Loewner energy via Brownian loop measure and action functional analogs of SLE/GFF couplings

Yilin Wang

ETH Zürich

January, 2019

IPAM UCLA

Contents

Introduction

- **Loewner's transform** [1923] consists of encoding the uniformizing conformal map of a simply connected domain $D \subset \mathbb{C}$ into evolution of conformal distortions that flatten out the boundary iteratively,

  non self-intersecting curve $\partial D \Leftrightarrow$ **real-valued driving function**.

  Main tool to solve Bieberbach's conjecture by De Branges in 1985.
- Random fractal non self-intersecting curves: the **Schramm-Loewner Evolution** introduced by Oded Schramm in 1999 which successfully describe interfaces in many statistical mechanics models.
- The **Loewner energy** is the action functional of SLE, also the large deviation rate function of SLE_κ as $\kappa \to 0$ [W. 2016].
- Loewner energy for Jordan curves (loops) on the Riemann's sphere, is non-negative, vanishing only on circles, and invariant under Möbius transformation [Rohde, W. 2017].
- **Weil-Petersson metric** is the unique homogeneous Kähler metric on the universal Teichmüller space. Loewner energy is Kähler potential of this metric.

Introduction

- **Loewner's transform** [1923] consists of encoding the uniformizing conformal map of a simply connected domain $D \subset \mathbb{C}$ into evolution of conformal distortions that flatten out the boundary iteratively,

  non self-intersecting curve $\partial D \Leftrightarrow$ **real-valued driving function**.

  Main tool to solve Bieberbach's conjecture by De Branges in 1985.
- Random fractal non self-intersecting curves: the **Schramm-Loewner Evolution** introduced by Oded Schramm in 1999 which successfully describe interfaces in many statistical mechanics models.
- The **Loewner energy** is the action functional of SLE, also the large deviation rate function of SLE_κ as $\kappa \to 0$ [W. 2016].
- Loewner energy for Jordan curves (loops) on the Riemann's sphere, is non-negative, vanishing only on circles, and invariant under Möbius transformation [Rohde, W. 2017].
- **Weil-Petersson metric** is the unique homogeneous Kähler metric on the universal Teichmüller space. Loewner energy is Kähler potential of this metric.

Introduction

- **Loewner's transform** [1923] consists of encoding the uniformizing conformal map of a simply connected domain $D \subset \mathbb{C}$ into evolution of conformal distortions that flatten out the boundary iteratively,

  non self-intersecting curve $\partial D \Leftrightarrow$ **real-valued driving function**.

  Main tool to solve Bieberbach's conjecture by De Branges in 1985.
- Random fractal non self-intersecting curves: the **Schramm-Loewner Evolution** introduced by Oded Schramm in 1999 which successfully describe interfaces in many statistical mechanics models.
- The **Loewner energy** is the action functional of SLE, also the large deviation rate function of SLE_κ as $\kappa \to 0$ [W. 2016].
- Loewner energy for Jordan curves (loops) on the Riemann's sphere, is non-negative, vanishing only on circles, and invariant under Möbius transformation [Rohde, W. 2017].
- **Weil-Petersson metric** is the unique homogeneous Kähler metric on the universal Teichmüller space. Loewner energy is Kähler potential of this metric.

Introduction

- **Loewner's transform** [1923] consists of encoding the uniformizing conformal map of a simply connected domain $D \subset \mathbb{C}$ into evolution of conformal distortions that flatten out the boundary iteratively,

  non self-intersecting curve $\partial D \Leftrightarrow$ **real-valued driving function**.

  Main tool to solve Bieberbach's conjecture by De Branges in 1985.
- Random fractal non self-intersecting curves: the **Schramm-Loewner Evolution** introduced by Oded Schramm in 1999 which successfully describe interfaces in many statistical mechanics models.
- The **Loewner energy** is the action functional of SLE, also the large deviation rate function of SLE_κ as $\kappa \to 0$ [W. 2016].
- Loewner energy for Jordan curves (loops) on the Riemann's sphere, is non-negative, vanishing only on circles, and invariant under Möbius transformation [Rohde, W. 2017].
- **Weil-Petersson metric** is the unique homogeneous Kähler metric on the universal Teichmüller space. Loewner energy is Kähler potential of this metric.

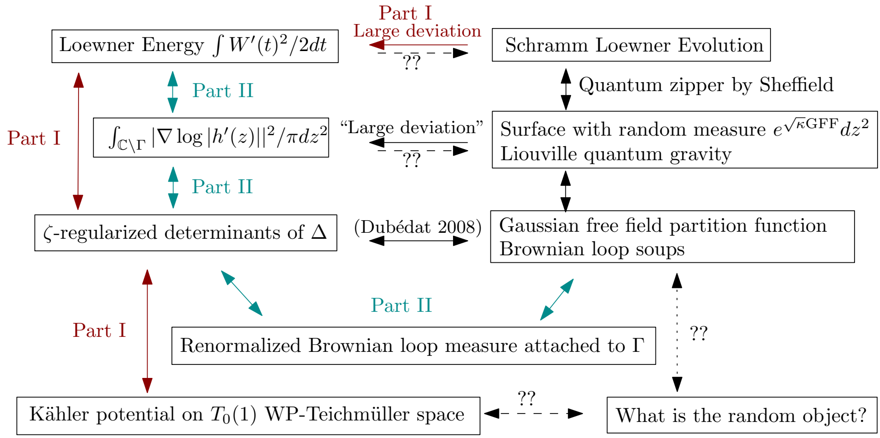
Introduction

- **Loewner's transform** [1923] consists of encoding the uniformizing conformal map of a simply connected domain $D \subset \mathbb{C}$ into evolution of conformal distortions that flatten out the boundary iteratively,

  non self-intersecting curve $\partial D \Leftrightarrow$ **real-valued driving function**.

  Main tool to solve Bieberbach's conjecture by De Branges in 1985.
- Random fractal non self-intersecting curves: the **Schramm-Loewner Evolution** introduced by Oded Schramm in 1999 which successfully describe interfaces in many statistical mechanics models.
- The **Loewner energy** is the action functional of SLE, also the large deviation rate function of SLE_κ as $\kappa \to 0$ [W. 2016].
- Loewner energy for Jordan curves (loops) on the Riemann's sphere, is non-negative, vanishing only on circles, and invariant under Möbius transformation [Rohde, W. 2017].
- **Weil-Petersson metric** is the unique homogeneous Kähler metric on the universal Teichmüller space. Loewner energy is Kähler potential of this metric.

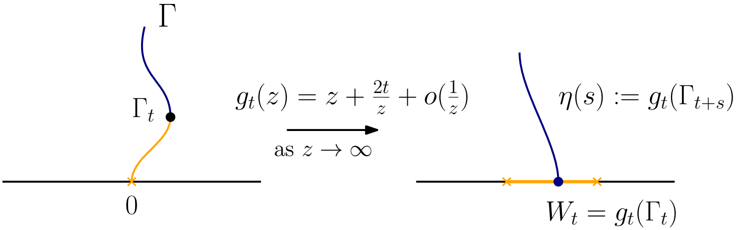
Action functionals vs. Random objects



Contents

Chordal Loewner chains

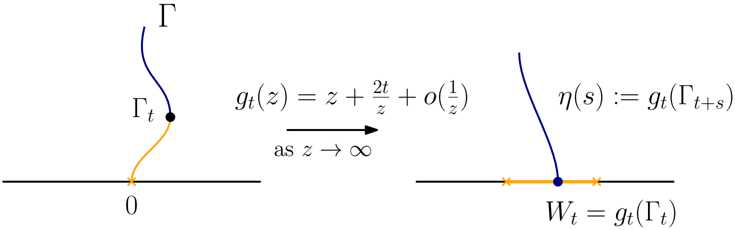
Let Γ be a simple chord in $\mathbb{H}$ from 0 to ∞.



- Γ is **capacity-parametrized** by $[0, \infty)$.
- $W : \mathbb{R}_+ \to \mathbb{R}$ is called the **driving function** of Γ.
- $W_0 = 0$.
- W is continuous.
- One can recover the curve Γ from W using Loewner's differential equation.
- We say that Γ is the **chordal Loewner chain** generated by W.
- The centered Loewner flow has the expansion $f_t(z) = g_t(z) - W_t = z - W_t + 2t/z + O(1/z)$.

Chordal Loewner chains

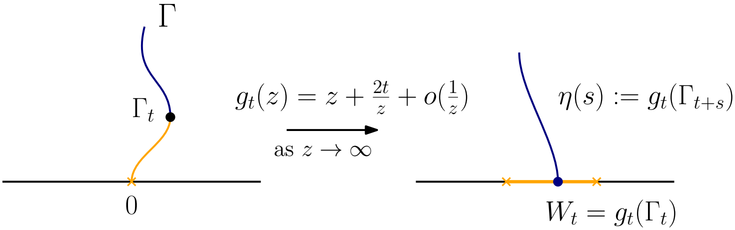
Let Γ be a simple chord in $\mathbb{H}$ from 0 to ∞.



- Γ is **capacity-parametrized** by $[0, \infty)$.
- $W : \mathbb{R}_+ \to \mathbb{R}$ is called the **driving function** of Γ.
- $W_0 = 0$.
- W is continuous.
- One can recover the curve Γ from W using Loewner's differential equation.
- We say that Γ is the **chordal Loewner chain** generated by W.
- The centered Loewner flow has the expansion $f_t(z) = g_t(z) - W_t = z - W_t + 2t/z + O(1/z)$.

Chordal Loewner chains

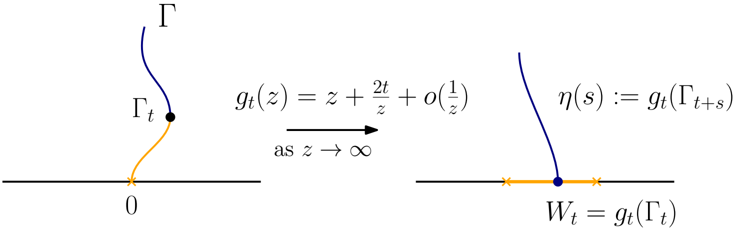
Let Γ be a simple chord in $\mathbb{H}$ from 0 to ∞.



- Γ is **capacity-parametrized** by $[0, \infty)$.
- $W : \mathbb{R}_+ \to \mathbb{R}$ is called the **driving function** of Γ.
- $W_0 = 0$.
- W is continuous.
- One can recover the curve Γ from W using Loewner's differential equation.
- We say that Γ is the **chordal Loewner chain** generated by W.
- The centered Loewner flow has the expansion $f_t(z) = g_t(z) - W_t = z - W_t + 2t/z + O(1/z)$.

Chordal Loewner chains

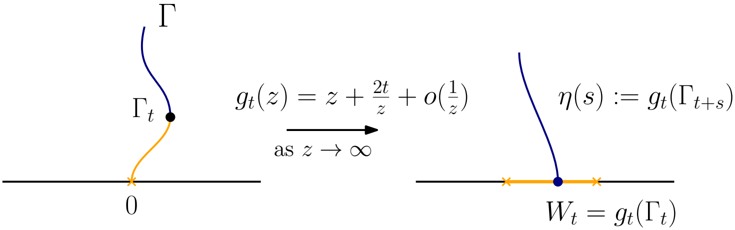
Let Γ be a simple chord in $\mathbb{H}$ from 0 to ∞.



- Γ is **capacity-parametrized** by $[0, \infty)$.
- $W : \mathbb{R}_+ \to \mathbb{R}$ is called the **driving function** of Γ.
- $W_0 = 0$.
- W is continuous.
- One can recover the curve Γ from W using Loewner's differential equation.
- We say that Γ is the **chordal Loewner chain** generated by W.
- The centered Loewner flow has the expansion $f_t(z) = g_t(z) - W_t = z - W_t + 2t/z + O(1/z)$.

Chordal Loewner chains

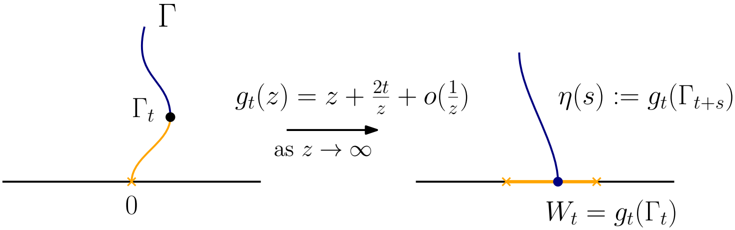
Let Γ be a simple chord in $\mathbb{H}$ from 0 to ∞.



- Γ is **capacity-parametrized** by $[0, \infty)$.
- $W : \mathbb{R}_+ \to \mathbb{R}$ is called the **driving function** of Γ.
- $W_0 = 0$.
- W is continuous.
- One can recover the curve Γ from W using Loewner's differential equation.
- We say that Γ is the **chordal Loewner chain** generated by W.
- The centered Loewner flow has the expansion $f_t(z) = g_t(z) - W_t = z - W_t + 2t/z + O(1/z)$.

Chordal Loewner chains

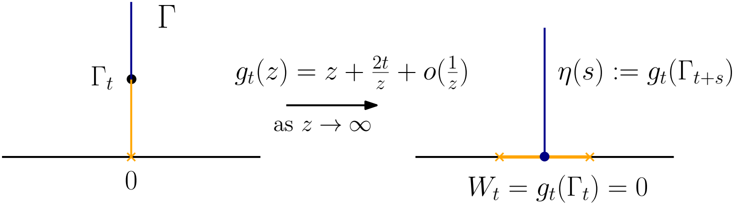
Let Γ be a simple chord in $\mathbb{H}$ from 0 to ∞.



