Loewner energy via Brownian loop measure and action functional analogs of SLE/GFF couplings

Yilin Wang

ETH Zürich

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IPAM UCLA
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4 What’s next?
Loewner's transform [1923] consists of encoding the uniformizing conformal map of a simply connected domain $D \subset \mathbb{C}$ into evolution of conformal distortions that flatten out the boundary iteratively,

non self-intersecting curve $\partial D \Leftrightarrow$ real-valued driving function.

Main tool to solve Bieberbach’s conjecture by De Branges in 1985.

Random fractal non self-intersecting curves: the Schramm-Loewner Evolution introduced by Oded Schramm in 1999 which successfully describe interfaces in many statistical mechanics models.

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Loewner energy for Jordan curves (loops) on the Riemann’s sphere, is non-negative, vanishing only on circles, and invariant under Möbius transformation [Rohde, W. 2017].

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Introduction

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Action functionals vs. Random objects

Loewner Energy $\int W'(t)^2/2dt$

Schramm Loewner Evolution

Surface with random measure $e^{\sqrt{\kappa}GFF}dz^2$

Liouville quantum gravity

“Large deviation”

Quantum zipper by Sheffield

Part I

Part II

Part I

Part II

Part I

Part II

Part I

Part II

Part I

Part II

Part I

Part II

Kähler potential on $T_0(1)$ WP-Teichmüller space

Renormalized Brownian loop measure attached to $\Gamma$

ζ-regularized determinants of $\Delta$

Loewner energy

What is the random object?

Part II

(Dubédat 2008)

Gaussian free field partition function

Brownian loop soups

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   - SLE and the Loewner energy
   - Zeta-regularized determinants of Laplacians
   - Weil-Petersson Teichmüller space

3 Part II: Applications

4 What’s next?
Chordal Loewner chains

Let \( \Gamma \) be a simple chord in \( \mathbb{H} \) from 0 to \( \infty \).

\[
g_t(z) = z + \frac{2t}{z} + o\left(\frac{1}{z}\right)
\]

as \( z \to \infty \)

\( \eta(s) := g_t(\Gamma_{t+s}) \)

\( W_t = g_t(\Gamma_t) \)

- \( \Gamma \) is capacity-parametrized by \([0, \infty)\).
- \( W : \mathbb{R}_+ \to \mathbb{R} \) is called the driving function of \( \Gamma \).
- \( W_0 = 0 \).
- \( W \) is continuous.
- One can recover the curve \( \Gamma \) from \( W \) using Loewner’s differential equation.
- We say that \( \Gamma \) is the chordal Loewner chain generated by \( W \).
- The centered Loewner flow has the expansion
  \[
f_t(z) = g_t(z) - W_t = z - W_t + 2t/z + O(1/z).
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$f_t(z) = g_t(z) - W_t = z - W_t + 2t/z + O(1/z)$. 
Chordal Loewner chain

- If $W \equiv 0$, then $\Gamma = i\mathbb{R}_+$.

\[ g_t(z) = z + \frac{2t}{z} + o\left(\frac{1}{z}\right) \quad \text{as } z \to \infty \]

When the curve is driven by $W = \sqrt{\kappa}B$ where $B$ is 1-d Brownian motion, the curve generated is the Schramm-Loewner Evolution of parameter $\kappa$ (SLE$_\kappa$).
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The chordal Loewner energy

\( D \subset \mathbb{C} \) a simply connected domain, \( a, b \) are two boundary points of \( D \).

\[ \phi : D \rightarrow \mathbb{H} \]
\[ \phi(a) = 0, \phi(b) = \infty \]

**Definition: Loewner energy**

We define the **Loewner energy of a simple chord** \( \Gamma \) in \( (D, a, b) \) to be

\[ I_{D,a,b}(\Gamma) := I_{\mathbb{H},0,\infty}(\phi(\Gamma)) := I(W) := \frac{1}{2} \int_0^\infty W'(t)^2 \, dt \]

where \( W \) is the driving function of \( \phi(\Gamma) \).
The Loewner energy is well-defined in \((D, a, b)\) since for \(c > 0\),

\[
l_{\mathbb{H}, 0, \infty} (\Gamma) = l_{\mathbb{H}, 0, \infty} (c\Gamma).
\]

\(l_{D, a, b}(\Gamma) = 0\) iff \(\Gamma\) is the hyperbolic geodesic connecting \(a\) and \(b\).

\(l_{D, a, b}(\Gamma) < \infty\), then \(\Gamma\) is rectifiable [Friz & Shekhar, 2015].
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Upper half-plane vs. other domains

Assume that ∂D is smooth in a neighborhood of b, a continuously parametrized chord Γ : [0, T] → D from a to b.

The capacity parametrization of Γ seen from b is chosen using the Schwarzian derivative of the mapping-out function:

$$\text{cap}(Γ[0, t]) := -\frac{S(g_t)(b)}{12}.$$ 

The driving function is given by

$$W_t = \frac{1}{2} \frac{g''_t(b)}{g'_t(b)}.$$ 

The Loewner energy is given by

$$I_{D,a,b}(Γ) = \sup_{0 \leq T_0 < T_1 < \cdots < T_n = T} \sum_{i=0}^{n-1} \frac{(W_{T_{i+1}} - W_{T_i})^2}{\text{cap}(Γ[0, T_{i+1}]) - \text{cap}(Γ[0, T_i])}.$$
Assume that $\partial D$ is smooth in a neighborhood of $b$, a continuously parametrized chord $\Gamma : [0, T] \to \overline{D}$ from $a$ to $b$.

\[ \begin{align*} 
  g_t : D \setminus \Gamma_{[0,t]} &\to D \\
  g_t(\Gamma_t) &= a, \quad g_t(b) = b \\
  g_t'(b) &= 1 
\end{align*} \]

The **capacity parametrization** of $\Gamma$ seen from $b$ is chosen using the Schwarzian derivative of the mapping-out function:

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The **Loewner energy** is given by

\[ I_{D,a,b}(\Gamma) = \sup_{0 \leq T_0 < T_1 < \cdots < T_n = T} \frac{1}{\text{cap}(\Gamma[0, T_{i+1}]) - \text{cap}(\Gamma[0, T_i])} \sum_{i=0}^{n-1} (W_{T_{i+1}} - W_{T_i})^2. \]
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SLE$_\kappa$ vs. Loewner energy

The Dirichlet energy $I(W)$ is the **action functional** of Brownian motion. Intuitively, the “Brownian path has the distribution on $C^0(\mathbb{R}_+, \mathbb{R})$ with density $\propto \exp(-I(W)) \, dW$.”

