathcal{F}, P)$: a probability space
$\{\varphi\}_{i\geq 1}$: independet random variables satisfying $\forall i \geq 1$, $\varphi_i \in L^2(\Omega, P)$,

$$\int_\Omega \varphi_i dP = 0 \quad \text{and} \quad \int_\Omega (\varphi_i)^2 dP = 1.$$

$\{\varphi_i\}_{i\geq 1}$: independent $\Rightarrow$ $\{\varphi_i\}_{i\geq 1}$ is an orthonormal system of $L^2(\Omega, P)$.

Second Ingredient: $(H, (\cdot, \cdot))$: a real Hilbert space,
$\{e_i\}_{i\geq 1}$: a complete orthonormal base of $(H, (\cdot, \cdot))$.

Define $\Phi : H \to L^2(\Omega, P)$ as, $\forall \{\alpha_i\}_{i\geq 1} \in \ell^2(\mathbb{R})$,

$$\Phi(\sum_{i=1}^{\infty} \alpha_i e_i) \underset{\text{def}}{=} \sum_{i=1}^{\infty} \alpha_i \varphi_i$$

$$\Phi \underset{\text{isometry}}{\longrightarrow} \overline{< \varphi_1, \varphi_2, \cdots >} \quad \text{i.e.} \quad (h,h) = \int_\Omega (\Phi(h))^2 dP.$$

Formally if $X_\omega = \sum_{i\geq 1} \varphi_i(\omega) e_i$, then

$$\Phi(h)(\omega) = (X_\omega, h).$$

Note that $X_\omega \notin H$!!

Additional Setting 1: Gaussian

Assume that the distribution of φ_i = the **centered normal Gaussian**;
$\forall$ Borel set $A \subseteq \mathbb{R}$,

$$P((\varphi_i)^{-1}(A)) = \int_A \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx.$$

$$\Downarrow$$

$$P(\Phi(h)^{-1}(A)) = \int_A \frac{1}{\sqrt{2\pi(h,h)}} e^{-\frac{x^2}{2(h,h)}} dx.$$

the distribution of $\Phi(h)$

Additional Setting 2: Dirichlet energy

$U \subseteq \mathbb{R}^n$: a bounded domain, μ_*: the Lebesgue measure
$H_0^1(U)$: the collection of H^1 functions with 0-bounday conditions.
$(\cdot,\cdot)_D$: the Dirichlet energy defined by

$$(f,g)_D \underset{\text{def}}{=} \int_U (\nabla f, \nabla g) d\mu_*$$

Fact: $(H_0^1(U), (\cdot,\cdot)_D)$ is a Hilbert space.

$\exists$ a complete orthonormal system $\{f_i\}_{i\geq 1}$ of $L^2(U, \mu_*)$ and $\{\lambda_i\}_{i\geq 1}$ such that $0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots$, $\lim_{i\to\infty} \lambda_i = +\infty$, $\forall i \geq 1$,

$$f_i \in H_0^1(U) \quad \text{and} \quad \Delta f_i = -\lambda_i f_i$$

$$\Downarrow$$

$\left\{e_i \underset{\text{def}}{=} \dfrac{f_i}{\sqrt{\lambda_i}}\right\}_{i\geq 1}$: an orothnormal base of $(H_0^1(U), (\cdot, \cdot)_D)$

Definition 0.1. $\Phi(H)$ is called the **Gaussian free field** over U associated with $(\Omega, \mathcal{F}, P)$ and $\{\varphi_i\}_{i\geq 1}$.

Formally

$$X_\omega(x) \underset{\text{def}}{=} \sum_{i \geq 1} \varphi_i(\omega) e_i(x) : \text{Gaussian random variable for each } x \in U$$

Then

$$\begin{aligned} E_P[X_\omega(x) X_\omega(y)] &= \int_\Omega X_\omega(x) X_\omega(y) P(d\omega) \\ &= \sum_{i=1}^{\infty} e_i(x) e_i(y) = \sum_{i=1}^{\infty} \frac{f_i(x) f_i(y)}{\lambda_i} = G_U(x, y), \end{aligned}$$

where $G_U(x, y)$ is the Green function of the domain U.