- Γ is **capacity-parametrized** by $[0, \infty)$.
- $W : \mathbb{R}_+ \to \mathbb{R}$ is called the **driving function** of Γ.
- $W_0 = 0$.
- W is continuous.
- One can recover the curve Γ from W using Loewner's differential equation.
- We say that Γ is the **chordal Loewner chain** generated by W.
- The centered Loewner flow has the expansion $f_t(z) = g_t(z) - W_t = z - W_t + 2t/z + O(1/z)$.

Chordal Loewner chain

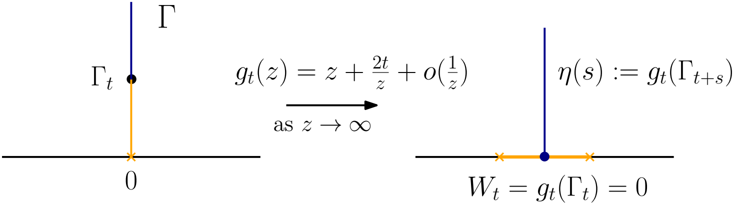
- If $W \equiv 0$, then $\Gamma = i\mathbb{R}_+$.



- When the curve is driven by $W = \sqrt{\kappa}B$ where B is 1-d Brownian motion, the curve generated is the **Schramm-Loewner Evolution of parameter** κ (SLE_κ).

Chordal Loewner chain

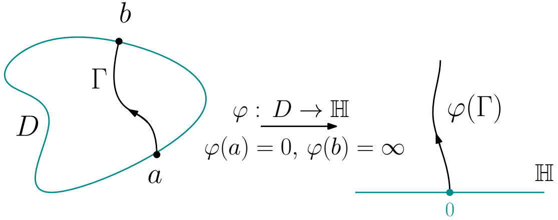
- If $W \equiv 0$, then $\Gamma = i\mathbb{R}_+$.



- When the curve is driven by $W = \sqrt{\kappa}B$ where B is 1-d Brownian motion, the curve generated is the **Schramm-Loewner Evolution of parameter** κ (SLE$_\kappa$).

The chordal Loewner energy

$D \subset \mathbb{C}$ a simply connected domain, a, b are two boundary points of D.



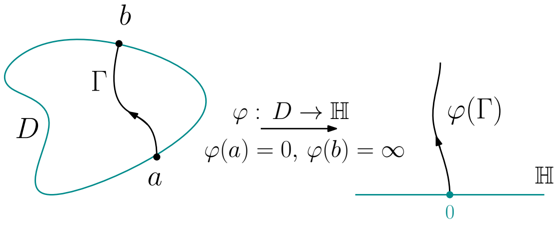
Definition: Loewner energy

We define the **Loewner energy of a simple chord Γ in** (D, a, b) to be

$$I_{D,a,b}(\Gamma) := I_{\mathbb{H},0,\infty}(\varphi(\Gamma)) := I(W) := \frac{1}{2}\int_0^\infty W'(t)^2 dt$$

where W is the driving function of $\varphi(\Gamma)$.

The chordal Loewner energy

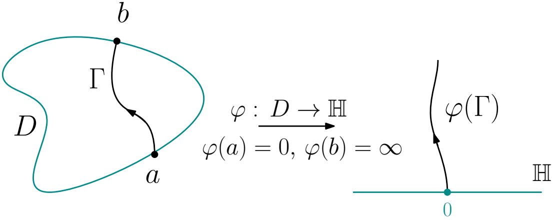


- The Loewner energy is well-defined in (D, a, b) since for $c > 0$,

$$I_{\mathbb{H},0,\infty}(\Gamma) = I_{\mathbb{H},0,\infty}(c\Gamma).$$

- $I_{D,a,b}(\Gamma) = 0$ *iff* Γ is the hyperbolic geodesic connecting a and b.
- $I_{D,a,b}(\Gamma) < \infty$, then Γ is rectifiable *[Friz & Shekhar, 2015]*.

The chordal Loewner energy

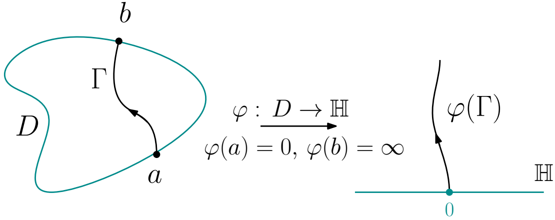


- The Loewner energy is well-defined in (D, a, b) since for $c > 0$,

$$I_{\mathbb{H},0,\infty}(\Gamma) = I_{\mathbb{H},0,\infty}(c\Gamma).$$

- $I_{D,a,b}(\Gamma) = 0$ *iff* Γ is the hyperbolic geodesic connecting a and b.
- $I_{D,a,b}(\Gamma) < \infty$, then Γ is rectifiable *[Friz & Shekhar, 2015]*.

The chordal Loewner energy



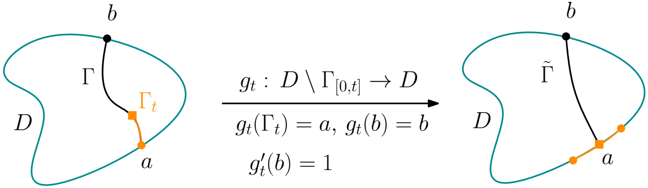
- The Loewner energy is well-defined in (D, a, b) since for $c > 0$,

$$I_{\mathbb{H},0,\infty}(\Gamma) = I_{\mathbb{H},0,\infty}(c\Gamma).$$

- $I_{D,a,b}(\Gamma) = 0$ *iff* Γ is the hyperbolic geodesic connecting a and b.
- $I_{D,a,b}(\Gamma) < \infty$, then Γ is rectifiable *[Friz & Shekhar, 2015]*.

Upper half-plane vs. other domains

Assume that ∂D is smooth in a neighborhood of b, a continuously parametrized chord $\Gamma : [0, T] \to \overline{D}$ from a to b.



The **capacity parametrization** of Γ seen from b is chosen using the Schwarzian derivative of the mapping-out function:

$$\mathrm{cap}(\Gamma[0,t]) := -\frac{S(g_t)(b)}{12}.$$

The **driving function** is given by

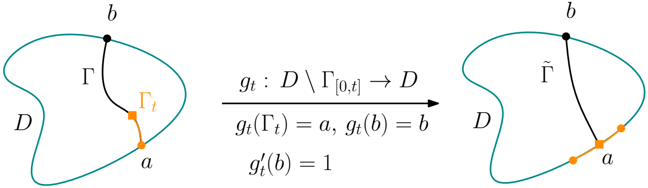
$$W_t = \frac{1}{2}\frac{g_t''(b)}{g_t'(b)}.$$

The **Loewner energy** is given by

$$I_{D,a,b}(\Gamma) = \sup_{0 \le T_0 < T_1 < \cdots < T_n = T} \sum_{i=0}^{n-1} \frac{(W_{T_{i+1}} - W_{T_i})^2}{\mathrm{cap}(\Gamma[0, T_{i+1}]) - \mathrm{cap}(\Gamma[0, T_i])}$$

Upper half-plane vs. other domains

Assume that ∂D is smooth in a neighborhood of b, a continuously parametrized chord $\Gamma : [0, T] \to \overline{D}$ from a to b.



The **capacity parametrization** of Γ seen from b is chosen using the Schwarzian derivative of the mapping-out function:

$$\mathrm{cap}(\Gamma[0, t]) := -\frac{S(g_t)(b)}{12}.$$

The **driving function** is given by

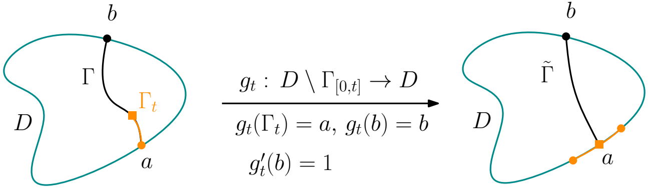
$$W_t = \frac{1}{2}\frac{g_t''(b)}{g_t'(b)}.$$

The **Loewner energy** is given by

$$I_{D,a,b}(\Gamma) = \sup_{0 \le T_0 < T_1 < \cdots < T_n = T} \sum_{i=0}^{n-1} \frac{(W_{T_{i+1}} - W_{T_i})^2}{\mathrm{cap}(\Gamma[0, T_{i+1}]) - \mathrm{cap}(\Gamma[0, T_i])}$$

Upper half-plane vs. other domains

Assume that ∂D is smooth in a neighborhood of b, a continuously parametrized chord $\Gamma : [0, T] \to \overline{D}$ from a to b.



The **capacity parametrization** of Γ seen from b is chosen using the Schwarzian derivative of the mapping-out function:

$$\mathrm{cap}(\Gamma[0, t]) := -\frac{S(g_t)(b)}{12}.$$

The **driving function** is given by

$$W_t = \frac{1}{2} \frac{g_t''(b)}{g_t'(b)}.$$

The **Loewner energy** is given by

$$I_{D,a,b}(\Gamma) = \sup_{0 \le T_0 < T_1 < \cdots < T_n = T} \sum_{i=0}^{n-1} \frac{(W_{T_{i+1}} - W_{T_i})^2}{\mathrm{cap}(\Gamma[0, T_{i+1}]) - \mathrm{cap}(\Gamma[0, T_i])}$$

SLE$_\kappa$ vs. Loewner energy

The Dirichlet energy $I(W)$ is the **action functional** of Brownian motion. Intuitively, the "Brownian path has the distribution on $C^0(\mathbb{R}_+, \mathbb{R})$ with density $\propto \exp(-I(W))\mathcal{D}W$."

However, $I(B) = \infty$ with probability 1.

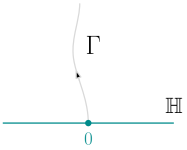
The Schilder's theorem states that $I(W)$ is also the **large deviation rate function** for Brownian motion $\sqrt{\kappa}B$ as $\kappa \to 0$. Loosely speaking,

$$\text{“}\mathrm{P}(\sqrt{\kappa}B \text{ stays close to } W) \approx \exp\left(-\frac{I(W)}{\kappa}\right).\text{”}$$

It should imply that the Loewner energy is the **large deviation rate function** of SLE$_\kappa$:

$$\text{“}\mathrm{P}(\text{SLE}_\kappa \text{ stays close to } \Gamma) \approx \exp\left(-\frac{I(\Gamma)}{\kappa}\right).\text{”} \tag{1}$$

The claim (1) is made precise in [W. 2016].



SLE$_\kappa$ vs. Loewner energy

The Dirichlet energy $I(W)$ is the **action functional** of Brownian motion. Intuitively, the "Brownian path has the distribution on $C^0(\mathbb{R}_+, \mathbb{R})$ with density $\propto \exp(-I(W))\mathcal{D}W$."

However, $I(B) = \infty$ *with probability* 1.

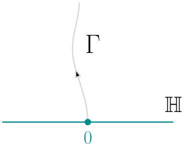
The Schilder's theorem states that $I(W)$ is also the **large deviation rate function** for Brownian motion $\sqrt{\kappa}B$ as $\kappa \to 0$. Loosely speaking,

$$\text{“}\mathrm{P}(\sqrt{\kappa}B \text{ stays close to } W) \approx \exp\left(-\frac{I(W)}{\kappa}\right).\text{”}$$

It should imply that the Loewner energy is the **large deviation rate function** of SLE$_\kappa$:

$$\text{“}\mathrm{P}(\mathrm{SLE}_\kappa \text{ stays close to } \Gamma) \approx \exp\left(-\frac{I(\Gamma)}{\kappa}\right).\text{”} \quad (1)$$

The claim (1) *is made precise in [W. 2016].*



SLE_κ vs. Loewner energy

The Dirichlet energy $I(W)$ is the **action functional** of Brownian motion. Intuitively, the "Brownian path has the distribution on $C^0(\mathbb{R}_+, \mathbb{R})$ with density $\propto \exp(-I(W))\mathcal{D}W$."

However, $I(B) = \infty$ *with probability* 1.

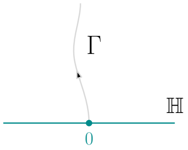
The Schilder's theorem states that $I(W)$ is also the **large deviation rate function** for Brownian motion $\sqrt{\kappa}B$ as $\kappa \to 0$. Loosely speaking,

$$\text{"}\mathrm{P}(\sqrt{\kappa}B \text{ stays close to } W) \approx \exp\left(-\frac{I(W)}{\kappa}\right)\text{."}$$

It should imply that the Loewner energy is the **large deviation rate function** of SLE_κ:

$$\text{"}\mathrm{P}(\text{SLE}_\kappa \text{ stays close to } \Gamma) \approx \exp\left(-\frac{I(\Gamma)}{\kappa}\right)\text{."} \tag{1}$$

The claim (1) *is made precise in [W. 2016].*



SLE_κ vs. Loewner energy

The Dirichlet energy $I(W)$ is the **action functional** of Brownian motion. Intuitively, the "Brownian path has the distribution on $C^0(\mathbb{R}_+, \mathbb{R})$ with density $\propto \exp(-I(W))\mathcal{D}W$."

However, $I(B) = \infty$ with probability 1.