However, $I(B) = \infty$ with probability 1.

The Schilder’s theorem states that $I(W)$ is also the **large deviation rate function** for Brownian motion $\sqrt{\kappa} B$ as $\kappa \to 0$. Loosely speaking,

\[ \text{“} \Pr(\sqrt{\kappa} B \text{ stays close to } W) \approx \exp\left( -\frac{I(W)}{\kappa} \right) \text{”} \]

It should imply that the Loewner energy is the large deviation rate function of SLE$_\kappa$:

\[ \text{“} \Pr(\text{SLE}_\kappa \text{ stays close to } \Gamma) \approx \exp\left( -\frac{I(\Gamma)}{\kappa} \right) \text{”} \quad (1) \]

The claim (1) is made precise in [W. 2016].
The Dirichlet energy $I(W)$ is the **action functional** of Brownian motion. Intuitively, the “Brownian path has the distribution on $C^0(\mathbb{R}_+,\mathbb{R})$ with density $\propto \exp(-I(W))DW$.”

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Reversibility of chordal Loewner energy

**Theorem (W. 2016)**

Let $\Gamma$ be a simple chord in $D$ connecting two boundary points $a$ and $b$, we have

$$I_{D,a,b}(\Gamma) = I_{D,b,a}(\Gamma).$$

The deterministic result is based on

**Theorem (Reversibility of SLE, Zhan 2008, Miller-Sheffield 2012)**

For $\kappa \leq 8$, the law of the trace of $\text{SLE}_{\kappa}$ in $(D, a, b)$, is the same as the law of $\text{SLE}_{\kappa}$ in $(D, b, a)$.

In fact, the decay rate as $\kappa \to 0$ of the probability of $\text{SLE}_{\kappa}$ stays close to $\Gamma$ is the same as the decay rate of being close to $-1/\Gamma$. 
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In fact, the decay rate as $\kappa \to 0$ of the probability of SLE$_\kappa$ stays close to $\Gamma$ is the same as the decay rate of being close to $-1/\Gamma$. 
In fact, the Loewner energy has more symmetries.
We define the **Loewner energy of a simple loop** $\Gamma : [0, 1] \mapsto \hat{\mathbb{C}}$ rooted at $\Gamma_0 = \Gamma_1$ to be

$$I^L(\Gamma, \Gamma_0) := \lim_{\varepsilon \to 0} I_{\hat{\mathbb{C}} \setminus \Gamma[0, \varepsilon], \Gamma_{\varepsilon}, \Gamma_0}(\Gamma[\varepsilon, 1]).$$

- $I^L(\Gamma, \Gamma_0) = 0$ if and only if $\Gamma$ is a (round) circle.
- If $\Gamma[0, s]$ is a circular arc (including line segments), then the RHS is constant for $\varepsilon \leq s$, and $I^L(\Gamma, \Gamma_0)$ equals to the chordal energy $I_{\hat{\mathbb{C}} \setminus \Gamma[s, 1], \Gamma_{s}, \Gamma_0}(\Gamma[s, 1])$. 

\[ \Gamma_0 \xrightarrow{\varepsilon \to 0} \Gamma_{\varepsilon} \]
Definition (Rohde, W., 2017)

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Root-invariance

Theorem (Rohde, W. 2017)
The Loewner loop energy is **independent** of the choice of root and orientation.

\[ I_L \] is invariant on the set of **free loops** under Möbius transformation;
\[ \implies \] The loop setting is more natural than the chordal setting.

*The proof is based on the reversibility of the chordal energy.*

Moreover,
- \( I_L(\Gamma) < \infty \), then \( \Gamma \) is a (rectifiable) quasicircle.
- If \( \Gamma \) is \( C^{1.5+\varepsilon} \) for some \( \varepsilon > 0 \), then \( I_L(\Gamma) < \infty \).
Theorem (Rohde, W. 2017)

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The Zeta-regularization of determinants is first introduced by Ray & Singer (1976). Hawking (1977) has pointed out that it allows to regularize quadratic path integrals. Osgood, Phillips & Sarnak (1988) have shown that the results obtained by comparing two functional determinants of Laplacian in the QFT formalism agree with the results obtained by the zeta-regularized determinant (Polyakov-Alvarez conformal anomaly formula).
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The functional $\mathcal{H}$

- $g_0(z) = \frac{4}{(1+|z|^2)^2} \, dz^2$ denotes the spherical metric;
- $g = e^{2\varphi} g_0$ be a metric conformally equivalent to $g_0$;
- $\Gamma$ a $C^\infty$ smooth simple loop in $\mathbb{C} \cup \{\infty\} \simeq S^2$;
- $D_1$ and $D_2$ two connected components $S^2 \setminus \Gamma$;
- $\Delta_g(D_i)$ the Laplace-Beltrami operator with Dirichlet boundary condition on $D_i$.

**Definition**

Let $\det_\zeta$ be the $\zeta$-regularized determinant, we introduce

$$\mathcal{H}(\Gamma, g) := \log \det_\zeta \Delta_g(S^2) - \log \text{Area}_g(S^2) - \log \det_\zeta \Delta_g(D_1) - \log \det_\zeta \Delta_g(D_2).$$
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**Theorem (W., 2018)**

If \( g = e^{2\varphi} g_0 \) is a metric conformally equivalent to the spherical metric \( g_0 \) on \( S^2 \), then:

1. \( \mathcal{H}(\cdot, g) = \mathcal{H}(\cdot, g_0) \)
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3. Let \( \Gamma \) be a smooth Jordan curve on \( S^2 \). We have the identity

\[
I^L(\Gamma, \Gamma(0)) = 12 \mathcal{H}(\Gamma, g) - 12 \mathcal{H}(S^1, g)
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\]

where \( D_1 \) and \( D_2 \) are two connected components of the complement of \( S^1 \).

In particular, the above identity gives already the parametrization independence of the Loewner loop energy for smooth loops.
\( \mathcal{H}(\Gamma, g) = \log \det_\zeta \Delta_g(S^2) - \log \text{Area}_g(S^2) - \log \det_\zeta \Delta_g(D_1) - \log \det_\zeta \Delta_g(D_2). \)

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The regularity assumption on the curve is due to the constraint from the zeta-regularization and its variation formula [OPS].