Gausiann free field to Liouville measure

Assume $n = 2$.

$\Phi(H)$: Gausssian free field over U associated wtih $(\Omega, \mathcal{F}, P)$ and $\{\varphi_i\}_{i\geq 1}$.

Define

$$X_{N,\omega}(z) \underset{\text{def}}{=} \sum_{i=1}^{N} \varphi_i(\omega) e_i(z).$$

$\Rightarrow \forall \omega \in \Omega, X_{N,\omega} \in H_0^1(U)$ and $\forall z \in U$, $X_{N,\omega}(x) \in L^2(\Omega, P)$.

In $L^2(\Omega, P)$,

$$\Phi(h) = \lim_{N\to\infty} (X_{N,\omega}, h)_D (= (X_\omega, h)_D, \text{ formally})$$

$X_{N,\omega}$: regularization of X_ω.

For $\gamma \geq 0$, define

$$\begin{aligned}\lambda_{N,\gamma,\omega}(z) &\underset{\text{def}}{=} \gamma X_{N,\omega}(z) - \frac{\gamma^2}{2} E_P[X_{N,\omega}(z)^2] \\ &= \gamma X_{N,\omega}(z) - \frac{\gamma^2}{2} \int_\Omega X_{N,\omega}(z)^2 P(d\omega) \\ &= \gamma \sum_{i=1}^{N} \varphi_i(\omega) e_i(z) - \frac{\gamma^2}{2} G_U^{(N)}(z, z),\end{aligned}$$

where $G_U^{(N)}(z, w) = \sum_{i=1}^{N} \frac{f_i(z) f_i(w)}{\lambda_i}$.

Define

$$M_{\gamma,N,\omega}(A) \underset{\text{def}}{=} \int_A e^{\lambda_{N,\gamma,\omega}(z)} \mu_*(dz)$$

for any Borel set $A \subseteq U$. Then

Theorem 0.2 (Duplantier and Sheffield[2])**.** $\forall \gamma \in [0,2)$, P-*a.s* $\omega \in \Omega$,

$$\boxed{M_{\gamma,N,\omega} \underset{weakly}{\rightarrow} M_{\gamma,\omega} \ \ as\ N \rightarrow \infty.}$$

Formally,

$$M_{\gamma,\omega}(dz) = e^{\lambda_{\gamma,\omega}(z)} \mu_*(dz), \text{where } \lambda_{\gamma,\omega}(z) = \gamma X_\omega(z) - \frac{\gamma^2}{2} E_P[X_\omega(z)^2]$$

$M_{\gamma,\omega}$: the **Liouville measure**, or, the **Liouville quantum gravity**

There is another way to construct the Liouville measure by means of **Gaussian multiplicative chaos** originated by Kahane[5] and recently developed by Robert and Vargas[8]. See also [7].

Liouville Brownian motion

By Garban, Rodes and Vargas [4, 3]

Time change of the Brownian motion with respect to $M_{\gamma,\omega}$ is possible.

$\exists p_{\gamma,\omega}(t, z_1, z_2)$: associated heat kernel which is jointly continuous

The time changed process is called **the Liouville Brownian motion**.

Asymptotic bahaviro of $p_{\gamma,\omega}(t, z_1, z_2)$:

Maillard, Rhodes, Vargas and Zeitouni [6] and Andres and Kajino[1].

By Andres and Kajino, P-a.e. $\omega \in \Omega$, $M_{\gamma,\omega}$-a.e. $z \in U$,

$$\lim_{t \downarrow 0} -\frac{\log p_{\gamma,\omega}(t, z, z)}{\log t} = 1 + \text{rough off-diagonal estimate}$$

Notable fact on the Liouville measure:

Proposition 0.3 ([4] + [1]). *Let* V *be a bounded subset of* U*. Then for* P*-a.e.* $\omega \in \Omega$, $\exists c_1, c_2, \alpha_1, \alpha_2 > 0, \exists R > 0, \forall x \in V, \forall r \in (0, R]$,

$$\boxed{c_1 r^{\alpha_1} \leq M_{\gamma,\omega}(B_*(x,r)) \leq c_2 r^{a_2}}.$$

Self-similarity of K: $q_1 = (0,0), q_2 = (1,0), q_3 = (1,1), q_4 = (0,1)$ and

$$F_i(x) = \frac{x - q_i}{2} + q_i$$

for $i = 1, 2, 3, 4$.

$$\begin{aligned} W_m &= \{1,2,3,4\}^m = \{w_1 \dots w_m | w_1, w_2, \dots, w_m \in \{1,2,3,4\}\} \\ W_* &= \bigcup_{m \geq 0} W_m \\ F_{w_1 \dots w_m} &= F_{w_1} \circ F_{w_2} \circ \dots \circ F_{w_m} \\ K_{w_1 \dots w_m} &= F_{w_1 \dots w_m}(K) \end{aligned}$$

$|w|$: length of $w \in W_* \underset{\text{def}}{=} m$ such that $w \in W_m$.

Random self-similar measure

$$\Delta = \{(p_1, p_2, p_3, p_4) | 0 < p_i < 1, p_1 + p_2 + p_3 + p_4 = 1\}$$

ν is a Bore regular probability measure on Δ.
Let $\{(\Delta_w, \nu_w)\}_{w \in W_*}$ be independent copies of (Δ, ν) and define $(\Omega, \mathbb{P}_\nu)$ as

$$\Omega = \prod_{w \in W_*} \Delta_w \quad \text{and} \quad \mathbb{P}_\nu = \prod_{w \in W_*} \nu_w.$$

Fact: $\mathbb{P}_\nu$-a.e. $\omega = \prod_{w \in W_*}(p_1(w,\omega), p_2(w,\omega), p_3(w,\omega), p_4(w,\omega))$,
$\exists$ Borel regular probability measure P^ν_ω such that

$$\boxed{P^\nu_\omega(K_{w_1 \dots w_m}) = p_{w_1}(\phi, \omega) p_{w_2}(w_1, \omega) \cdots p_{w_m}(w_1 \dots w_{m-1}, \omega)}$$

P^ν_ω: a random self-similar measure on K.

Theorem 0.4. *Assume that* $\exists q > 0$ *such that* $\forall i = 1, 2, 3, 4$,

$$\int_\Delta (p_i)^{-q} d\nu < +\infty.$$

Then for $\mathbb{P}_\nu$*-a.e.* ω, P^ν_ω *has weak exponential decay.*

the reflected Brownian motion on the square $K = [0,1]^2$

$$\boxed{\mathcal{E}(u,v) = \int_K \sum_{i=1}^{2} \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i} dx} : \ \textbf{Dirichlet energy}$$

for $u, v \in H^{1,2}(K)$: the Sobolev space.

$$\mathcal{E}(u,v) = -\int_K u(Lv)dx,$$

where $\boxed{Lv = \sum_{i=1}^{2} \frac{\partial^2 v}{\partial {x_i}^2}}$: **Laplacian**.

Time Change = Inhomogeneous media

Introduce μ: **Density of the media**, a Radon measure on K.
Assume that $\mu(A) = 0$ whenever $\mathrm{Cap}(\mathrm{A}) = 0$.

$$\boxed{\mu \text{ need not to be absolutely continuous to } dx}$$

Then by taking a proper modification $\mathcal{F}_\mu$ of the original domain $H^{1,2}(K)$,

$(\mathcal{E}, \mathcal{F}_\mu)$: a local regular Dirichlet form on $L^2(K, \mu)$

$\longrightarrow$ same pathes but different speeds

Define

$$h(x,y) \underset{\text{def}}{=} \max\{-\log|x-y|, 1\}$$

$$h_\mu(w) \underset{\text{def}}{=} \sup_{x\in K_w} \int_{K_w} h(x,y)\mu(dy)$$

$h(x,y) \asymp$ the 0-Green function

Proposition 0.5. *μ: weakly exponential decay* $\Rightarrow \exists C > 0, \forall w \in W_*,$

$$h_\mu(w) \le C|w|^2\mu(K_w).$$

Furthermore, time change is possible with respect to μ.