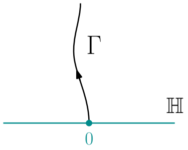
The Schilder's theorem states that $I(W)$ is also the **large deviation rate function** for Brownian motion $\sqrt{\kappa}B$ as $\kappa \to 0$. Loosely speaking,

$$\text{"}\mathrm{P}(\sqrt{\kappa}B \text{ stays close to } W) \approx \exp\left(-\frac{I(W)}{\kappa}\right).\text{"}$$

It should imply that the Loewner energy is the **large deviation rate function** of SLE_κ:

$$\text{"}\mathrm{P}(\text{SLE}_\kappa \text{ stays close to } \Gamma) \approx \exp\left(-\frac{I(\Gamma)}{\kappa}\right).\text{"} \quad (1)$$

The claim (1) is made precise in [W. 2016].

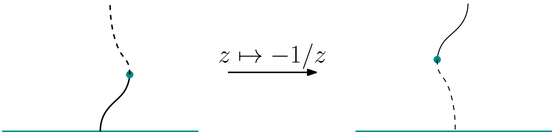


Reversibility of chordal Loewner energy

Theorem (W. 2016)

Let Γ be a simple chord in D connecting two boundary points a and b, we have

$$I_{D,a,b}(\Gamma) = I_{D,b,a}(\Gamma).$$



The deterministic result is based on

Theorem (Reversibility of SLE, Zhan 2008, Miller-Sheffield 2012)

For $\kappa \leq 8$, the law of the trace of SLE_κ in (D, a, b), is the same as the law of SLE_κ in (D, b, a).

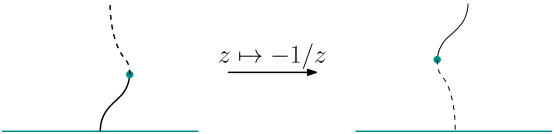
In fact, the decay rate as $\kappa \to 0$ of the probability of SLE_κ stays close to Γ is the same as the decay rate of being close to $-1/\Gamma$.

Reversibility of chordal Loewner energy

Theorem (W. 2016)

Let Γ be a simple chord in D connecting two boundary points a and b, we have

$$I_{D,a,b}(\Gamma) = I_{D,b,a}(\Gamma).$$



The deterministic result is based on

Theorem (Reversibility of SLE, Zhan 2008, Miller-Sheffield 2012)

For $\kappa \leq 8$, the law of the trace of SLE_κ in (D, a, b), is the same as the law of SLE_κ in (D, b, a).

In fact, the decay rate as $\kappa \to 0$ of the probability of SLE_κ stays close to Γ is the same as the decay rate of being close to $-1/\Gamma$.

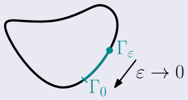
In fact, the Loewner energy has more symmetries.

Loewner loop energy

Definition (Rohde, W., 2017)

We define the **Loewner energy of a simple loop** $\Gamma : [0,1] \mapsto \hat{\mathbb{C}}$ rooted at $\Gamma_0 = \Gamma_1$ to be

$$I^L(\Gamma, \Gamma_0) := \lim_{\varepsilon \to 0} I_{\hat{\mathbb{C}} \setminus \Gamma[0,\varepsilon], \Gamma_\varepsilon, \Gamma_0}(\Gamma[\varepsilon, 1]).$$



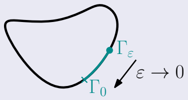
- $I^L(\Gamma, \Gamma_0) = 0$ if and only if Γ is a (round) circle.
- If $\Gamma[0, s]$ is a circular arc (including line segments), then the RHS is constant for $\varepsilon \leq s$, and $I^L(\Gamma, \Gamma_0)$ equals to the chordal energy $I_{\hat{\mathbb{C}} \setminus \Gamma[0,s], \Gamma_s, \Gamma_0}(\Gamma[s, 1])$.

Loewner loop energy

Definition (Rohde, W., 2017)

We define the **Loewner energy of a simple loop** $\Gamma : [0,1] \mapsto \hat{\mathbb{C}}$ rooted at $\Gamma_0 = \Gamma_1$ to be

$$I^L(\Gamma, \Gamma_0) := \lim_{\varepsilon \to 0} I_{\hat{\mathbb{C}} \setminus \Gamma[0,\varepsilon], \Gamma_\varepsilon, \Gamma_0}(\Gamma[\varepsilon, 1]).$$



- $I^L(\Gamma, \Gamma_0) = 0$ if and only if Γ is a (round) circle.
- If $\Gamma[0,s]$ is a circular arc (including line segments), then the RHS is constant for $\varepsilon \leq s$, and $I^L(\Gamma, \Gamma_0)$ equals to the chordal energy $I_{\hat{\mathbb{C}} \setminus \Gamma[0,s], \Gamma_s, \Gamma_0}(\Gamma[s,1])$.

Root-invariance

Theorem (Rohde, W. 2017)

The Loewner loop energy is **independent** of the choice of root and orientation.

$\Longrightarrow$ I^L is invariant on the set of **free loops** under Möbius transformation;

$\Longrightarrow$ The loop setting is more natural than the chordal setting.

The proof is based on the reversibility of the chordal energy.

Moreover,

- $I^L(\Gamma) < \infty$, then Γ is a (rectifiable) quasicircle.
- If Γ is $C^{1.5+\varepsilon}$ for some $\varepsilon > 0$, then $I^L(\Gamma) < \infty$.

Root-invariance

Theorem (Rohde, W. 2017)

The Loewner loop energy is **independent** of the choice of root and orientation.

$\Longrightarrow$ I^L is invariant on the set of **free loops** under Möbius transformation;

$\Longrightarrow$ The loop setting is more natural than the chordal setting.

The proof is based on the reversibility of the chordal energy.

Moreover,

- $I^L(\Gamma) < \infty$, then Γ is a (rectifiable) quasicircle.
- If Γ is $C^{1.5+\varepsilon}$ for some $\varepsilon > 0$, then $I^L(\Gamma) < \infty$.

Root-invariance

Theorem (Rohde, W. 2017)

The Loewner loop energy is **independent** of the choice of root and orientation.

$\Longrightarrow$ I^L is invariant on the set of **free loops** under Möbius transformation;
$\Longrightarrow$ The loop setting is more natural than the chordal setting.
The proof is based on the reversibility of the chordal energy.

Moreover,

- $I^L(\Gamma) < \infty$, then Γ is a (rectifiable) quasicircle.
- If Γ is $C^{1.5+\varepsilon}$ for some $\varepsilon > 0$, then $I^L(\Gamma) < \infty$.

Root-invariance

Theorem (Rohde, W. 2017)

The Loewner loop energy is **independent** of the choice of root and orientation.

$\Longrightarrow$ I^L is invariant on the set of **free loops** under Möbius transformation;
$\Longrightarrow$ The loop setting is more natural than the chordal setting.
The proof is based on the reversibility of the chordal energy.

Moreover,

- $I^L(\Gamma) < \infty$, then Γ is a (rectifiable) quasicircle.
- If Γ is $C^{1.5+\varepsilon}$ for some $\varepsilon > 0$, then $I^L(\Gamma) < \infty$.

Zeta-regularizated determinants

- The Zeta-regularization of determinants is first introduced by *Ray & Singer (1976)*.
- *Hawking (1977)* has pointed out that it allows to regularize quadratic path integrals.
- *Osgood, Phillips & Sarnak (1988)* have shown that the results obtained by comparing two functional determinants of Laplacian in the QFT formalism agree with the results obtained by the zeta-regularized determinant (**Polyakov-Alvarez conformal anomaly formula**).

Zeta-regularizated determinants

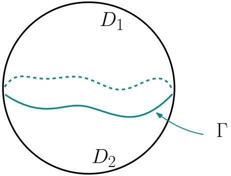
- The Zeta-regularization of determinants is first introduced by *Ray & Singer (1976)*.
- *Hawking (1977)* has pointed out that it allows to regularize quadratic path integrals.
- *Osgood, Phillips & Sarnak (1988)* have shown that the results obtained by comparing two functional determinants of Laplacian in the QFT formalism agree with the results obtained by the zeta-regularized determinant (**Polyakov-Alvarez conformal anomaly formula**).

Zeta-regularizated determinants

- The Zeta-regularization of determinants is first introduced by *Ray & Singer (1976)*.
- *Hawking (1977)* has pointed out that it allows to regularize quadratic path integrals.
- *Osgood, Phillips & Sarnak (1988)* have shown that the results obtained by comparing two functional determinants of Laplacian in the QFT formalism agree with the results obtained by the zeta-regularized determinant (**Polyakov-Alvarez conformal anomaly formula**).

The functional $\mathcal{H}$

- $g_0(z) = \frac{4}{(1+|z|^2)^2}\,\mathrm{d}z^2$ denotes the spherical metric;
- $g = e^{2\varphi} g_0$ be a metric conformally equivalent to g_0;
- Γ a C^∞ **smooth** simple loop in $\mathbb{C} \cup \{\infty\} \simeq S^2$;
- D_1 and D_2 two connected components $S^2 \backslash \Gamma$;
- $\Delta_g(D_i)$ the Laplace-Beltrami operator with Dirichlet boundary condition on D_i.



Definition

Let $\det_\zeta$ be the ζ-regularized determinant, we introduce

$$\mathcal{H}(\Gamma, g) := \log \det{}_\zeta \Delta_g(S^2) - \log \mathrm{Area}_g(S^2) - \log \det{}_\zeta \Delta_g(D_1) - \log \det{}_\zeta \Delta_g(D_2).$$

Loewner Energy vs. Determinants

$$\mathcal{H}(\Gamma, g) = \log \det_\zeta \Delta_g(S^2) - \log \text{Area}_g(S^2) - \log \det_\zeta \Delta_g(D_1) - \log \det_\zeta \Delta_g(D_2).$$

Theorem (W., 2018)

If $g = e^{2\varphi} g_0$ is a metric conformally equivalent to the spherical metric g_0 on S^2, then:

1. $\mathcal{H}(\cdot, g) = \mathcal{H}(\cdot, g_0)$
2. Circles minimize $\mathcal{H}(\cdot, g)$ among all C^∞ smooth Jordan curves.
3. Let Γ be a smooth Jordan curve on S^2. We have the identity

$$\begin{aligned} I^L(\Gamma, \Gamma(0)) &= 12\mathcal{H}(\Gamma, g) - 12\mathcal{H}(S^1, g) \\ &= 12 \log \frac{\det_\zeta(-\Delta_g(\mathbb{D}_1))\det_\zeta(-\Delta_g(\mathbb{D}_2))}{\det_\zeta(-\Delta_g(D_1))\det_\zeta(-\Delta_g(D_2))}, \end{aligned}$$

   where $\mathbb{D}_1$ and $\mathbb{D}_2$ are two connected components of the complement of S^1.

In particular, the above identity gives already the parametrization independence of the Loewner loop energy for smooth loops.

Loewner Energy vs. Determinants

$$\mathcal{H}(\Gamma, g) = \log \det_\zeta \Delta_g(S^2) - \log \mathrm{Area}_g(S^2) - \log \det_\zeta \Delta_g(D_1) - \log \det_\zeta \Delta_g(D_2).$$

Theorem (W., 2018)

If $g = e^{2\varphi} g_0$ is a metric conformally equivalent to the spherical metric g_0 on S^2, then:

1. $\mathcal{H}(\cdot, g) = \mathcal{H}(\cdot, g_0)$
2. Circles minimize $\mathcal{H}(\cdot, g)$ among all C^∞ smooth Jordan curves.
3. Let Γ be a smooth Jordan curve on S^2. We have the identity

$$\begin{aligned} I^L(\Gamma, \Gamma(0)) &= 12\mathcal{H}(\Gamma, g) - 12\mathcal{H}(S^1, g) \\ &= 12 \log \frac{\det_\zeta(-\Delta_g(\mathbb{D}_1))\det_\zeta(-\Delta_g(\mathbb{D}_2))}{\det_\zeta(-\Delta_g(D_1))\det_\zeta(-\Delta_g(D_2))}, \end{aligned}$$

   where $\mathbb{D}_1$ and $\mathbb{D}_2$ are two connected components of the complement of S^1.

In particular, the above identity gives already the parametrization independence of the Loewner loop energy for smooth loops.

Remarks

- The regularity assumption on the curve is due to the constraint from the zeta-regularization and its variation formula [OPS].
- Picking different metrics g provide a wide range of identities with the Loewner energy that usually look different in their expression involving scalar curvatures, geodesic curvatures, conformal maps $D_1 \to \mathbb{D}_1$, etc., (but of course they are equal).
- One of the identities links to the Weil-Petersson class of the universal Teichmüller space.

Remarks

- The regularity assumption on the curve is due to the constraint from the zeta-regularization and its variation formula [OPS].
- Picking different metrics g provide a wide range of identities with the Loewner energy that usually look different in their expression involving scalar curvatures, geodesic curvatures, conformal maps $D_1 \to \mathbb{D}_1$, etc., (but of course they are equal).
- One of the identities links to the Weil-Petersson class of the universal Teichmüller space.

Remarks

- The regularity assumption on the curve is due to the constraint from the zeta-regularization and its variation formula [OPS].
- Picking different metrics g provide a wide range of identities with the Loewner energy that usually look different in their expression involving scalar curvatures, geodesic curvatures, conformal maps $D_1 \to \mathbb{D}_1$, etc., (but of course they are equal).
- One of the identities links to the Weil-Petersson class of the universal Teichmüller space.

Universal Teichmüller space

- $QS(S^1)$ the group of quasisymmetric sense-preserving homeomorphism of S^1;

A sense-preserving homeomorphism $\varphi : S^1 \to S^1$ is **quasisymmetric** if there exists $M \geq 1$ such that for all $\theta \in \mathbb{R}$ and $t \in (0, \pi)$,

$$\frac{1}{M} \leq \left| \frac{\varphi(e^{i(\theta+t)}) - \varphi(e^{i\theta})}{\varphi(e^{i\theta}) - \varphi(e^{i(\theta-t)})} \right| \leq M.$$

- $\text{Möb}(S^1) \simeq \text{PSL}(2, \mathbb{R})$ the subgroup of Möbius function of S^1.