Picking different metrics $g$ provide a wide range of identities with the Loewner energy that usually look different in their expression involving scalar curvatures, geodesic curvatures, conformal maps $D_1 \to \mathbb{D}_1$, etc., (but of course they are equal).

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Remarks

- The regularity assumption on the curve is due to the constraint from the zeta-regularization and its variation formula [OPS].
- Picking different metrics $g$ provide a wide range of identities with the Loewner energy that usually look different in their expression involving scalar curvatures, geodesic curvatures, conformal maps $D_1 \to \mathbb{D}_1$, etc., (but of course they are equal).
- One of the identities links to the Weil-Petersson class of the universal Teichmüller space.
Universal Teichmüller space

- \( QS(S^1) \) the group of quasisymmetric sense-preserving homeomorphism of \( S^1 \);

A sense-preserving homeomorphism \( \varphi : S^1 \to S^1 \) is quasisymmetric if there exists \( M \geq 1 \) such that for all \( \theta \in \mathbb{R} \) and \( t \in (0, \pi) \),

\[
\frac{1}{M} \leq \left| \frac{\varphi(e^{i(\theta+t)}) - \varphi(e^{i\theta})}{\varphi(e^{i\theta}) - \varphi(e^{i(\theta-t)})} \right| \leq M.
\]

- \( \text{Möb}(S^1) \cong \text{PSL}(2, \mathbb{R}) \) the subgroup of Möbius function of \( S^1 \).

The universal Teichmüller space is

\[
T(1) := QS(S^1)/\text{Möb}(S^1) \cong \{ \varphi \in QS(S^1), \ \varphi \text{ fixes } -1, -i \text{ and } 1 \}.
\]

It can be modeled by Beltrami coefficients as well:

\[
T(1) = L^\infty(\mathbb{D}, \mathbb{C})_1/ \sim,
\]

where

\[
\|\mu\|_\infty < 1, \|\nu\|_\infty < 1, \quad \mu \sim \nu \iff w_\mu|_{S^1} = w_\nu|_{S^1}
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\( w_\mu \) is the normalized solution (fixes \(-1, -i, 1\)) \( \mathbb{D} \to \mathbb{D} \) to the Beltrami equation

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\overline{\partial}w_\mu(z) = \mu(z)\partial w_\mu(z).
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Associate $\Gamma$ with its **welding function** $\varphi$:

$$\varphi := g^{-1} \circ f|_{S^1}$$

\[\text{[Rohde, W. 2017]: } I^L(\Gamma) < \infty \Rightarrow \Gamma \text{ is a quasicircle } \iff \varphi \in QS(S^1).\]

“$\iff$” is not true, there are quasicircles with $\infty$ Loewner energy.

**Question**

What is the class of finite energy loops in $T(1)$?
Welding function

- Associate $\Gamma$ with its **welding function** $\varphi$:

\[
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\]

\[g(\infty) = \infty\]

\[f : D \to D\]
\[g : D^* \to D^*\]

\[\Gamma\]
\[D\]
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The homogeneous space of $\mathcal{C}^\infty$-smooth diffeomorphisms

$$M := \text{Diff}(S^1)/\text{M"ob}(S^1) \subset T(1)$$

has a Kähler structure [Witten, Bowick, Rajeev, etc.].

There is a unique homogeneous Kähler metric (up to constant factor): the Weil-Petersson metric.
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There is a unique homogeneous Kähler metric (up to constant factor): the Weil-Petersson metric.
Weil-Petersson metric

The tangent space at \( id \) of \( M \) consists of \( C^\infty \) vector fields on \( S^1 \):

\[
v = v(\theta) \frac{\partial}{\partial \theta} = \sum_{m \in \mathbb{Z} \setminus \{-1, 0, 1\}} v_m e^{im\theta} \frac{\partial}{\partial \theta}, \text{ where } v_{-m} = \overline{v_m}.
\]

The almost complex structure \( J^2 = -Id \) is given by the Hilbert transform:

\[
J(v)_m = -i \text{sgn}(m)v_m, \text{ for } m \in \mathbb{Z} \setminus \{-1, 0, 1\}.
\]

In particular,

\[
J \left( \cos(m\theta) \frac{\partial}{\partial \theta} \right) = \sin(m\theta) \frac{\partial}{\partial \theta}; \quad J \left( \sin(m\theta) \frac{\partial}{\partial \theta} \right) = -\cos(m\theta) \frac{\partial}{\partial \theta}.
\]

The Weil-Petersson symplectic form \( \omega(\cdot, \cdot) \) and the Riemannian metric \( \langle \cdot, \cdot \rangle_{WP} \) is given at the origin by

\[
\omega(v, w) = i \sum_{m \in \mathbb{Z} \setminus \{-1, 0, 1\}} (m^3 - m)v_m w_{-m},
\]

\[
\langle v, w \rangle_{WP} = \omega(v, J(w)) = \sum_{m=2}^{\infty} (m^3 - m) \text{Re}(v_m w_{-m}).
\]
**Weil-Petersson Class**

- **Weil-Petersson Teichmüller space** $T_0(1)$ is the closure of $\text{Diff}(S^1)/\text{Möb}(S^1) \subset T(1)$ under the WP-metric. **Weil-Petersson class** $\text{WP}(S^1) \subset QS(S^1)$ are homeomorphisms representing points in $T_0(1)$.

- The above description and many other characterizations are provided by [Nag, Verjovski, Sullivan, Cui, Takhtajan, Teo, Shen, etc].

