Loop measures, partition functions, and multiple SLEs

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(Discrete time) Random Walk Loop Measure (in $\mathbb{Z}^2$)

- A rooted loop is a nearest neighbor path $l = [\ell_0, \ell_1, \ldots, \ell_{2n}]$ with $\ell_0 = \ell_{2n}$.
- An (unrooted) loop ℓ is an equivalence class of rooted loops under the relation

$$[\ell_0, \ell_1, \ldots, \ell_{2n}] \sim [\ell_1, \ell_2, \ldots, \ell_{2n}, \ell_1] \sim [\ell_2, \ell_3, \ldots, \ell_1, \ell_2] \sim \cdots .$$

- The rooted loop measure $\tilde{m}$ gives measure $(2n)^{-1}\, 4^{-2n}$ to each rooted loop of $2n$ steps ($n > 0$).
- The (unrooted) loop measure m gives each unrooted loop l measure

$$m(\ell) = \sum_{l \in \ell} \tilde{m}(l) = \frac{K(\ell)}{2n\, 4^{2n}},$$

where $K(\ell)$ is the number of representatives of ℓ. $K(\ell)$ divides $2n$ but could be smaller.

- If $A \subset \mathbb{Z}^2$, m_A is m restricted to loops lying in A.
- $$\exp\left\{\sum m_A(\ell)\right\} = \frac{1}{\det(I - P_A)}.$$

  Here $P_A - I$ is the usual random walk Laplacian on A with Dirichlet boundary conditions.
- Poissonian realizations from the random walk measure are called loop soups.
- The scaling limit of the random walk loop measure is called the Brownian loop measure which gives Brownian loop soups. (L-Werner,-Trujillo Ferreras)
- The Brownian loop measure is a conformal invariant.
- The loop measure was discovered in trying to understand the (generalized) restriction property for SLE (L-W-Schramm)

- There is also a continuous-time, discrete space version (Le Jan)
- The loop measures and soups can be extended to much wider classes of graphs and weights on paths. (Lupu and metric graphs/cable systems)
- Close relation between loop measure/soups and Gaussian free field (Le Jan, Lupu)
- These ideas can be extend to measures that are not positive giving Gaussian fields with negative covariances (L-Perlman, Panov) and applications to loop-erased random walk (Kenyon, L-Beneš-Viklund).

λ-SAW (Kozdron -L)

- The model contains two parameters, one lattice-independent λ and one lattice-dependent β which we choose to be critical.
- In fact, $\lambda = -\mathbf{c}/2$ where $\mathbf{c} \in (-\infty, 1]$ is the central charge.
- For any SAW η (or collection of nonintersecting SAWS, $\boldsymbol{\eta} = (\eta^1, \ldots, \eta^n)$) in A give measure

$$e^{-\beta|\boldsymbol{\eta}|} \exp\left\{-\frac{\mathbf{c}}{2} \sum_{\boldsymbol{\eta}\cap\ell\neq\emptyset} m_A(\ell)\right\}.$$

which can also be written as

$$\det(I - P_A)^{-\mathbf{c}/2}\, e^{-\beta|\boldsymbol{\eta}|} \left[\frac{\det(I - P_{A\setminus\boldsymbol{\eta}})}{\det(I - P_A)}\right]^{-\mathbf{c}/2}.$$

Wild meta-conjectures

- This measure on paths converges (somehow to some form of) SLE_κ, $\kappa \leq 4$ with

$$\mathbf{c} = \frac{(6-\kappa)\,(3\kappa-8)}{2\kappa}.$$

- If we add another term

$$e^{\rho N(\boldsymbol{\eta})}\, e^{-\beta|\boldsymbol{\eta}|}\, \exp\left\{-\frac{\mathbf{c}}{2}\sum_{\boldsymbol{\eta}\cap\ell\neq\emptyset} m_A(\ell)\right\}.$$

where $N(\boldsymbol{\eta})$ denotes the number of "boundary edges", then converges to SLE_κ for $4 < \kappa < 8$. (Since this measure would be reversible would not be true for $\kappa > 8$.)