The **universal Teichmüller space** is

$$T(1) := QS(S^1)/\text{Möb}(S^1) \simeq \{\varphi \in QS(S^1), \varphi \text{ fixes } -1, -i \text{ and } 1\}.$$

It can be modeled by **Beltrami coefficients** as well:

$$T(1) = L^\infty(\mathbb{D}, \mathbb{C})_1 / \sim,$$

where

$$\|\mu\|_\infty < 1, \|\nu\|_\infty < 1, \quad \mu \sim \nu \Leftrightarrow w_\mu|_{S^1} = w_\nu|_{S^1}$$

w_μ is the normalized solution (fixes $-1, -i, 1$) $\mathbb{D} \to \mathbb{D}$ to the Beltrami equation

$$\overline{\partial} w_\mu(z) = \mu(z) \partial w_\mu(z).$$

Universal Teichmüller space

- $QS(S^1)$ the group of quasisymmetric sense-preserving homeomorphism of S^1;

A sense-preserving homeomorphism $\varphi : S^1 \to S^1$ is **quasisymmetric** if there exists $M \geq 1$ such that for all $\theta \in \mathbb{R}$ and $t \in (0, \pi)$,

$$\frac{1}{M} \leq \left| \frac{\varphi(e^{i(\theta+t)}) - \varphi(e^{i\theta})}{\varphi(e^{i\theta}) - \varphi(e^{i(\theta-t)})} \right| \leq M.$$

- $\text{Möb}(S^1) \simeq \text{PSL}(2, \mathbb{R})$ the subgroup of Möbius function of S^1.

The **universal Teichmüller space** is

$$T(1) := QS(S^1)/\text{Möb}(S^1) \simeq \{\varphi \in QS(S^1),\ \varphi \text{ fixes } -1, -i \text{ and } 1\}.$$

It can be modeled by **Beltrami coefficients** as well:

$$T(1) = L^\infty(\mathbb{D}, \mathbb{C})_1 / \sim,$$

where

$$\|\mu\|_\infty < 1, \|\nu\|_\infty < 1, \quad \mu \sim \nu \Leftrightarrow w_\mu|_{S^1} = w_\nu|_{S^1}$$

w_μ is the normalized solution (fixes $-1, -i, 1$) $\mathbb{D} \to \mathbb{D}$ to the Beltrami equation

$$\bar{\partial} w_\mu(z) = \mu(z) \partial w_\mu(z).$$

Universal Teichmüller space

- $QS(S^1)$ the group of quasisymmetric sense-preserving homeomorphism of S^1;

A sense-preserving homeomorphism $\varphi : S^1 \to S^1$ is **quasisymmetric** if there exists $M \geq 1$ such that for all $\theta \in \mathbb{R}$ and $t \in (0, \pi)$,

$$\frac{1}{M} \leq \left| \frac{\varphi(e^{i(\theta+t)}) - \varphi(e^{i\theta})}{\varphi(e^{i\theta}) - \varphi(e^{i(\theta-t)})} \right| \leq M.$$

- $\text{Möb}(S^1) \simeq \text{PSL}(2, \mathbb{R})$ the subgroup of Möbius function of S^1.

The **universal Teichmüller space** is

$$T(1) := QS(S^1)/\text{Möb}(S^1) \simeq \{\varphi \in QS(S^1),\ \varphi \text{ fixes } -1, -i \text{ and } 1\}.$$

It can be modeled by **Beltrami coefficients** as well:

$$T(1) = L^\infty(\mathbb{D}, \mathbb{C})_1 / \sim,$$

where

$$\|\mu\|_\infty < 1, \|\nu\|_\infty < 1, \quad \mu \sim \nu \Leftrightarrow w_\mu|_{S^1} = w_\nu|_{S^1}$$

w_μ is the normalized solution (fixes $-1, -i, 1$) $\mathbb{D} \to \mathbb{D}$ to the Beltrami equation

$$\overline{\partial} w_\mu(z) = \mu(z) \partial w_\mu(z).$$

Universal Teichmüller space

- $QS(S^1)$ the group of quasisymmetric sense-preserving homeomorphism of S^1;

A sense-preserving homeomorphism $\varphi : S^1 \to S^1$ is **quasisymmetric** if there exists $M \geq 1$ such that for all $\theta \in \mathbb{R}$ and $t \in (0, \pi)$,

$$\frac{1}{M} \leq \left| \frac{\varphi(e^{i(\theta+t)}) - \varphi(e^{i\theta})}{\varphi(e^{i\theta}) - \varphi(e^{i(\theta-t)})} \right| \leq M.$$

- $\text{Möb}(S^1) \simeq \text{PSL}(2, \mathbb{R})$ the subgroup of Möbius function of S^1.

The **universal Teichmüller space** is

$$T(1) := QS(S^1)/\text{Möb}(S^1) \simeq \{\varphi \in QS(S^1),\, \varphi \text{ fixes } -1, -i \text{ and } 1\}.$$

It can be modeled by **Beltrami coefficients** as well:

$$T(1) = L^\infty(\mathbb{D}, \mathbb{C})_1 / \sim,$$

where

$$\|\mu\|_\infty < 1, \|\nu\|_\infty < 1, \quad \mu \sim \nu \Leftrightarrow w_\mu|_{S^1} = w_\nu|_{S^1}$$

w_μ is the normalized solution (fixes $-1, -i, 1$) $\mathbb{D} \to \mathbb{D}$ to the Beltrami equation

$$\bar\partial w_\mu(z) = \mu(z) \partial w_\mu(z).$$

Universal Teichmüller space

- $QS(S^1)$ the group of quasisymmetric sense-preserving homeomorphism of S^1;

A sense-preserving homeomorphism $\varphi : S^1 \to S^1$ is **quasisymmetric** if there exists $M \geq 1$ such that for all $\theta \in \mathbb{R}$ and $t \in (0, \pi)$,

$$\frac{1}{M} \leq \left| \frac{\varphi(e^{i(\theta+t)}) - \varphi(e^{i\theta})}{\varphi(e^{i\theta}) - \varphi(e^{i(\theta-t)})} \right| \leq M.$$

- $\text{Möb}(S^1) \simeq \text{PSL}(2, \mathbb{R})$ the subgroup of Möbius function of S^1.

The **universal Teichmüller space** is

$$T(1) := QS(S^1)/\text{Möb}(S^1) \simeq \{\varphi \in QS(S^1),\, \varphi \text{ fixes } -1, -i \text{ and } 1\}.$$

It can be modeled by **Beltrami coefficients** as well:

$$T(1) = L^\infty(\mathbb{D}, \mathbb{C})_1 / \sim,$$

where

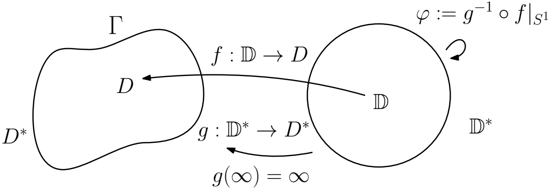
$$\|\mu\|_\infty < 1, \|\nu\|_\infty < 1, \quad \mu \sim \nu \Leftrightarrow w_\mu|_{S^1} = w_\nu|_{S^1}$$

w_μ is the normalized solution (fixes $-1, -i, 1$) $\mathbb{D} \to \mathbb{D}$ to the Beltrami equation

$$\bar{\partial} w_\mu(z) = \mu(z) \partial w_\mu(z).$$

Welding function

- Associate Γ with its **welding function** φ:



[Rohde, W. 2017]: $I^L(\Gamma) < \infty \Rightarrow \Gamma$ is a quasicircle $\Leftrightarrow \varphi \in QS(S^1)$.

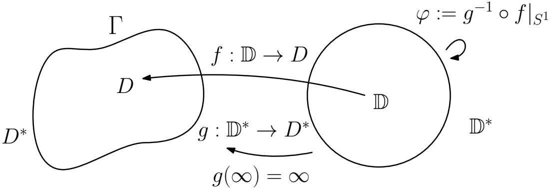
- "$\Leftarrow$" is not true, there are quasicircles with ∞ Loewner energy.

Question

What is the class of finite energy loops in $T(1)$?

Welding function

- Associate Γ with its **welding function** φ:



[Rohde, W. 2017]: $I^L(\Gamma) < \infty \Rightarrow \Gamma$ is a quasicircle $\Leftrightarrow \varphi \in QS(S^1)$.

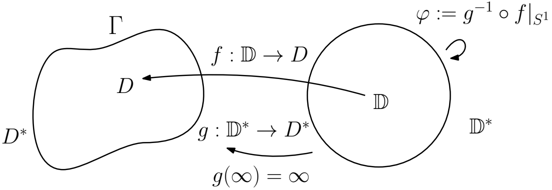
- "$\Leftarrow$" is not true, there are quasicircles with ∞ Loewner energy.

Question

What is the class of finite energy loops in $T(1)$?

Welding function

- Associate Γ with its **welding function** φ:



[Rohde, W. 2017]: $I^L(\Gamma) < \infty \Rightarrow \Gamma$ is a quasicircle $\Leftrightarrow \varphi \in QS(S^1)$.

- "$\Leftarrow$" is not true, there are quasicircles with ∞ Loewner energy.

Question

What is the class of finite energy loops in $T(1)$?

Weil-Petersson Class

- The homogeneous space of C^∞-smooth diffeomorphisms

$$M := \text{Diff}(S^1)/\text{Möb}(S^1) \subset T(1)$$

  has a Kähler structure *[Witten, Bowick, Rajeev, etc.]*.
- There is a **unique** homogeneous Kähler metric (up to constant factor): the **Weil-Petersson metric**.

- The homogeneous space of C^∞-smooth diffeomorphisms

$$M := \mathrm{Diff}(S^1)/\mathrm{M\ddot{o}b}(S^1) \subset T(1)$$

has a Kähler structure *[Witten, Bowick, Rajeev, etc.]*.
- There is a **unique** homogeneous Kähler metric (up to constant factor): the **Weil-Petersson metric**.

Weil-Petersson metric

The tangent space at *id* of M consists of C^∞ vector fields on S^1:

$$v = v(\theta)\frac{\partial}{\partial\theta} = \sum_{m\in\mathbb{Z}\setminus\{-1,0,1\}} v_m e^{im\theta}\frac{\partial}{\partial\theta}, \text{ where } v_{-m} = \overline{v_m}.$$

The almost complex structure $J^2 = -Id$ is given by the Hilbert transform:

$$J(v)_m = -i\text{sgn}(m)v_m, \text{ for } m \in \mathbb{Z}\setminus\{-1,0,1\}.$$

In particular,

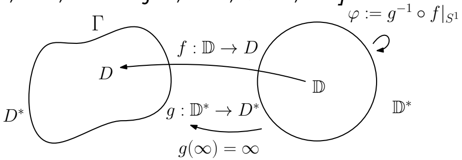
$$J\left(\cos(m\theta)\frac{\partial}{\partial\theta}\right) = \sin(m\theta)\frac{\partial}{\partial\theta}; \quad J\left(\sin(m\theta)\frac{\partial}{\partial\theta}\right) = -\cos(m\theta)\frac{\partial}{\partial\theta}.$$

The Weil-Petersson symplectic form $\omega(\cdot,\cdot)$ and the Riemannian metric $\langle\cdot,\cdot\rangle_{WP}$ is given at the origin by

$$\omega(v,w) = i\sum_{m\in\mathbb{Z}\setminus\{-1,0,1\}} (m^3 - m)v_m w_{-m},$$

$$\langle v, w\rangle_{WP} = \omega(v, J(w)) = \sum_{m=2}^{\infty}(m^3 - m)\,\mathrm{Re}(v_m w_{-m}).$$

- **Weil-Petersson Teichmüller space** $T_0(1)$ is the closure of $\mathrm{Diff}(S^1)/\mathrm{M\ddot{o}b}(S^1) \subset T(1)$ under the WP-metric. **Weil-Petersson class** $\mathrm{WP}(S^1) \subset QS(S^1)$ are homeomorphisms representing points in $T_0(1)$.
- The above description and many other characterizations are provided by *[Nag, Verjovski, Sullivan, Cui, Takhtajan, Teo, Shen, etc]*.



Theorem (Takhtajan & Teo, 2006)

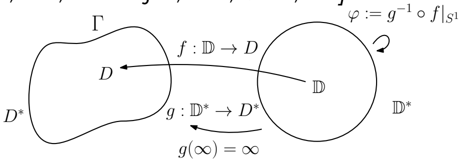
The **universal Liouville action** $\mathbf{S}_1 : T_0(1) \to \mathbb{R}$,

$$\mathbf{S}_1([\varphi]) := \int_{\mathbb{D}} \left| \frac{f''}{f'}(z) \right|^2 \mathrm{d}z^2 + \int_{\mathbb{D}^*} \left| \frac{g''}{g'}(z) \right|^2 \mathrm{d}z^2 + 4\pi \log \left| \frac{f'(0)}{g'(\infty)} \right|$$

is a Kähler potential of the Weil-Petersson metric, where

$$g'(\infty) = \lim_{z\to\infty} g'(z) = \tilde{g}'(0)^{-1} \text{ and } \tilde{g}(z) = 1/g(1/z).$$

- **Weil-Petersson Teichmüller space** $T_0(1)$ is the closure of $\mathrm{Diff}(S^1)/\mathrm{M\ddot{o}b}(S^1) \subset T(1)$ under the WP-metric. **Weil-Petersson class** $\mathrm{WP}(S^1) \subset QS(S^1)$ are homeomorphisms representing points in $T_0(1)$.
- The above description and many other characterizations are provided by *[Nag, Verjovski, Sullivan, Cui, Takhtajan, Teo, Shen, etc]*.