![Diagram of Weil-Petersson Teichmüller space](image)

**Theorem (Takhtajan & Teo, 2006)**

The universal Liouville action $S_1 : T_0(1) \to \mathbb{R}$,

$$S_1([\varphi]) := \int_D \left| \frac{f''}{f'}(z) \right|^2 \, dz^2 + \int_{D^*} \left| \frac{g''}{g'}(z) \right|^2 \, dz^2 + 4\pi \log \left| \frac{f'(0)}{g'(-\infty)} \right|$$

is a Kähler potential of the Weil-Petersson metric, where

$$g'(\infty) = \lim_{z \to \infty} g'(z) = \tilde{g}'(0)^{-1} \text{ and } \tilde{g}(z) = 1/g(1/z).$$
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\[ \Gamma \]

\[ f : \mathbb{D} \to D \]

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Theorem (W. 2018)

A bounded simple loop $\Gamma$ in $\hat{\mathbb{C}}$ has finite Loewner energy if and only if $[\varphi] \in T_0(1)$. Moreover,

$$l^L(\Gamma) = S_1([\varphi])/\pi.$$
Loewner Energy vs. Weil-Petersson Class

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Characterizations of the WP-Class (an incomplete list)

[Nag, Verjovsky, Sullivan, Cui, Taktajan, Teo, Shen, etc.] The following are equivalent:

- The welding function $\varphi$ is in Weil-Petersson class;
- $\int_{\mathbb{D}} |\nabla \log |f'(z)||^2 \, dz^2 = \int_{\mathbb{D}} |f''(z)/f'(z)|^2 \, dz^2 < \infty$;
- $\int_{\mathbb{D}^*} |g''(z)/g'(z)|^2 \, dz^2 < \infty$;
- $\int_{\mathbb{D}} |S(f)|^2 \rho^{-1}(z) \, dz^2 < \infty$;
- $\int_{\mathbb{D}^*} |S(g)|^2 \rho^{-1}(z) \, dz^2 < \infty$;
- $\varphi$ has quasiconformal extension to $\mathbb{D}$, whose complex dilation $\mu = \partial_z \varphi / \partial \bar{z} \varphi$ satisfies
  $$\int_{\mathbb{D}} |\mu(z)|^2 \rho(z) \, dz^2 < \infty;$$
- $\varphi$ is absolutely continuous with respect to arc-length measure, such that $\log |\varphi'|$ belongs to the Sobolev space $H^{1/2}(S^1)$;
- Grunsky operator associated to $f$ or $g$ is Hilbert-Schmidt,

where $\rho(z) \, dz^2 = 1/(1 - |z|^2)^2 \, dz^2$ is the hyperbolic metric on $\mathbb{D}$ or $\mathbb{D}^*$ and

$$S(f) = \frac{f'''}{f'} - \frac{3}{2} \left( \frac{f''}{f'} \right)^2$$

is the Schwarzian derivative of $f$. 
Contents

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3 Part II: Applications
   - Brownian loop measure interpretation
   - Action functional analogs of SLE/GFF couplings

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Part I Large deviation

Schramm Loewner Evolution

Quantum zipper by Sheffield

Surface with random measure $e^{\sqrt{\kappa}GFF}dz^2$
Liouville quantum gravity

“Large deviation”

Part II

Surface with random measure $e^{\sqrt{\kappa}GFF}dz^2$
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Part I

Part II

ζ-regularized determinants of $\Delta$

(Dubédat 2008)

Gaussian free field partition function
Brownian loop soups

Renormalized Brownian loop measure attached to $\Gamma$

Kähler potential on $T_0(1)$ WP-Teichmüller space

What is the random object?
Recall \( \mathcal{H}(\Gamma, g) = \log \det_\zeta \Delta_g(S^2) - \log \text{Area}_g(S^2) - \log \det_\zeta \Delta_g(D_1) - \log \det_\zeta \Delta_g(D_2). \)

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where \( D_1 \) and \( D_2 \) are two connected components of the complement of \( S^1 \).
Zeta-regularized determinants

- $\Delta_g(S^2)$ is non-negative, essentially self-adjoint for the $L^2$ product.
- The spectrum is
  
  \begin{align*}
  0 = \lambda_0 < \lambda_1 \leq \lambda_2 \cdots
  \end{align*}

- Define the Zeta-function
  
  \begin{align*}
  \zeta_{\Delta}(s) := \sum_{i \geq 1} \lambda_i^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty \text{Tr}(e^{-t\Delta})t^{s-1}dt,
  \end{align*}

  it can be analytically continued to a neighborhood of 0.

- Define (following Ray & Singer 1976)
  
  \begin{align*}
  \log \det'(\Delta_g(S^2)) := -\zeta_{\Delta}'(0)
  \end{align*}

  \begin{align*}
  \text{"} = \sum_{i \geq 1} \log(\lambda_i)\lambda_i^{-s} |_{s=0} = \log(\prod_{i \geq 1} \lambda_i).\"
  \end{align*}
Proof of the identity (sketch)

\[ i^L(\Gamma, \Gamma(0)) = 12 \log \frac{\det_\zeta(-\Delta_{\mathbb{D}_1, g_0}) \det_\zeta(-\Delta_{\mathbb{D}_2, g_0})}{\det_\zeta(-\Delta_{\mathbb{D}_1, g_0}) \det_\zeta(-\Delta_{\mathbb{D}_2, g_0})} \]

- When \( \Gamma \) passes through \( \infty \), we show

\[ i^L(\Gamma, \infty) = D_{\mathbb{H} \cup \mathbb{H}^*}(\log |h'|) := \frac{1}{\pi} \left( \int_{\mathbb{H} \cup \mathbb{H}^*} \left| \nabla \log |h'(z)| \right|^2 \, dz^2 \right), \]

where \( h \) maps conformally \( \mathbb{H} \cup \mathbb{H}^* \) to the complement of \( \Gamma \) and fixes \( \infty \).

The right-hand side does not involve Loewner iteration of conformal maps.

- Use the Polyakov-Alvarez conformal anomaly formula to compare determinants of Laplacians.
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\[ \Gamma \]
\[ \begin{array}{c}
H \\
\cap
\end{array} \\
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\cup
\end{array} \]

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Polyakov-Alvarez conformal anomaly formula

Take $g = e^{2\sigma}g_0$ a metric conformally equivalent to $g_0$. (Here think $\sigma = \log |h'|$.)


For a compact surface $M$ without boundary,

$$
\left( \log \text{det}_{\zeta}'(-\Delta_g) - \log \text{vol}_g(M) \right) - \left( \log \text{det}_{\zeta}'(-\Delta_0) - \log \text{vol}_0(M) \right) = -\frac{1}{6\pi} \left[ \frac{1}{2} \int_M |\nabla_0 \sigma|^2 \, d\text{vol}_0 + \int_M K_0 \sigma \, d\text{vol}_0. \right]
$$

The analogue for a compact surface $D$ with smooth boundary is:

$$
\log \text{det}_{\zeta}(-\Delta_g) - \log \text{det}_{\zeta}(-\Delta_0) = -\frac{1}{6\pi} \left[ \frac{1}{2} \int_D |\nabla_0 \sigma|^2 \, d\text{vol}_0 + \int_D K_0 \sigma \, d\text{vol}_0 + \int_{\partial D} k_0 \sigma \, d\text{l}_0 \right] - \frac{1}{4\pi} \int_{\partial D} \partial_n \sigma \, d\text{l}_0.
$$

“Taking $g_0 = dz^2$”, we have $K_0 \equiv 0$ and $k_0 \equiv 0$. We get:

$$
I^L(\Gamma, \Gamma(0)) = \frac{1}{\pi} \left( \int_{\mathbb{H} \cup \mathbb{H}^*} |\nabla \log |h'(z)||^2 \, d\text{z}^2 \right) = 12\mathcal{H}(\Gamma, g_0) - 12\mathcal{H}(S^1, g_0).
$$
Polyakov-Alvarez conformal anomaly formula

Take \( g = e^{2\sigma} g_0 \) a metric conformally equivalent to \( g_0 \). (Here think \( \sigma = \log |h'| \).)