μ: the volume doubling property $\Rightarrow \exists C > 0, \forall w \in W_*$

$$h_\mu(w) \le C|w|\mu(K_w).$$

Poincaré inequality

Classical Poincaré inequality:

$$\boxed{Cr^2 \int_{B_*(x,r)} \sum_{i=1}^{2} \left(\frac{\partial u}{\partial x_i} \right)^2 dx \geq \int_{B_*(x,r)} |u(y) - u|_{B_*(x,r)}|^2 dy},$$

where $B_*(x,r) = \{y||x-y| < r\}$.

$\Rightarrow$ Estimate of the second eigenvalue of the Neumann Laplacian λ_2, i.e.

$$\lambda_2 \geq \frac{C}{r^2}.$$

Poincaré inequality in general

(X,d): a metric space, μ: a Radon measure on X.
$(\mathcal{E},\mathcal{F})$: a local regular Dirichlet form on $L^2(X,\mu)$.
U,V: open subsets of X, $\overline{V}\subseteq U\subseteq X$.
$g_U(x,y)$: 0-order Green function associated with $\mathcal{E}$ on U.

Assumption 0.6. $\exists\alpha>0$, $\forall x,y\in V$, $z\in U$ with $d(x,z)\geq d(x,y)/2$,

$$g_U(x,z)\leq \alpha g_U(x,y)$$

The following result is an adaptation/modification of that in
R. Bass, A stability theorem for elliptic Harnack inequalities, J. Eur. Math. Soc. 17(2013), 856–876

Theorem 0.7. *Under Assumption 0.6, $\forall f \in \mathcal{F}$,*

$$\mathcal{E}(f,f) \geq \frac{1}{2^6} \frac{\mu(V) \inf_{x,y\in V} g_U(x,y)}{\alpha^2 \left(\sup_{x\in V} \int_U g_U(x,z)\mu(dz) \right)^2} \int_V (f(x) - (f)_V)^2 \mu(dx)$$

Remark.

$$E_x(\tau_U) = \int_U g_U(x,z)\mu(dz) : \ \textbf{Exit time form } U$$

Theorem 0.8. *$\exists C > 0$ such that μ: Time change is possible $\Rightarrow$ $\lambda_2(\mu) \geq \frac{C}{h_\mu(\phi)^2}$, i.e.*

$$\boxed{\mathcal{E}(u,u) \geq \frac{C}{h_\mu(\phi)^2} \int_K (u(x) - (u)_\mu)^2 \mu(dx)},$$

where $(u)_\mu \underset{\text{def}}{=} \int_K u(x)\mu(dx)$.

Note that

$$\mathcal{E}(u,u) = \int_K \left(\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 \right) dxdy$$

Weak exponential decay + Poincaré inequality $\Rightarrow$ Nash inequality

Define μ_w as $\mu_w(A) = \mu(F_w(A))/\mu(K_w)$.

$$\begin{aligned}\mathcal{E}(u,u) &= \sum_{w\in\Lambda} \mathcal{E}(u\circ F_w, u\circ F_w) \\ &\geq \sum_{w\in\Lambda} \frac{C}{h_{\mu_w}(\phi)^2} \int_K \Big(u\circ F_w - (u\circ F_w)_{\mu_w}\Big)^2 d\mu_w \\ &= \sum_{w\in\Lambda} \frac{C}{h_{\mu_w}(\phi)^2} \left(\frac{1}{\mu(K_w)} \int_{K_w} u^2 d\mu - \frac{1}{\mu(K_w)^2} \Big(\int_{K_w} u d\mu \Big)^2 \right),\end{aligned}$$

where Λ is a **partition**.