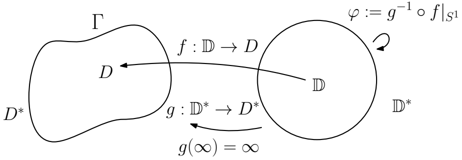
Theorem (Takhtajan & Teo, 2006)

The **universal Liouville action $\mathbf{S}_1$** : $T_0(1) \to \mathbb{R}$,

$$\mathbf{S}_1([\varphi]) := \int_{\mathbb{D}} \left| \frac{f''}{f'}(z) \right|^2 \mathrm{d}z^2 + \int_{\mathbb{D}^*} \left| \frac{g''}{g'}(z) \right|^2 \mathrm{d}z^2 + 4\pi \log \left| \frac{f'(0)}{g'(\infty)} \right|$$

is a Kähler potential of the Weil-Petersson metric, where

$$g'(\infty) = \lim_{z\to\infty} g'(z) = \tilde{g}'(0)^{-1} \text{ and } \tilde{g}(z) = 1/g(1/z).$$



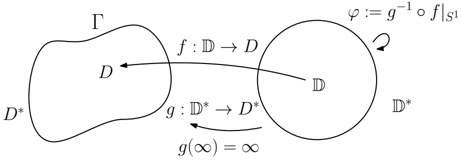
Theorem (W. 2018)

A bounded simple loop Γ in $\hat{\mathbb{C}}$ has finite Loewner energy *if and only if* $[\varphi] \in T_0(1)$. Moreover,

$$I^L(\Gamma) = \mathbf{S_1}([\varphi])/\pi.$$

- There is no regularity assumption on the loop for the identity to hold.
- This gives a new characterization of the WP-Class, and a new viewpoint on the Kähler potential on $T_0(1)$ (or alternatively a way to look at the Loewner energy).

Loewner Energy vs. Weil-Petersson Class



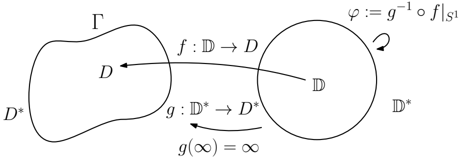
Theorem (W. 2018)

A bounded simple loop Γ in $\hat{\mathbb{C}}$ has finite Loewner energy *if and only if* $[\varphi] \in T_0(1)$. Moreover,

$$I^L(\Gamma) = \mathbf{S}_1([\varphi])/\pi.$$

- There is no regularity assumption on the loop for the identity to hold.
- This gives a new characterization of the WP-Class, and a new viewpoint on the Kähler potential on $T_0(1)$ (or alternatively a way to look at the Loewner energy).

Loewner Energy vs. Weil-Petersson Class



Theorem (W. 2018)

A bounded simple loop Γ in $\hat{\mathbb{C}}$ has finite Loewner energy *if and only if* $[\varphi] \in T_0(1)$. Moreover,

$$I^L(\Gamma) = \mathbf{S_1}([\varphi])/\pi.$$

- There is no regularity assumption on the loop for the identity to hold.
- This gives a new characterization of the WP-Class, and a new viewpoint on the Kähler potential on $T_0(1)$ (or alternatively a way to look at the Loewner energy).

Characterizations of the WP-Class (an incomplete list)

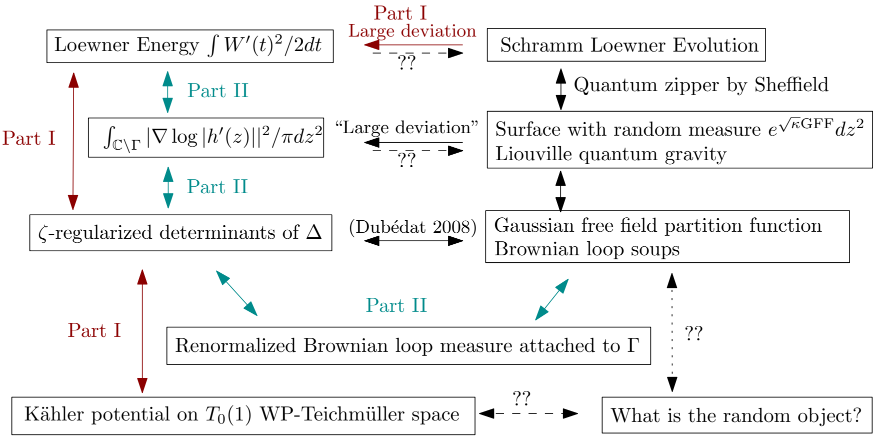
[Nag, Verjovsky, Sullivan, Cui, Taktajan, Teo, Shen, etc.] The following are equivalent:

- The welding function φ is in Weil-Petersson class;
- $\int_{\mathbb{D}} |\nabla \log |f'(z)||^2 \, \mathrm{d}z^2 = \int_{\mathbb{D}} |f''(z)/f'(z)|^2 \, \mathrm{d}z^2 < \infty$;
- $\int_{\mathbb{D}^*} |g''(z)/g'(z)|^2 \, \mathrm{d}z^2 < \infty$;
- $\int_{\mathbb{D}} |\mathcal{S}(f)|^2 \, \rho^{-1}(z) \, \mathrm{d}z^2 < \infty$;
- $\int_{\mathbb{D}^*} |\mathcal{S}(g)|^2 \, \rho^{-1}(z) \, \mathrm{d}z^2 < \infty$;
- φ has quasiconformal extension to $\mathbb{D}$, whose complex dilation $\mu = \partial_{\bar{z}}\varphi / \partial_z \varphi$ satisfies
$$\int_{\mathbb{D}} |\mu(z)|^2 \, \rho(z) \, \mathrm{d}z^2 < \infty;$$
- φ is absolutely continuous with respect to arc-length measure, such that $\log |\varphi'|$ belongs to the Sobolev space $H^{1/2}(S^1)$;
- Grunsky operator associated to f or g is Hilbert-Schmidt,

where $\rho(z) \, \mathrm{d}z^2 = 1/(1 - |z|^2)^2 \, \mathrm{d}z^2$ is the hyperbolic metric on $\mathbb{D}$ or $\mathbb{D}^*$ and
$$\mathcal{S}(f) = \frac{f'''}{f'} - \frac{3}{2} \left(\frac{f''}{f'} \right)^2$$
is the Schwarzian derivative of f.

Contents

Action functionals vs. Random objects



Loewner Energy vs. Determinants

Recall $\mathcal{H}(\Gamma, g) = \log \det_\zeta \Delta_g(S^2) - \log \mathrm{Area}_g(S^2) - \log \det_\zeta \Delta_g(D_1) - \log \det_\zeta \Delta_g(D_2)$.

Theorem (W., 2018)

If $g = e^{2\varphi} g_0$ is a metric conformally equivalent to the spherical metric g_0 on S^2, then:

1. $\mathcal{H}(\cdot, g) = \mathcal{H}(\cdot, g_0)$
2. Circles minimize $\mathcal{H}(\cdot, g)$ among all C^∞ smooth Jordan curves.
3. Let Γ be a smooth Jordan curve on S^2. We have the identity

$$I^L(\Gamma, \Gamma(0)) = 12\mathcal{H}(\Gamma, g) - 12\mathcal{H}(S^1, g)$$
$$= 12 \log \frac{\det_\zeta(-\Delta_g(\mathbb{D}_1))\det_\zeta(-\Delta_g(\mathbb{D}_2))}{\det_\zeta(-\Delta_g(D_1))\det_\zeta(-\Delta_g(D_2))},$$

where $\mathbb{D}_1$ and $\mathbb{D}_2$ are two connected components of the complement of S^1.

Zeta-regularizated determinants

- $\Delta_g(S^2)$ is non-negative, essentially self-adjoint for the L^2 product.
- The spectrum is
$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \cdots$$
- Define the Zeta-function
$$\zeta_\Delta(s) := \sum_{i \geq 1} \lambda_i^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty \mathrm{Tr}(e^{-t\Delta}) t^{s-1} dt,$$
it can be analytically continued to a neighborhood of 0.
- Define (following Ray & Singer 1976)
$$\log \det'_\zeta(\Delta_g(S^2)) := -\zeta'_\Delta(0)$$
$$\text{“} = \sum_{i \geq 1} \log(\lambda_i) \lambda_i^{-s}|_{s=0} = \log(\prod_{i \geq 1} \lambda_i).\text{”}$$

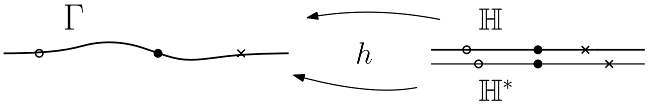
Proof of the identity (sketch)

$$I^L(\Gamma, \Gamma(0)) = 12 \log \frac{\det_\zeta(-\Delta_{\mathbb{D}_1, g_0}) \det_\zeta(-\Delta_{\mathbb{D}_2, g_0})}{\det_\zeta(-\Delta_{D_1, g_0}) \det_\zeta(-\Delta_{D_2, g_0})}$$

- When Γ passes through ∞, we show

$$I^L(\Gamma, \infty) = \mathcal{D}_{\mathbb{H}\cup\mathbb{H}^*}(\log |h'|) := \frac{1}{\pi} \left(\int_{\mathbb{H}\cup\mathbb{H}^*} \left|\nabla \log |h'(z)|\right|^2 \mathrm{d}z^2 \right),$$

where h maps conformally $\mathbb{H} \cup \mathbb{H}^*$ to the complement of Γ and fixes ∞.



The right-hand side does not involve Loewner iteration of conformal maps.

- Use the Polyakov-Alvarez conformal anomaly formula to compare determinants of Laplacians.

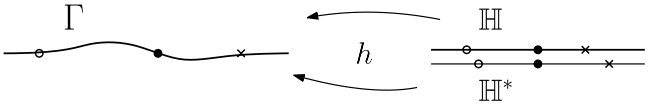
Proof of the identity (sketch)

$$I^L(\Gamma, \Gamma(0)) = 12 \log \frac{\det_\zeta(-\Delta_{\mathbb{D}_1, g_0})\det_\zeta(-\Delta_{\mathbb{D}_2, g_0})}{\det_\zeta(-\Delta_{D_1, g_0})\det_\zeta(-\Delta_{D_2, g_0})}$$

- When Γ passes through ∞, we show

$$I^L(\Gamma, \infty) = \mathcal{D}_{\mathbb{H}\cup\mathbb{H}^*}(\log |h'|) := \frac{1}{\pi}\left(\int_{\mathbb{H}\cup\mathbb{H}^*} \left|\nabla \log |h'(z)|\right|^2 \mathrm{d}z^2\right),$$

where h maps conformally $\mathbb{H} \cup \mathbb{H}^*$ to the complement of Γ and fixes ∞.



The right-hand side does not involve Loewner iteration of conformal maps.

- Use the Polyakov-Alvarez conformal anomaly formula to compare determinants of Laplacians.

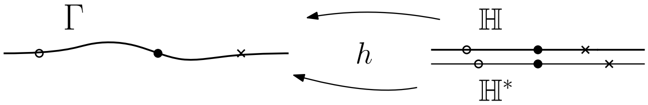
Proof of the identity (sketch)

$$I^L(\Gamma, \Gamma(0)) = 12 \log \frac{\det_\zeta(-\Delta_{\mathbb{D}_1, g_0}) \det_\zeta(-\Delta_{\mathbb{D}_2, g_0})}{\det_\zeta(-\Delta_{D_1, g_0}) \det_\zeta(-\Delta_{D_2, g_0})}$$

- When Γ passes through ∞, we show

$$I^L(\Gamma, \infty) = \mathcal{D}_{\mathbb{H}\cup\mathbb{H}^*}(\log |h'|) := \frac{1}{\pi} \left(\int_{\mathbb{H}\cup\mathbb{H}^*} \left| \nabla \log |h'(z)| \right|^2 \mathrm{d}z^2 \right),$$

where h maps conformally $\mathbb{H} \cup \mathbb{H}^*$ to the complement of Γ and fixes ∞.



The right-hand side does not involve Loewner iteration of conformal maps.

- Use the Polyakov-Alvarez conformal anomaly formula to compare determinants of Laplacians.

Polyakov-Alvarez conformal anomaly formula

Take $g = e^{2\sigma} g_0$ a metric conformally equivalent to g_0. (Here think $\sigma = \log|h'|$.)