For a compact surface \( M \) without boundary,

\[
\left( \log \det'_{\zeta} (-\Delta_g) - \log \text{vol}_g (M) \right) - \left( \log \det'_{\zeta} (-\Delta_0) - \log \text{vol}_0 (M) \right) \\
= - \frac{1}{6\pi} \left[ \frac{1}{2} \int_M |\nabla_0 \sigma|^2 \, d\text{vol}_0 + \int_M K_0 \sigma \, d\text{vol}_0 \right]
\]

The analogue for a compact surface \( D \) with smooth boundary is:

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l^L(\Gamma, \Gamma(0)) = \frac{1}{\pi} \left( \int_{\mathbb{H} \cup \mathbb{H}^*} \left| \nabla \log |h'(z)| \right|^2 \, dz^2 \right) = 12\mathcal{H}(\Gamma, g_0) - 12\mathcal{H}(S^1, g_0).
\]
Brownian loop measure

Introduced by Greg Lawler and Wendelin Werner.

[Following J. Dubédat] Let $x \in M$, $t > 0$, consider the sub-probability measure $\mathcal{W}_x^t$ on the path of Brownian motion (diffusion generated by $-\Delta_M$) on $M$ started from $x$ on the time interval $[0, t]$, killed if it hits the boundary of $M$.

The measures $\mathcal{W}_x^t \rightarrow y$ on paths from $x$ to $y$ are obtained from the disintegration of $\mathcal{W}_x^t$ according to its endpoint $y$:

$$\mathcal{W}_x^t = \int_M \mathcal{W}_{x \rightarrow y}^t \, d\text{vol}(y).$$

Define the Brownian loop measure on $M$:

$$\mu_M^{\text{loop}} := \int_0^\infty \frac{dt}{t} \int_M \mathcal{W}_{x \rightarrow x}^t \, d\text{vol}(x).$$

In particular,

$$|\mathcal{W}_{x \rightarrow x}^t| = p_t(x, x).$$

We consider $\mu_M^{\text{loop}}$ as measure on unrooted Brownian loops by forgetting the starting point.
Brownian loop measure

Introduced by Greg Lawler and Wendelin Werner.

[Following J. Dubédat] Let $x \in M$, $t > 0$, consider the sub-probability measure $W_x^t$ on the path of Brownian motion (diffusion generated by $-\Delta_M$) on $M$ started from $x$ on the time interval $[0, t]$, killed if it hits the boundary of $M$.

The measures $W_{x \to y}^t$ on paths from $x$ to $y$ are obtained from the disintegration of $W_x^t$ according to its endpoint $y$:

$$W_x^t = \int_M W_{x \to y}^t \, d\text{vol}(y).$$

Define the Brownian loop measure on $M$:

$$\mu_{\text{loop}}^M := \int_0^\infty \frac{dt}{t} \int_M W_{x \to x}^t \, d\text{vol}(x).$$

In particular,

$$|W_{x \to x}^t| = p_t(x, x).$$

We consider $\mu_{\text{loop}}^M$ as measure on unrooted Brownian loops by forgetting the starting point.
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The measures \( W^t_{x \to y} \) on paths from \( x \) to \( y \) are obtained from the disintegration of \( W^t_x \) according to its endpoint \( y \):

\[
W^t_x = \int_M W^t_{x \to y} \, d\text{vol}(y).
\]

Define the **Brownian loop measure** on \( M \):

\[
\mu_{M}^{\text{loop}} := \int_0^\infty \frac{dt}{t} \int_M W^t_{x \to x} \, d\text{vol}(x).
\]

In particular,

\[
|W^t_{x \to x}| = p_t(x, x).
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We consider \( \mu_{M}^{\text{loop}} \) as measure on **unrooted** Brownian loops by forgetting the starting point.
Brownian loop measure

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Define the **Brownian loop measure** on $M$:

$$\mu^\text{loop}_M := \int_0^\infty \frac{dt}{t} \int_M \mathbb{W}^t_{x \to x} \, d\text{vol}(x).$$

In particular,

$$\left|\mathbb{W}^t_{x \to x}\right| = p_t(x, x).$$

We consider $\mu^\text{loop}_M$ as measure on **unrooted** Brownian loops by forgetting the starting point.
The Brownian loop measure satisfies the following two remarkable properties

- **(Restriction property)** If $M' \subset M$, then
  \[ d\mu_{\text{loop}}^{M'}(\delta) = 1_{\delta \in M'} d\mu_{\text{loop}}^{M}(\delta). \]

- **(Conformal invariance)** On the surfaces $M_1 = (M, g)$ and $M_2 = (M, e^{2\sigma} g)$ be two conformally equivalent Riemann surface, where $\sigma \in C^\infty(M, \mathbb{R})$, then
  \[ \mu_{M_1}^{\text{loop}} = \mu_{M_2}^{\text{loop}}. \]
"\[ |\mu^\text{loop}_M| = - \log \det_\zeta(\Delta).\]"

If we compute formally, the total mass of \(\mu^\text{loop}_M\) is given by

"\[ |\mu^\text{loop}_M| = \int_0^\infty \frac{dt}{t} \int_M p_t(x, x) \, d\mathrm{vol}(x) = \int_0^\infty t^{-1} \text{Tr} \left( e^{-t\Delta} \right) \, dt.\]"

On the other hand, \(1/\Gamma(s)\) is analytic and has the expansion near 0 as

\[ 1/\Gamma(s) = s + O(s^2). \]

Therefore for any analytic function \(f\) in a neighborhood of 0,

\[ \left. \left( \frac{f(s)}{\Gamma(s)} \right)' \right|_{s=0} = f(0). \]