Note that if μ is **volume doubling** or **self-similar**, then $h_{\mu_w}(\phi) \leq C'$

By the weak exopnential property,

$$h_{\mu_w}(\phi)^2 \leq c_1(\log \mu(K_w)^4 + c_2).$$

Set

$$\varphi(t) = t(c_1(\log t)^4 + c_2).$$

Then

$$\begin{aligned}\mathcal{E}(u,u) &\geq \sum_{w\in\Lambda} \frac{C}{h_{\mu_w}(\phi)^2}\left(\frac{1}{\mu(K_w)}\int_{K_w} u^2 d\mu - \frac{1}{\mu(K_w)^2}\Big(\int_{K_w} u d\mu\Big)^2\right) \\ &\geq \sum_{w\in\Lambda} \frac{C}{\varphi(\mu(K_w))}\left(\int_{K_w} u^2 d\mu - \frac{1}{\mu(K_w)}\Big(\int_{K_w} u d\mu\Big)^2\right)\end{aligned}$$

Consequently,

$$\mathcal{E}(u,u) + C\sum_{w\in\Lambda}\frac{1}{\mu(K_w)\varphi(\mu(K_w))}\Big(\int_{K_w} ud\mu\Big)^2 \geq C\sum_{w\in\Lambda}\frac{1}{\varphi(\mu(K_w))}\int_{K_w} u^2 d\mu$$

Let Λ is the partition of W_* where $\varphi(\mu(K_w)) \approx t$. (**Cheating**!!)
Set $\xi = \varphi^{-1}$. Then $\forall t \in (0,1]$,

$$\boxed{\mathcal{E}(u,u) + \frac{C_1}{t\xi(t)}||u||_1^2 \geq \frac{C_2}{t}||u||_2^2}$$ – **Nash type inequality**.

$\Rightarrow \forall t \in (0,1]$,

$$\boxed{||T_t u||_\infty \leq C\max\{1, \xi(t)^{-1}\}||u||_1}$$

where T_t is the semigroup assocoiated with the time change.
Note that if μ is **voulme doubling** or **self-similar**, then $\xi(t) = t$.

Consequently,

Theorem 0.9. *If μ has weak exponential decay, then*
(1) H_μ *=associated self-adjoint operator: compact resolvent*
(2) *The associated semigroup* $\{T_t\}_{t>0}$: *ultraconatractive*
(3) $\exists$ *jointly continuous heat kernel* $p_\mu(t,x,y)$ *and* $\exists \alpha, c > 0$

$$p_\mu(t,x,x) \leq ct^{-\alpha}$$

for $t \in (0,1]$.

Assuem that μ has weak exponential decay.

General lower heat kernel estimate:

$\exists \gamma > 0$, μ-a.e. $x \in K$, $\exists T_x > 0$, $\forall t \in (0, T_x]$,

$$\frac{c}{t|\log t|^9} \leq p_\mu(t, x, x)$$

With a weak doubling condition:

$\exists f : (0, \infty) \to [1, \infty)$: non-increasing, $\forall x \in K, r > 0$

$$\mu(B_*(x, 2r)) \leq f(r)\mu(B_*(x, r)) \quad \text{and} \quad \lim_{r \downarrow 0} \frac{\log f(r)}{\log r} = 0.$$

Then $\forall x \in K$,

$$\lim_{t \downarrow 0} -\frac{\log p_\mu(t, x, x)}{\log t} = 1$$

Volume doubling case:
Assume that μ has the volume doubling property with respect to the Euclidean metric, i.e.

$$\mu(B_*(x,2r)) \leq C\mu(B_*(x,r)).$$

Theorem 0.10 (Sub-Gaussian)**.**
μ has the volume doubling property with respect to the Eucliden metric
$\Leftrightarrow$
$\exists d_\mu$: a metric quasisymmetric to the Euclidean metric, $\exists \beta \geq 2$, $\exists c_1, c_2, c_3, c_4 > 0$, $\forall x, y, t$,

$$p_\mu(t,x,y) \leq \frac{c_1}{t} \exp\left(- c_2 \Big(\frac{d_\mu(x,y)^\beta}{t} \Big)^{\frac{1}{\beta-1}} \right).$$

and if $d_\mu(x,y) \leq c_4 t$, then

$$\frac{c_3}{t} \leq p_\mu(t,x,y).$$

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