Theorem ([Polyakov 1981], [Alvarez 1983], [Osgood, et al. 1988])

For a compact surface M without boundary,

$$\left(\log \det_\zeta'(-\Delta_g) - \log \mathrm{vol}_g(M)\right) - \left(\log \det_\zeta'(-\Delta_0) - \log \mathrm{vol}_0(M)\right)$$
$$= -\frac{1}{6\pi}\left[\frac{1}{2}\int_M |\nabla_0 \sigma|^2 \,\mathrm{dvol}_0 + \int_M K_0 \sigma \,\mathrm{dvol}_0.\right]$$

The analogue for a compact surface D with smooth boundary is:

$$\log \det_\zeta(-\Delta_g) - \log \det_\zeta(-\Delta_0)$$
$$= -\frac{1}{6\pi}\left[\frac{1}{2}\int_D |\nabla_0 \sigma|^2 \,\mathrm{dvol}_0 + \int_D K_0 \sigma \,\mathrm{dvol}_0 + \int_{\partial D} k_0 \sigma \,\mathrm{dl}_0\right] - \frac{1}{4\pi}\int_{\partial D} \partial_n \sigma \,\mathrm{dl}_0.$$

"Taking $g_0 = dz^2$", we have $K_0 \equiv 0$ and $k_0 \equiv 0$. We get:

$$I^L(\Gamma, \Gamma(0)) = \frac{1}{\pi}\left(\int_{\mathbb{H}\cup\mathbb{H}^*} \left|\nabla \log|h'(z)|\right|^2 \,\mathrm{d}z^2\right) = 12\mathcal{H}(\Gamma, g_0) - 12\mathcal{H}(S^1, g_0). \quad \square$$

Polyakov-Alvarez conformal anomaly formula

Take $g = e^{2\sigma} g_0$ a metric conformally equivalent to g_0. (Here think $\sigma = \log |h'|$.)

Theorem ([Polyakov 1981], [Alvarez 1983], [Osgood, et al. 1988])

For a compact surface M without boundary,

$$\left(\log \det'_\zeta(-\Delta_g) - \log \mathrm{vol}_g(M)\right) - \left(\log \det'_\zeta(-\Delta_0) - \log \mathrm{vol}_0(M)\right)$$
$$= -\frac{1}{6\pi}\left[\frac{1}{2}\int_M |\nabla_0 \sigma|^2 \, \mathrm{dvol}_0 + \int_M K_0 \sigma \, \mathrm{dvol}_0.\right]$$

The analogue for a compact surface D with smooth boundary is:

$$\log \det_\zeta(-\Delta_g) - \log \det_\zeta(-\Delta_0)$$
$$= -\frac{1}{6\pi}\left[\frac{1}{2}\int_D |\nabla_0 \sigma|^2 \, \mathrm{dvol}_0 + \int_D K_0 \sigma \, \mathrm{dvol}_0 + \int_{\partial D} k_0 \sigma \, \mathrm{dl}_0\right] - \frac{1}{4\pi}\int_{\partial D} \partial_n \sigma \, \mathrm{dl}_0.$$

"Taking $g_0 = dz^2$", we have $K_0 \equiv 0$ and $k_0 \equiv 0$. We get:

$$I^L(\Gamma, \Gamma(0)) = \frac{1}{\pi}\left(\int_{\mathbb{H}\cup\mathbb{H}^*} \left|\nabla \log |h'(z)|\right|^2 \mathrm{d}z^2\right) = 12\mathcal{H}(\Gamma, g_0) - 12\mathcal{H}(S^1, g_0). \quad \square$$

Brownian loop measure

Introduced by Greg Lawler and Wendelin Werner.

[Following J. Dubédat] Let $x \in M$, $t > 0$, consider the sub-probability measure $\mathbb{W}_x^t$ on the path of Brownian motion (diffusion generated by $-\Delta_M$) on M started from x on the time interval $[0, t]$, killed if it hits the boundary of M.
The measures $\mathbb{W}_{x\to y}^t$ on paths from x to y are obtained from the disintegration of $\mathbb{W}_x^t$ according to its endpoint y:

$$\mathbb{W}_x^t = \int_M \mathbb{W}_{x\to y}^t \,\mathrm{dvol}(y).$$

Define the **Brownian loop measure** on M:

$$\mu_M^{loop} := \int_0^\infty \frac{\mathrm{d}t}{t} \int_M \mathbb{W}_{x\to x}^t \,\mathrm{dvol}(x).$$

In particular,

$$\left|\mathbb{W}_{x\to x}^t\right| = p_t(x,x).$$

We consider μ_M^{loop} as measure on **unrooted** Brownian loops by forgetting the starting point.

Brownian loop measure

Introduced by Greg Lawler and Wendelin Werner.

[Following J. Dubédat] Let $x \in M$, $t > 0$, consider the sub-probability measure $\mathbb{W}_x^t$ on the path of Brownian motion (diffusion generated by $-\Delta_M$) on M started from x on the time interval $[0, t]$, killed if it hits the boundary of M.
The measures $\mathbb{W}_{x \to y}^t$ on paths from x to y are obtained from the disintegration of $\mathbb{W}_x^t$ according to its endpoint y:

$$\mathbb{W}_x^t = \int_M \mathbb{W}_{x \to y}^t \, \mathrm{dvol}(y).$$

Define the **Brownian loop measure** on M:

$$\mu_M^{loop} := \int_0^\infty \frac{\mathrm{d}t}{t} \int_M \mathbb{W}_{x \to x}^t \, \mathrm{dvol}(x).$$

In particular,

$$\left| \mathbb{W}_{x \to x}^t \right| = p_t(x, x).$$

We consider μ_M^{loop} as measure on **unrooted** Brownian loops by forgetting the starting point.

Brownian loop measure

Introduced by Greg Lawler and Wendelin Werner.

[Following J. Dubédat] Let $x \in M$, $t > 0$, consider the sub-probability measure $\mathbb{W}_x^t$ on the path of Brownian motion (diffusion generated by $-\Delta_M$) on M started from x on the time interval $[0, t]$, killed if it hits the boundary of M.
The measures $\mathbb{W}_{x \to y}^t$ on paths from x to y are obtained from the disintegration of $\mathbb{W}_x^t$ according to its endpoint y:

$$\mathbb{W}_x^t = \int_M \mathbb{W}_{x \to y}^t \, \mathrm{dvol}(y).$$

Define the **Brownian loop measure** on M:

$$\mu_M^{loop} := \int_0^\infty \frac{\mathrm{d}t}{t} \int_M \mathbb{W}_{x \to x}^t \, \mathrm{dvol}(x).$$

In particular,

$$\left| \mathbb{W}_{x \to x}^t \right| = p_t(x, x).$$

We consider μ_M^{loop} as measure on **unrooted** Brownian loops by forgetting the starting point.

Brownian loop measure

Introduced by Greg Lawler and Wendelin Werner.

[Following J. Dubédat] Let $x \in M$, $t > 0$, consider the sub-probability measure $\mathbb{W}_x^t$ on the path of Brownian motion (diffusion generated by $-\Delta_M$) on M started from x on the time interval $[0, t]$, killed if it hits the boundary of M.
The measures $\mathbb{W}_{x \to y}^t$ on paths from x to y are obtained from the disintegration of $\mathbb{W}_x^t$ according to its endpoint y:

$$\mathbb{W}_x^t = \int_M \mathbb{W}_{x \to y}^t \, \mathrm{dvol}(y).$$

Define the **Brownian loop measure** on M:

$$\mu_M^{loop} := \int_0^\infty \frac{\mathrm{d}t}{t} \int_M \mathbb{W}_{x \to x}^t \, \mathrm{dvol}(x).$$

In particular,

$$\left| \mathbb{W}_{x \to x}^t \right| = p_t(x, x).$$

We consider μ_M^{loop} as measure on **unrooted** Brownian loops by forgetting the starting point.

Property of Brownian loop measure

The Brownian loop measure satisfies the following two remarkable properties

- *(Restriction property)* If $M' \subset M$, then

$$d\mu_{M'}^{loop}(\delta) = 1_{\delta \in M'} d\mu_{M}^{loop}(\delta).$$

- *(Conformal invariance)* On the surfaces $M_1 = (M, g)$ and $M_2 = (M, e^{2\sigma} g)$ be two conformally equivalent Riemann surface, where $\sigma \in C^\infty(M, \mathbb{R})$, then

$$\mu_{M_1}^{loop} = \mu_{M_2}^{loop}.$$

Loop measure vs. determinant of Laplacian

$$\text{``}\left|\mu_M^{loop}\right| = -\log \det_\zeta(\Delta).\text{''}$$

If we compute formally, the total mass of μ_M^{loop} is given by

$$\text{``}\left|\mu_M^{loop}\right| = \int_0^\infty \frac{\mathrm{d}t}{t} \int_M p_t(x,x)\,\mathrm{dvol}(x) = \int_0^\infty t^{-1}\mathrm{Tr}\left(e^{-t\Delta}\right)\,\mathrm{d}t.\text{''}$$

On the other hand, $1/\Gamma(s)$ is analytic and has the expansion near 0 as

$$1/\Gamma(s) = s + O(s^2).$$

Therefore for any analytic function f in a neighborhood of 0,

$$\left(\frac{f(s)}{\Gamma(s)}\right)'\bigg|_{s=0} = f(0).$$

Take formally $f(s) = \int_0^\infty t^{s-1}\mathrm{Tr}(e^{-t\Delta})\,\mathrm{d}t$, we have

$$\text{``}-\log\det_\zeta(\Delta) = \zeta_\Delta'(0) = \left(\frac{f(s)}{\Gamma(s)}\right)'\bigg|_{s=0} = \int_0^\infty t^{-1}\mathrm{Tr}(e^{-t\Delta})\,\mathrm{d}t = \left|\mu_M^{loop}\right|\text{''}. \qquad (2)$$

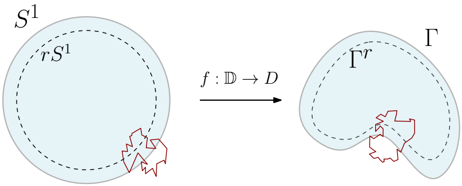
Loop measure vs. Loewner energy (heuristic)

$$“\left|\mu_M^{loop}\right| = -\log \det_\zeta(\Delta).”$$

The determinant expression of Loewner energy suggests that we have formally

$$\begin{aligned} “\frac{1}{12} I^L(\Gamma) &= \log \frac{\det_\zeta(\Delta_{\mathbb{D}_1,g})\det_\zeta(\Delta_{\mathbb{D}_2,g})}{\det_\zeta(\Delta_{D_1,g})\det_\zeta(\Delta_{D_2,g})} \\ &= \left|\mu_{D_1}^{loop}\right| + \left|\mu_{D_2}^{loop}\right| - \left|\mu_{\mathbb{D}_1}^{loop}\right| - \left|\mu_{\mathbb{D}_2}^{loop}\right| + \left|\mu_{S^2}^{loop}\right| - \left|\mu_{S^2}^{loop}\right| \\ &= \mu_{S^2}^{loop}(\{\delta; \delta \cap S^1 \neq \emptyset\}) - \mu_{S^2}^{loop}(\{\delta; \delta \cap \Gamma \neq \emptyset\}).” \end{aligned}$$

However, both terms diverge due to the small and large Brownian loops (from the conformal invariance).



For a Brownian loop $\delta \subset D$, where $D \subset \mathbb{D}$ is simply connected, we denote δ^{out} its outer boundary (therefore of $SLE_{8/3}$ type).
Let $A, B \subset \mathbb{C}$ be disjoint compact sets,

$$\mathcal{W}(A, B; D) := \left|\mu^{loop}\{\delta \subset D; \delta^{out} \text{ intersects both } A \text{ and } B\}\right| < \infty.$$

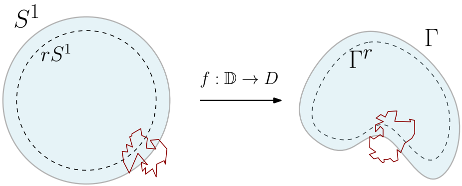
Introduced by W. Werner.

Theorem (W., 2018)

For all Jordan curve Γ (no regularity assumption),

$$\frac{1}{12} I^L(\Gamma) = \lim_{r \to 1} \mathcal{W}(S^1, rS^1; \mathbb{C}) - \mathcal{W}(\Gamma, \Gamma^r; \mathbb{C}).$$

Loop measure vs. Loewner energy



For a Brownian loop $\delta \subset D$, where $D \subset \mathbb{D}$ is simply connected, we denote δ^{out} its outer boundary (therefore of $\mathrm{SLE}_{8/3}$ type).
Let $A, B \subset \mathbb{C}$ be disjoint compact sets,

$$\mathcal{W}(A, B; D) := \left|\mu^{loop}\{\delta \subset D; \delta^{out} \text{ intersects both } A \text{ and } B\}\right| < \infty.$$

Introduced by W. Werner.

Theorem (W., 2018)

For all Jordan curve Γ (no regularity assumption),

$$\frac{1}{12} I^L(\Gamma) = \lim_{r \to 1} \mathcal{W}(S^1, rS^1; \mathbb{C}) - \mathcal{W}(\Gamma, \Gamma^r; \mathbb{C}).$$

Proof: Chordal Conformal restriction

Lemma 1: Chordal Conformal restriction

Let (D, a, b) and (D', a, b) be two simply connected domains in $\mathbb{C}$ coinciding in a neighborhood of a and b, and Γ a simple curve in both (D, a, b) and (D', a, b). Then we have

$$\begin{aligned} I_{D',a,b}(\Gamma) - I_{D,a,b}(\Gamma) =& I_{D,a,b}(\psi(\Gamma)) - I_{D,a,b}(\Gamma) \\ =& 3 \log \left|\psi'(a)\psi'(b)\right| + 12\mathcal{W}(\Gamma, D\backslash D'; D) - 12\mathcal{W}(\Gamma, D'\backslash D; D'), \end{aligned}$$

where $\psi : D' \to D$ is a conformal map fixing a and b.