Take formally \(f(s) = \int_0^\infty t^{s-1} \text{Tr} \left( e^{-t\Delta} \right) \, dt\), we have

"\[ - \log \det_\zeta(\Delta) = \zeta'_\Delta(0) = \left. \left( \frac{f(s)}{\Gamma(s)} \right)' \right|_{s=0} = \int_0^\infty t^{-1} \text{Tr} \left( e^{-t\Delta} \right) \, dt = |\mu^\text{loop}_M|. \]" (2)
The determinant expression of Loewner energy suggests that we have formally

\[
\frac{1}{12} l^L(\Gamma) = \log \frac{\det \zeta(\Delta_{D_1, g}) \det \zeta(\Delta_{D_2, g})}{\det \zeta(\Delta_{D_1, g}) \det \zeta(\Delta_{D_2, g})} = |\mu_{D_1}^{\text{loop}}| + |\mu_{D_2}^{\text{loop}}| - |\mu_{D_1}^{\text{loop}}| - |\mu_{D_2}^{\text{loop}}| + |\mu_{S_2}^{\text{loop}}| - |\mu_{S_2}^{\text{loop}}| = \mu_{S_2}^{\text{loop}}(\{\delta; \delta \cap S^1 \neq \emptyset\}) - \mu_{S_2}^{\text{loop}}(\{\delta; \delta \cap \Gamma \neq \emptyset\}).
\]

However, both terms diverge due to the small and large Brownian loops (from the conformal invariance).

" \mid \mu_M^{\text{loop}} \mid = - \log \det \zeta(\Delta)."
For a Brownian loop $\delta \subset D$, where $D \subset \mathbb{D}$ is simply connected, we denote $\delta^{\text{out}}$ its outer boundary (therefore of $\text{SLE}_{8/3}$ type).

Let $A, B \subset \mathbb{C}$ be disjoint compact sets,

$$\mathcal{W}(A, B; D) := \left| \mu^{\text{loop}} \{ \delta \subset D; \delta^{\text{out}} \text{ intersects both } A \text{ and } B \} \right| < \infty.$$ 

Introduced by W. Werner.

**Theorem (W., 2018)**

For all Jordan curve $\Gamma$ (no regularity assumption),

$$\frac{1}{12} I^L(\Gamma) = \lim_{r \to 1} \mathcal{W}(S^1, rS^1; \mathbb{C}) - \mathcal{W}(\Gamma, \Gamma^r; \mathbb{C}).$$
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$$\frac{1}{12} l^L(\Gamma) = \lim_{r \to 1} \mathcal{W}(S^1, rS^1; \mathbb{C}) - \mathcal{W}(\Gamma, \Gamma^r; \mathbb{C}).$$
Proof: Chordal Conformal restriction

Lemma 1: Chordal Conformal restriction

Let \((D, a, b)\) and \((D', a, b)\) be two simply connected domains in \(\mathbb{C}\) coinciding in a neighborhood of \(a\) and \(b\), and \(\Gamma\) a simple curve in both \((D, a, b)\) and \((D', a, b)\). Then we have

\[
I_{D', a, b}(\Gamma) - I_{D, a, b}(\Gamma) = I_{D, a, b}(\psi(\Gamma)) - I_{D, a, b}(\Gamma)
\]

\[
= 3 \log |\psi'(a)\psi'(b)| + 12W(\Gamma, D \setminus D'; D) - 12W(\Gamma, D' \setminus D; D'),
\]

where \(\psi : D' \to D\) is a conformal map fixing \(a\) and \(b\).

**Deterministic proof, similar computation as in SLE conformal restriction.**

**Intuition:** The SLE partition function is

\[
\mathcal{Z}^{\text{SLE}_{\kappa}}_{(D, a, b)} = H_D(a, b) B \det \zeta(\Delta)^{-c/2},
\]

where as \(\kappa \to 0\),

\[
\beta = \frac{6 - \kappa}{2\kappa} \sim \frac{3}{\kappa}, \quad c = \frac{(3\kappa - 8)(6 - \kappa)}{2\kappa} \sim -\frac{24}{\kappa}.
\]

The Energy = “\(-\kappa \log(\cdot)\)”
Proof: Loop Conformal restriction

**Lemma 2: Loop conformal restriction**

If $\eta$ is a Jordan curve with finite energy and $\Gamma = f(\eta)$, where $f : A \to \tilde{A}$ is conformal on a neighborhood $A$ of $\eta$, then

$$I_L(\Gamma) - I_L(\eta) = 12W(\eta, A^c; \mathbb{C}) - 12W(\Gamma, \tilde{A}^c; \mathbb{C}).$$

**Proof of Lemma 2:**

Let $b \to a$. 

- $K := A^c \cup T$
- $\tilde{K} := \tilde{A}^c \cup \tilde{T}$
- $\psi : \mathcal{C}\setminus((ab)_\eta \cup K) \to \mathcal{C}\setminus(ab)_\eta$
- $\tilde{\psi} : \mathcal{C}\setminus((\tilde{a}\tilde{b})_\Gamma \cup \tilde{K}) \to \mathcal{C}\setminus(\tilde{a}\tilde{b})_\Gamma$
- $g : \mathcal{C}\setminus(ab)_\eta \to \mathcal{C}\setminus(\tilde{a}\tilde{b})_\Gamma$
Proof: Equipotentials

When $\eta = rS^1$, $\Gamma' = f(rS^1)$ is the equipotential, and $A = \mathbb{D}$.

We deduce

$$I^L(\Gamma') = 12\mathcal{W}(rS^1, S^1; \mathbb{C}) - 12\mathcal{W}(\Gamma', \Gamma; \mathbb{C}).$$

Lemma 3

We have: $I^L(\Gamma') \xrightarrow{r \to 1} I^L(\Gamma)$.

In fact, $r \mapsto I^L(\Gamma')$ is increasing if $I^L(\Gamma) > 0$, namely when $\Gamma$ is not a circle. It will follow from the flow-line coupling for finite energy curve [Viklund, W. 2019+].
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**Lemma 3**

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SLE/GFF coupling analogs: A Dictionary

*Work in progress with F. Viklund.* With $\gamma = \sqrt{\kappa}$, $\chi = \gamma/2 - 2/\gamma$:

**Random Conformal Geometry $\leftrightarrow$ Action Functional Analogs**

- Neumann GFF on $\mathbb{H} \leftrightarrow 2u_1 : \mathbb{H} \to \mathbb{R}$ with finite Dirichlet energy;
- Neumann GFF on $\mathbb{H}^* \leftrightarrow 2u_2 : \mathbb{H}^* \to \mathbb{R}$ with finite Dirichlet energy;
- $\gamma$-LQG measure on $\mathbb{H}$, $e^{\gamma \text{GFF}} \, dz^2 \leftrightarrow e^{2u_1(z)} \, dz^2$;
- $\gamma$-LQG boundary measure on $\mathbb{R} = \partial \mathbb{H} \leftrightarrow e^{u_1(z)} \, |dz|$, $u_1|_{\mathbb{R}} \in H^{1/2}(\mathbb{R})$;
- “SLE$_\kappa$ loop” $\leftrightarrow$ finite energy loop $\Gamma$;
- $\gamma$-LQG on $\mathbb{C} \leftrightarrow e^{2\varphi(z)} \, dz^2$;
- $\gamma$-quantum chaos wrt. $\leftrightarrow$ trace of $\varphi$ on $\Gamma \in H^{1/2}(\Gamma)$;

natural parametrization on SLE loop

independent couple $\leftrightarrow$ sum up their rate functions;

$e^{i\text{GFF}/\chi} \leftrightarrow e^{i\varphi(z)}$ unit vector field;
Isometric conformal welding

Let $D_1, D_2 \subset \mathbb{C}$ be Jordan domains bounded respectively by rectifiable curves $\Gamma_1$ and $\Gamma_2$ of same total length. Let $\psi : \Gamma_1 \rightarrow \Gamma_2$ be an isometry (preserves the arc-length).

- [Huber 1976] The solution does not always exist.
- [Bishop 1990] Even if the solution exists, $\Gamma$ can be a curve of positive area $\Rightarrow$ non-uniqueness of solution.
- [David 1982, Zinsmeister 1982... ] If $D_1$ and $D_2$ are chord-arc, then the solution exists and is unique, which is an quasi-circle. [Bishop 1990] The Hausdorff dimension of $\Gamma$ can take any value in $1 < d < 2$.
- [David 1982] If the chord-arc constant of domains are close enough to 1, $\Gamma$ is also chord-arc.
- [Viklund, W. 2019+] We will see that isometric welding of two finite energy domains has also finite energy (solution exists and is unique).
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- [Viklund, W. 2019+] We will see that isometric welding of two finite energy domains has also finite energy (solution exists and is unique).
Welding coupling identity

Let $\varphi \in W^{1,2}_{loc}(\mathbb{C})$ with finite Dirichlet energy:

$$D_{\mathbb{C}}(\varphi) := \frac{1}{\pi} \int_{\mathbb{C}} |\nabla \varphi(z)|^2 \, dz^2 < \infty,$$

$\Gamma$ an infinite Jordan curve, $f, g$ the conformal maps from $\mathbb{H}, \mathbb{H}^*$ onto $H, H^*$, respectively.

**Theorem (Welding coupling 2019+)**

We have $e^{2\varphi} \in L^1_{loc}(\mathbb{C})$, so the measure $e^{2\varphi} \, dz^2$ is well-defined and locally finite. The pull-back measures $e^{2u_1}$ by $f$ on $\mathbb{H}$ (resp. $e^{2u_2}$ by $g$ on $\mathbb{H}^*$) satisfy

$$u_1(z) = \varphi \circ f(z) + \log |f'(z)|, \quad u_2(z) = \varphi \circ g(z) + \log |g'(z)|.$$

We have the identity

$$D_{\mathbb{H}}(u_1) + D_{\mathbb{H}^*}(u_2) = l^L(\Gamma) + D_{\mathbb{C}}(\varphi).$$
Theorem (Welding-coupling uniqueness, 2019+)

Suppose $u_1$ and $u_2$ with finite Dirichlet energy are given. Then there exist unique $\Gamma$, $\varphi$, $f$, and $g$ such that the following holds:

1. $\Gamma$ is an infinite Jordan curve passing through 0 and 1;
2. If $H$ and $H^*$ are the connected components of $\mathbb{C}\setminus\Gamma$, then $f : \mathbb{H} \to H$ is the conformal map fixing 0, 1 and $\infty$ and $g : \mathbb{H}^* \to H^*$ is the conformal map fixing 0, $\infty$;
3. $\varphi \in W_{\text{loc}}^{1,2}(\mathbb{C})$ and $D_{\mathbb{C}}(\varphi) < \infty$;
4. $u_1(z) = \varphi \circ f(z) + \log |f'(z)|$, $z \in \mathbb{H}$;
5. $u_2(z) = \varphi \circ g(z) + \log |g'(z)|$, $z \in \mathbb{H}^*$.

In fact, $\Gamma$ is obtained from the isometric conformal welding of $\mathbb{H}$ and $\mathbb{H}^*$ according to the boundary lengths $e^{u_1}|dz|$ and $e^{u_2}|dz|$. Moreover, $I^L(\Gamma) < \infty$. 
Isometric welding of finite energy domains

Assume $I^L(\Gamma_1) < \infty, I^L(\Gamma_2) < \infty$, both curves pass through $\infty$.

**Corollary**

The isometric conformal welding of Euclidean domain $H_1$ bounded by $\Gamma_1$ and $H_2$ bounded by $\Gamma_2$ has a unique solution $\Gamma$ up to Möbius transformation. Moreover,

$$I^L(\Gamma) < I^L(\Gamma_1) + I^L(\Gamma_2)$$

if $I^L(\Gamma_1) + I^L(\Gamma_2) \neq 0$.

In fact, let $u_1 = \log |f'_1|, u_2 = \log |g'_2|, \quad D(u_1) \leq I^L(\Gamma_1), \quad I^L(\Gamma) \leq D(u_1) + D(u_2) \leq I^L(\Gamma_1) + I^L(\Gamma_2)$.

The first equality holds only when $I^L(\Gamma_1) = 0$. 
Assume $I^L(\Gamma_1) < \infty$, $I^L(\Gamma_2) < \infty$, both curves pass through $\infty$.

**Corollary**

The isometric conformal welding of Euclidean domain $H_1$ bounded by $\Gamma_1$ and $H_2$ bounded by $\Gamma_2$ has a unique solution $\Gamma$ up to Möbius transformation. Moreover,

$$I^L(\Gamma) < I^L(\Gamma_1) + I^L(\Gamma_2)$$

if $I^L(\Gamma_1) + I^L(\Gamma_2) \neq 0$.