Deterministic proof, similar computation as in SLE conformal restriction.
Intuition: The SLE partition function is

$$\mathcal{Z}^{\mathrm{SLE}_\kappa}_{(D,a,b)} = H_D(a, b)^\beta \det_\zeta(\Delta)^{-c/2},$$

where as $\kappa \to 0$,

$$\beta = \frac{6-\kappa}{2\kappa} \sim \frac{3}{\kappa}, \quad c = \frac{(3\kappa-8)(6-\kappa)}{2\kappa} \sim -\frac{24}{\kappa}.$$

The Energy = "$-\kappa \log(\cdot)$"

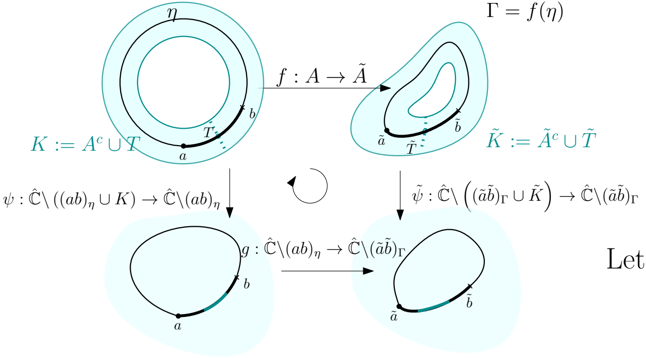
Proof: Loop Conformal restriction

Lemma 2: Loop conformal restriction

If η is a Jordan curve with finite energy and $\Gamma = f(\eta)$, where $f : A \to \tilde{A}$ is conformal on a neighborhood A of η, then

$$I^L(\Gamma) - I^L(\eta) = 12\mathcal{W}(\eta, A^c; \mathbb{C}) - 12\mathcal{W}(\Gamma, \tilde{A}^c; \mathbb{C}).$$

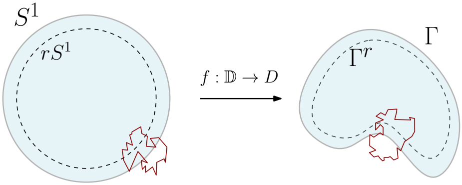
Proof of Lemma 2:



Let $b \to a$.

Proof: Equipotentials

When $\eta = rS^1$, $\Gamma^r = f(rS^1)$ is the equipotential, and $A = \mathbb{D}$.



We deduce

$$I^L(\Gamma^r) = 12\mathcal{W}(rS^1, S^1; \mathbb{C}) - 12\mathcal{W}(\Gamma^r, \Gamma; \mathbb{C}).$$

Lemma 3

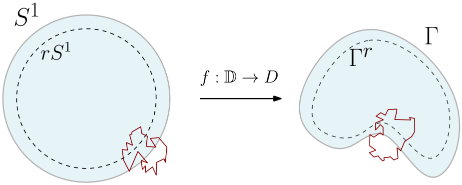
We have: $I^L(\Gamma^r) \xrightarrow{r \to 1} I^L(\Gamma)$.

□

In fact, $r \mapsto I^L(\Gamma^r)$ is increasing if $I^L(\Gamma) > 0$, namely when Γ is not a circle. It will follow from the flow-line coupling for finite energy curve [Viklund, W. 2019+].

Proof: Equipotentials

When $\eta = rS^1$, $\Gamma^r = f(rS^1)$ is the equipotential, and $A = \mathbb{D}$.



We deduce

$$I^L(\Gamma^r) = 12\mathcal{W}(rS^1, S^1; \mathbb{C}) - 12\mathcal{W}(\Gamma^r, \Gamma; \mathbb{C}).$$

Lemma 3

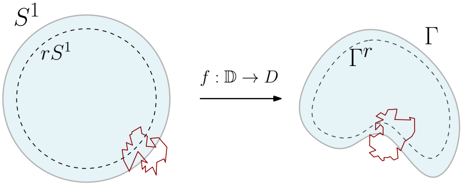
We have: $I^L(\Gamma^r) \xrightarrow{r \to 1} I^L(\Gamma)$.

□

In fact, $r \mapsto I^L(\Gamma^r)$ is increasing if $I^L(\Gamma) > 0$, namely when Γ is not a circle. It will follow from the flow-line coupling for finite energy curve [Viklund, W. 2019+].

Proof: Equipotentials

When $\eta = rS^1$, $\Gamma^r = f(rS^1)$ is the equipotential, and $A = \mathbb{D}$.



We deduce

$$I^L(\Gamma^r) = 12\mathcal{W}(rS^1, S^1; \mathbb{C}) - 12\mathcal{W}(\Gamma^r, \Gamma; \mathbb{C}).$$

Lemma 3

We have: $I^L(\Gamma^r) \xrightarrow{r \to 1} I^L(\Gamma)$.

□

In fact, $r \mapsto I^L(\Gamma^r)$ is increasing if $I^L(\Gamma) > 0$, namely when Γ is not a circle. It will follow from the flow-line coupling for finite energy curve [Viklund, W. 2019+].

SLE/GFF coupling analogs: A Dictionary

Work in progress with F. Viklund. With $\gamma = \sqrt{\kappa}$, $\chi = \gamma/2 - 2/\gamma$:

Random Conformal Geometry		**Action Functional Analogs**
Neumann GFF on $\mathbb{H}$	$\longleftrightarrow$	$2u_1 : \mathbb{H} \to \mathbb{R}$ with finite Dirichlet energy;
Neumann GFF on $\mathbb{H}^*$	$\longleftrightarrow$	$2u_2 : \mathbb{H}^* \to \mathbb{R}$ with finite Dirichlet energy;
γ-LQG measure on $\mathbb{H}$, $e^{\gamma GFF} dz^2$	$\longleftrightarrow$	$e^{2u_1(z)} dz^2$;
γ-LQG boundary measure on $\mathbb{R} = \partial\mathbb{H}$	$\longleftrightarrow$	$e^{u_1(z)} \lvert dz \rvert$, $\quad u_1\rvert_{\mathbb{R}} \in H^{1/2}(\mathbb{R})$;
"SLE_κ loop"	$\longleftrightarrow$	finite energy loop Γ;
γ-LQG on $\mathbb{C}$	$\longleftrightarrow$	$e^{2\varphi(z)} dz^2$;
γ-quantum chaos wrt. natural parametrization on SLE loop	$\longleftrightarrow$	trace of φ on $\Gamma \in H^{1/2}(\Gamma)$;
independent couple	$\longleftrightarrow$	sum up their rate functions;
$e^{iGFF/\chi}$	$\longleftrightarrow$	$e^{i\varphi(z)}$ unit vector field;

Isometric conformal welding

Let $D_1, D_2 \subset \mathbb{C}$ be Jordan domains bounded respectively by rectifiable curves Γ_1 and Γ_2 of same total length. Let $\psi : \Gamma_1 \to \Gamma_2$ be an isometry (preserves the arc-length).

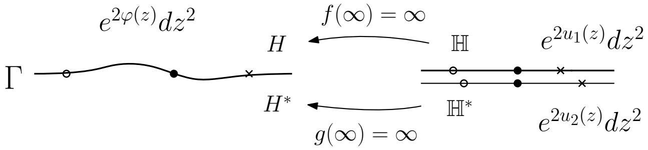
- *[Huber 1976]* The solution does not always exist.
- *[Bishop 1990]* Even if the solution exists, Γ can be a curve of positive area $\Longrightarrow$ non-uniqueness of solution.
- *[David 1982, Zinsmeister 1982...]* If D_1 and D_2 are chord-arc, then the solution exists and is unique, which is an quasi-circle. *[Bishop 1990]* The Hausdorff dimension of Γ can take any value in $1 < d < 2$.
- *[David 1982]* If the chord-arc constant of domains are close enough to 1, Γ is also chord-arc.
- *[Viklund, W. 2019+]* We will see that isometric welding of two finite energy domains has also finite energy (solution exists and is unique).

Isometric conformal welding

Let D_1, $D_2 \subset \mathbb{C}$ be Jordan domains bounded respectively by rectifiable curves Γ_1 and Γ_2 of same total length. Let $\psi : \Gamma_1 \to \Gamma_2$ be an isometry (preserves the arc-length).

- *[Huber 1976]* The solution does not always exist.
- *[Bishop 1990]* Even if the solution exists, Γ can be a curve of positive area $\Longrightarrow$ non-uniqueness of solution.
- *[David 1982, Zinsmeister 1982...]* If D_1 and D_2 are chord-arc, then the solution exists and is unique, which is an quasi-circle. *[Bishop 1990]* The Hausdorff dimension of Γ can take any value in $1 < d < 2$.
- *[David 1982]* If the chord-arc constant of domains are close enough to 1, Γ is also chord-arc.
- *[Viklund, W. 2019+]* We will see that isometric welding of two finite energy domains has also finite energy (solution exists and is unique).

Welding coupling identity



Let $\varphi \in W^{1,2}_{loc}(\mathbb{C})$ with finite Dirichlet energy:

$$\mathcal{D}_{\mathbb{C}}(\varphi) := \frac{1}{\pi} \int_{\mathbb{C}} |\nabla \varphi(z)|^2 \, dz^2 < \infty,$$

Γ an infinite Jordan curve, f, g the conformal maps from $\mathbb{H}, \mathbb{H}^*$ onto H, H^*, respectively.

Theorem (Welding coupling 2019+)

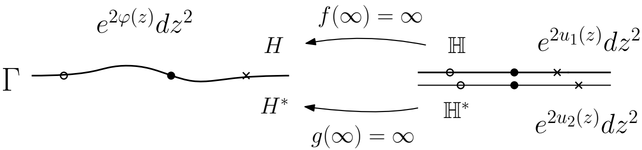
We have $e^{2\varphi} \in L^1_{loc}(\mathbb{C})$, so the measure $e^{2\varphi} dz^2$ is well-defined and locally finite. The pull-back measures e^{2u_1} by f on $\mathbb{H}$ (resp. e^{2u_2} by g on $\mathbb{H}^*$) satisfy

$$u_1(z) = \varphi \circ f(z) + \log \left| f'(z) \right|, \quad u_2(z) = \varphi \circ g(z) + \log \left| g'(z) \right|.$$

We have the identity

$$\mathcal{D}_{\mathbb{H}}(u_1) + \mathcal{D}_{\mathbb{H}^*}(u_2) = I^L(\Gamma) + \mathcal{D}_{\mathbb{C}}(\varphi).$$

Welding-coupling uniqueness



Theorem (Welding-coupling uniqueness, 2019+)

Suppose u_1 and u_2 with finite Dirichlet energy are given. Then there exist unique Γ, φ, f, and g such that the following holds:

1. Γ is an infinite Jordan curve passing through 0 and 1;
2. If H and H^* are the connected components of $\mathbb{C}\setminus\Gamma$, then $f : \mathbb{H} \to H$ is the conformal map fixing $0, 1$ and ∞ and $g : \mathbb{H}^* \to H^*$ is the conformal map fixing $0, \infty$;
3. $\varphi \in W^{1,2}_{loc}(\mathbb{C})$ and $\mathcal{D}_{\mathbb{C}}(\varphi) < \infty$;
4. $u_1(z) = \varphi \circ f(z) + \log |f'(z)|$, $z \in \mathbb{H}$;
5. $u_2(z) = \varphi \circ g(z) + \log |g'(z)|$, $z \in \mathbb{H}^*$.

In fact, Γ is obtained from the isometric conformal welding of $\mathbb{H}$ and $\mathbb{H}^*$ according to the boundary lengths $e^{u_1}|dz|$ and $e^{u_2}|dz|$. Moreover, $I^L(\Gamma) < \infty$.

Isometric welding of finite energy domains

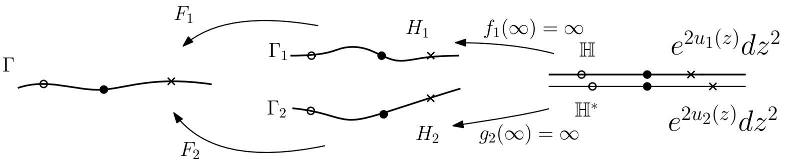
Assume $I^L(\Gamma_1) < \infty, I^L(\Gamma_2) < \infty$, both curves pass through ∞.

Corollary

The isometric conformal welding of Euclidean domain H_1 bounded by Γ_1 and H_2 bounded by Γ_2 has a unique solution Γ up to Möbius transformation. Moreover,

$$I^L(\Gamma) < I^L(\Gamma_1) + I^L(\Gamma_2)$$

if $I^L(\Gamma_1) + I^L(\Gamma_2) \neq 0$.



In fact, let $u_1 = \log|f_1'|$, $u_2 = \log|g_2'|$,

$$\mathcal{D}(u_1) \le I^L(\Gamma_1), \quad I^L(\Gamma) \le \mathcal{D}(u_1) + \mathcal{D}(u_2) \le I^L(\Gamma_1) + I^L(\Gamma_2).$$

The first equality holds only when $I^L(\Gamma_1) = 0$.

Isometric welding of finite energy domains

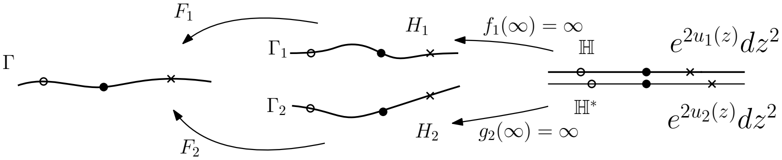
Assume $I^L(\Gamma_1) < \infty, I^L(\Gamma_2) < \infty$, both curves pass through ∞.

Corollary

The isometric conformal welding of Euclidean domain H_1 bounded by Γ_1 and H_2 bounded by Γ_2 has a unique solution Γ up to Möbius transformation. Moreover,

$$I^L(\Gamma) < I^L(\Gamma_1) + I^L(\Gamma_2)$$

if $I^L(\Gamma_1) + I^L(\Gamma_2) \neq 0$.