In fact, let $u_1 = \log|f_1'|$, $u_2 = \log|g_2'|$,

$$D(u_1) \leq I^L(\Gamma_1), \quad I^L(\Gamma) \leq D(u_1) + D(u_2) \leq I^L(\Gamma_1) + I^L(\Gamma_2).$$

The first equality holds only when $I^L(\Gamma_1) = 0$. 
Recall that $u_1(z) = \varphi \circ f(z) + \log |f'(z)|$, $u_2(z) = \varphi \circ g(z) + \log |g'(z)|$.

Use the identity $I^L(\Gamma) = D_H(\log |f'|) + D_{H^*}(\log |g'|)$.

Prove that the cross-terms cancel out.
Notice that since the harmonic conjugate arg($f'$) has the same Dirichlet energy as log |$f'$|.
We have the identity

$$I^L(\Gamma) = D_\mathbb{H}(\text{arg } f') + D_{\mathbb{H}^*}(\text{arg } g').$$

⇒ the analog to the forward SLE/GFF coupling (flow-line coupling).
Notice that since the harmonic conjugate $\arg(f')$ has the same Dirichlet energy as $\log|f'|$. We have the identity

$$l^L(\Gamma) = D_H(\arg f') + D_H^*(\arg g').$$

⇒ the analog to the forward SLE/GFF coupling (flow-line coupling).
Analog to flow-line coupling

Let $\eta$ be a bounded $C^1$ Jordan curve and $\Gamma := \mu(\eta)$, where $\mu$ is a Möbius function mapping one point of $\eta$ to $\infty$.

For $z = \Gamma(s)$, define the function $\tau : \Gamma \to \mathbb{R}$ such that $\tau$ is continuous and

$$\tau(z) := \arg(\Gamma'(s)) = -\arg(f^{-1})'(z).$$

We denote by $P[\tau](z) = -\arg(f^{-1})'(z)$ the Poisson integral of $\tau$ in $\mathbb{C}$ (defined from both sides of $\Gamma$).

**Theorem (Flowline coupling analog 2019+)**

We have the identity

$$I^L(\Gamma) = D_C(P[\tau]) = \min_{\phi, \phi|\Gamma=\tau} D_C(\phi).$$

Conversely, under regularity condition of $\phi$ and $D_C(\phi) < \infty$, then for all $z_0 \in \mathbb{C}$, the solution to the differential equation

$$\Gamma'(t) = \exp(i\phi(\Gamma(t))), \forall t \in \mathbb{R} \quad \text{and} \quad \Gamma(0) = z_0$$

is an infinite arclength parametrized simple curve and

$$I^L(\Gamma) \leq D_C(\phi).$$
Analog to flow-line coupling

Let \( \eta \) be a bounded \( C^1 \) Jordan curve and \( \Gamma := \mu(\eta) \), where \( \mu \) is a M"obius function mapping one point of \( \eta \) to \( \infty \).

For \( z = \Gamma(s) \), define the function \( \tau : \Gamma \to \mathbb{R} \) such that \( \tau \) is continuous and

\[
\tau(z) := \arg(\Gamma'(s)) = -\arg(f^{-1})'(z).
\]

We denote by \( \mathcal{P}[\tau](z) = -\arg(f^{-1})'(z) \) the Poisson integral of \( \tau \) in \( \mathbb{C} \) (defined from both sides of \( \Gamma \)).

**Theorem (Flowline coupling analog 2019+)**

We have the identity

\[
I^L(\Gamma) = \mathcal{D}_\mathbb{C}(\mathcal{P}[\tau]) = \min_{\varphi, \varphi|_{\Gamma=\tau}} \mathcal{D}_\mathbb{C}(\varphi).
\]

Conversely, under regularity condition of \( \varphi \) and \( \mathcal{D}_\mathbb{C}(\varphi) < \infty \), then for all \( z_0 \in \mathbb{C} \), the solution to the differential equation

\[
\Gamma'(t) = \exp(i\varphi(\Gamma(t))), \forall t \in \mathbb{R} \quad \text{and} \quad \Gamma(0) = z_0
\]

is an infinite arclength parametrized simple curve and

\[
I^L(\Gamma) \leq \mathcal{D}_\mathbb{C}(\varphi).
\]
Equipotential energy decrease

\[ \Gamma^y := f(\mathbb{R} + iy) \]

\[ f(\infty) = \infty \]

Corollary

We have \( I^L(\Gamma^y) \leq I^L(\Gamma) \). The equality holds if and only if \( I^L(\Gamma) = 0 \).

Proof: Since on \( \Gamma^y \), \( \tau^y = \mathcal{P}[\tau] \). We have

\[ I^L(\Gamma^y) = \mathcal{D}_C(\mathcal{P}[\tau^y]) \leq \mathcal{D}_C(\mathcal{P}[\tau]) = I^L(\Gamma). \]

Yilin Wang (ETH Zürich)
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- Quantum zipper by Sheffield
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Part I

Part II

(Dubédat 2008)
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  Malliavin’s measure on diffeomorphisms of the circle?
- In which space does the random welding belong to? (What analytic framework beyond quasiconformal geometry?)
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- How is the Kähler structure on the WP-Teichmüller space encoded in the Loewner’s driving function? Why there is such a coincidence?
- Topological group structure on WP-Teichmüller space $\Rightarrow$ what meaning in the Loewner setting?
- Use driving function to find purely geometric characterization of WP-quasicircles? (Jones’ Conjecture)
- \([TT06]\) WP-quasicircle $\Leftrightarrow$ associated Grunsky operator $G$ is Hilbert-Schmidt. Moreover,
  \[ I^L(\Gamma) \propto \log \det_F (I - G^* G), \]
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  Is it a better object to look at than zeta-regularized determinant of Laplacian? Interpretation of Grunsky operator?
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- Multiple-chord Loewner energy, large deviation of multiple SLE (work in progress with E. Peltola).
- Energy of (multiple) loops in higher genus surface?
- Probabilistic interpretation of Weil-Petersson metric on Teichmüller space of compact surfaces (genus $\geq 2$)? Natural measure on Teichmüller/moduli space?
- Conformal field theory (SLE, statistical mechanics models) $\Rightarrow$ String theory (Kähler geometry on universal Teichmüller space)???
Thanks for your attention!