In fact, let $u_1 = \log |f_1'|$, $u_2 = \log |g_2'|$,

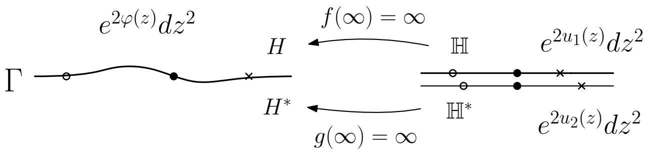
$$\mathcal{D}(u_1) \leq I^L(\Gamma_1), \quad I^L(\Gamma) \leq \mathcal{D}(u_1) + \mathcal{D}(u_2) \leq I^L(\Gamma_1) + I^L(\Gamma_2).$$

The first equality holds only when $I^L(\Gamma_1) = 0$.

Elements of proof of welding coupling identity

Welding coupling identity

$$\mathcal{D}_{\mathbb{H}}(u_1) + \mathcal{D}_{\mathbb{H}^*}(u_2) = I^L(\Gamma) + \mathcal{D}_{\mathbb{C}}(\varphi).$$



- Recall that $u_1(z) = \varphi \circ f(z) + \log|f'(z)|, \quad u_2(z) = \varphi \circ g(z) + \log|g'(z)|.$
- Use the identity $I^L(\Gamma) = \mathcal{D}_{\mathbb{H}}(\log|f'|) + \mathcal{D}_{\mathbb{H}^*}(\log|g'|)$.
- Prove that the cross-terms cancel out. □

Notice that since the harmonic conjugate $\arg(f')$ has the same Dirichlet energy as $\log |f'|$. We have the identity

$$I^L(\Gamma) = \mathcal{D}_{\mathbb{H}}(\arg f') + \mathcal{D}_{\mathbb{H}^*}(\arg g').$$

$\Rightarrow$ the analog to the forward SLE/GFF coupling (flow-line coupling).

Notice that since the harmonic conjugate $\arg(f')$ has the same Dirichlet energy as $\log|f'|$. We have the identity

$$I^L(\Gamma) = \mathcal{D}_{\mathbb{H}}(\arg f') + \mathcal{D}_{\mathbb{H}^*}(\arg g').$$

$\Rightarrow$ the analog to the forward SLE/GFF coupling (flow-line coupling).

Analog to flow-line coupling

Let η be a bounded C^1 Jordan curve and $\Gamma := \mu(\eta)$, where μ is a Möbius function mapping one point of η to ∞.
For $z = \Gamma(s)$, define the function $\tau : \Gamma \to \mathbb{R}$ such that τ is continuous and

$$\tau(z) := \arg(\Gamma'(s)) = -\arg(f^{-1})'(z).$$

We denote by $\mathcal{P}[\tau](z) = -\arg(f^{-1})'(z)$ the Poisson integral of τ in $\mathbb{C}$ (defined from both sides of Γ).

Theorem (Flowline coupling analog 2019+)

We have the identity

$$I^L(\Gamma) = \mathcal{D}_{\mathbb{C}}(\mathcal{P}[\tau]) = \min_{\varphi, \varphi|_\Gamma = \tau} \mathcal{D}_{\mathbb{C}}(\varphi).$$

Conversely, under regularity condition of φ and $\mathcal{D}_{\mathbb{C}}(\varphi) < \infty$, then for all $z_0 \in \mathbb{C}$, the solution to the differential equation

$$\Gamma'(t) = \exp\left(i\varphi(\Gamma(t))\right), \forall t \in \mathbb{R} \quad \text{and} \quad \Gamma(0) = z_0$$

is an infinite arclength parametrized simple curve and

$$I^L(\Gamma) \le \mathcal{D}_{\mathbb{C}}(\varphi).$$

Analog to flow-line coupling

Let η be a bounded C^1 Jordan curve and $\Gamma := \mu(\eta)$, where μ is a Möbius function mapping one point of η to ∞.
For $z = \Gamma(s)$, define the function $\tau : \Gamma \to \mathbb{R}$ such that τ is continuous and

$$\tau(z) := \arg(\Gamma'(s)) = -\arg(f^{-1})'(z).$$

We denote by $\mathcal{P}[\tau](z) = -\arg(f^{-1})'(z)$ the Poisson integral of τ in $\mathbb{C}$ (defined from both sides of Γ).

Theorem (Flowline coupling analog 2019+)

We have the identity

$$I^L(\Gamma) = \mathcal{D}_{\mathbb{C}}(\mathcal{P}[\tau]) = \min_{\varphi, \varphi|_\Gamma = \tau} \mathcal{D}_{\mathbb{C}}(\varphi).$$

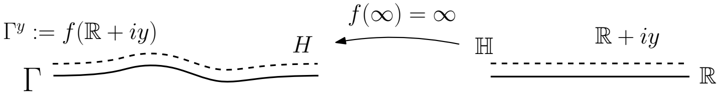
Conversely, under regularity condition of φ and $\mathcal{D}_{\mathbb{C}}(\varphi) < \infty$, then for all $z_0 \in \mathbb{C}$, the solution to the differential equation

$$\Gamma'(t) = \exp\left(i\varphi(\Gamma(t))\right), \forall t \in \mathbb{R} \quad \text{and} \quad \Gamma(0) = z_0$$

is an infinite arclength parametrized simple curve and

$$I^L(\Gamma) \le \mathcal{D}_{\mathbb{C}}(\varphi).$$

Equipotential energy decrease

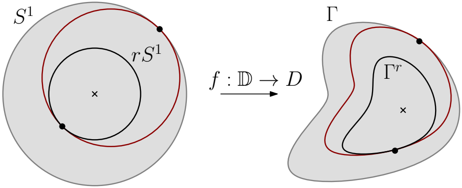


Corollary

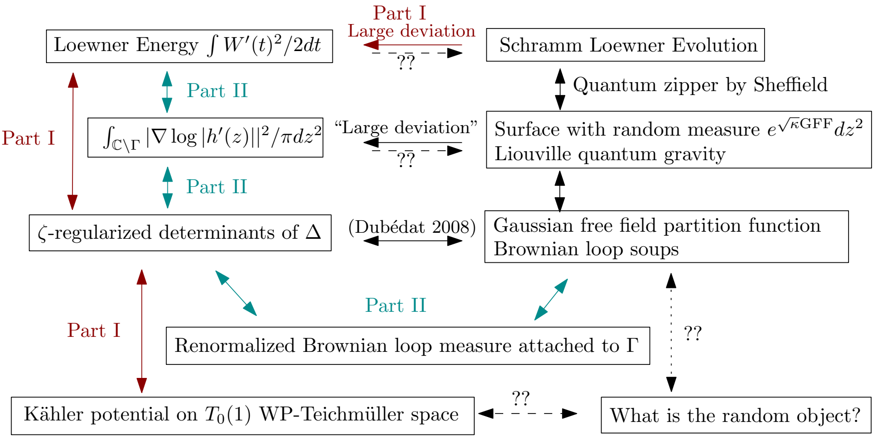
We have $I^L(\Gamma^y) \le I^L(\Gamma)$. The equality holds if and only if $I^L(\Gamma) = 0$.

Proof: Since on Γ^y, $\tau^y = \mathcal{P}[\tau]$. We have

$$I^L(\Gamma^y) = \mathcal{D}_{\mathbb{C}}(\mathcal{P}[\tau^y]) \le \mathcal{D}_{\mathbb{C}}(\mathcal{P}[\tau]) = I^L(\Gamma). \quad \square$$



Contents


Part I
Large deviation
??
Loewner Energy $\int W'(t)^2/2dt$
Schramm Loewner Evolution
Part II
Quantum zipper by Sheffield
Part I
$\int_{\mathbb{C}\setminus\Gamma} |\nabla \log |h'(z)||^2/\pi dz^2$
"Large deviation"
??
Surface with random measure $e^{\sqrt{\kappa}\mathrm{GFF}}dz^2$
Liouville quantum gravity
Part II
ζ-regularized determinants of Δ
(Dubédat 2008)
Gaussian free field partition function
Brownian loop soups
Part II
Part I
Renormalized Brownian loop measure attached to Γ
??
Kähler potential on $T_0(1)$ WP-Teichmüller space
??
What is the random object?

What random model?

- What is the random model naturally associated to the WP-Teichmüller space? Malliavin's measure on diffeomorphisms of the circle?
- In which space does the random welding belong to? (What analytic framework beyond quasiconformal geometry?)
- What is the gradient flow of the Loewner energy and what meaning in Loewner's framework? Other natural dynamics? Stochastic gradient flow?
- Random model $\Longrightarrow$ an intrinsic description of SLE loop ($\kappa \leq 4$)? $\Longrightarrow$ Reversibility?

What random model?

- What is the random model naturally associated to the WP-Teichmüller space? Malliavin's measure on diffeomorphisms of the circle?
- In which space does the random welding belong to? (What analytic framework beyond quasiconformal geometry?)
- What is the gradient flow of the Loewner energy and what meaning in Loewner's framework? Other natural dynamics? Stochastic gradient flow?
- Random model $\Longrightarrow$ an intrinsic description of SLE loop ($\kappa \leq 4$)? $\Longrightarrow$ Reversibility?

What random model?

- What is the random model naturally associated to the WP-Teichmüller space? Malliavin's measure on diffeomorphisms of the circle?
- In which space does the random welding belong to? (What analytic framework beyond quasiconformal geometry?)
- What is the gradient flow of the Loewner energy and what meaning in Loewner's framework? Other natural dynamics? Stochastic gradient flow?
- Random model $\Longrightarrow$ an intrinsic description of SLE loop ($\kappa \leq 4$)? $\Longrightarrow$ Reversibility?

What random model?

- What is the random model naturally associated to the WP-Teichmüller space? Malliavin's measure on diffeomorphisms of the circle?
- In which space does the random welding belong to? (What analytic framework beyond quasiconformal geometry?)
- What is the gradient flow of the Loewner energy and what meaning in Loewner's framework? Other natural dynamics? Stochastic gradient flow?
- Random model $\implies$ an intrinsic description of SLE loop ($\kappa \leq 4$)? $\implies$ Reversibility?

Exploring the connection

- How is the Kähler structure on the WP-Teichmüller space encoded in the Loewner's driving function? Why there is such a coincidence?
- Topological group structure on WP-Teichmüller space $\Longrightarrow$ what meaning in the Loewner setting?
- Use driving function to find purely geometric characterization of WP-quasicircles? (Jones' Conjecture)
- *[TT06]* WP-quasicircle $\Leftrightarrow$ associated Grunsky operator G is Hilbert-Schmidt. Moreover,

$$I^L(\Gamma) \propto \log \det{}_F(I - G^* G),$$

where $\det_F$ is the Fredholm determinant (only well-defined when G is HS).
Is it a better object to look at than zeta-regularized determinant of Laplacian?
Interpretation of Grunsky operator?

Exploring the connection

- How is the Kähler structure on the WP-Teichmüller space encoded in the Loewner's driving function? Why there is such a coincidence?
- Topological group structure on WP-Teichmüller space $\Longrightarrow$ what meaning in the Loewner setting?
- Use driving function to find purely geometric characterization of WP-quasicircles? (Jones' Conjecture)
- *[TT06]* WP-quasicircle $\Leftrightarrow$ associated Grunsky operator G is Hilbert-Schmidt. Moreover,

$$I^L(\Gamma) \propto \log \det{}_F(I - G^* G),$$

  where $\det_F$ is the Fredholm determinant (only well-defined when G is HS). Is it a better object to look at than zeta-regularized determinant of Laplacian? Interpretation of Grunsky operator?

Exploring the connection

- How is the Kähler structure on the WP-Teichmüller space encoded in the Loewner's driving function? Why there is such a coincidence?
- Topological group structure on WP-Teichmüller space $\implies$ what meaning in the Loewner setting?
- Use driving function to find purely geometric characterization of WP-quasicircles? (Jones' Conjecture)
- *[TT06]* WP-quasicircle $\Leftrightarrow$ associated Grunsky operator G is Hilbert-Schmidt. Moreover,

$$I^L(\Gamma) \propto \log \det{}_F(I - G^* G),$$

where $\det_F$ is the Fredholm determinant (only well-defined when G is HS). Is it a better object to look at than zeta-regularized determinant of Laplacian? Interpretation of Grunsky operator?

Exploring the connection

- How is the Kähler structure on the WP-Teichmüller space encoded in the Loewner's driving function? Why there is such a coincidence?
- Topological group structure on WP-Teichmüller space $\Longrightarrow$ what meaning in the Loewner setting?
- Use driving function to find purely geometric characterization of WP-quasicircles? (Jones' Conjecture)
- *[TT06]* WP-quasicircle $\Leftrightarrow$ associated Grunsky operator G is Hilbert-Schmidt. Moreover,

$$I^L(\Gamma) \propto \log \det{}_F(I - G^* G),$$

where $\det_F$ is the Fredholm determinant (only well-defined when G is HS). Is it a better object to look at than zeta-regularized determinant of Laplacian? Interpretation of Grunsky operator?

Other topology?

- Multiple-chord Loewner energy, large deviation of multiple SLE (work in progress with E. Peltola).
- Energy of (multiple) loops in higher genus surface?
- Probabilistic interpretation of Weil-Petersson metric on Teichmüller space of compact surfaces (genus ≥ 2)? Natural measure on Teichmüller/moduli space?
- Conformal field theory (SLE, statistical mechanics models) $\Longrightarrow$ String theory (Kähler geometry on universal Teichmüller space)???

Thanks for your attention!