More precise (wild) conjectures, $\kappa \leq 4$

- Let $D = \{x + iy \in \mathbb{C} : -1 < |x|, |y| < 1\}$ and let $A_N = \{x + iy \in \mathbb{Z} + i\mathbb{Z} : -N < |x|, |y| < N\}$.
- Consider the measure restricted to SAWs starting at $-N$ going to N and otherwise in A_N.
- There exists a scaling exponent b such that the total mass of the measure is asymptotic to $\Psi_D(-1,1)\, N^{-2b}$.
- There exists scaling dimension d such that we can normalize paths such that

$$\eta^N(t) = N^{-1}\, \eta(tN^d).$$

- If we normalize and multiply the measure on scaled paths by N^{2b}, then the limit exists and is a measure on continuous paths $\mu_D(-1,1)$ of total mass $\Psi_D(-1,1)$. (partition function)

- The family of measures $\mu_D(z, w)$ is conformally covariant

$$f \circ \mu_D(z, w) = |f'(z)|^b \, |f'(w)|^b \mu_{f(D)}(f(z), f(w)).$$

  The probability measure $\mu_D^{\#}(z, w)$ is conformally invariant.
- Here the pushforward is defined with appropriate change of parametrization. The image of $\gamma[s, t]$ is $f(\gamma[s, t])$ and the time to traverse it is

$$\int_s^t |f'(\gamma(r))|^d \, dr.$$

- The construction shows that we would expect $\mu_D^{\#}(z, w)$ to satisfy the domain Markov property, and hence (by Schramm) be SLE_κ for some κ.
- Similar construction if z, w are interior points but a different scaling exponent $\tilde{b}$ appears.

- The limit should satisfy the (generalized) restriction or boundary perturbation property. If $D' \subset D$ and D', D agree near z, w, then $\mu_{D'}(z,w) \ll \mu_D(z,w)$ with Radon-Nikodym derivative

$$Y(\gamma) = 1\{\gamma \subset D'\} \, \exp\left\{ \frac{\mathbf{c}}{2} \, m_D(\gamma, D \setminus D') \right\},$$

  where now $m_D(V_1, V_2)$ denotes the Brownian loop measure of loops in D that intersect both V_1 and V_2.
- The expected value of Y with respect to $\mu_D^{\#}(z,w)$ is the conformal invariant

$$\frac{\Psi_{D'}(z,w)}{\Psi_D(z,w)}.$$

- Restriction property true for $SLE_\kappa, \kappa \leq 4$ with

$$\frac{13 - \mathbf{c} - \sqrt{(\mathbf{c}-1)(\mathbf{c}-25)}}{3}.$$

$$d = 1 + \frac{\kappa}{8}, \quad b = \frac{6-\kappa}{2\kappa}, \quad \tilde{b} = b\,\frac{\kappa-2}{4}.$$

- Natural parametrization by d-dimensional Minkowski content.
- The limit exists in a strong sense for $\mathbf{c} = -2, \kappa = 2$ where λ-SAW is same as loop-erased random walk
- For $\mathbf{c} = 0, \kappa = 8/3$ can compute (assuming existence of conformally invariant limit) critical exponents for self-avoiding walk
- NOT the same as some other models with SLE scaling limits (although the $\mathbf{c} = 6, \kappa = 0$ on triangular lattice agrees with percolation exploration)
- For $0 < \mathbf{c} \leq 1$ related to construction of CLE using Brownian loop clusters.

SLE_κ partition function $\kappa < 8$

- Satisfies scaling rule

$$f \circ \Psi_D(z,w) = |f'(z)|^{b'} \, |f'(w)|^{b'} \, \Psi_{f(D)}(f(z), f(w)),$$

where $b' = b$ or $\tilde{b}$ depending on if z, w is boundary or interior point. Normalized so that

$$\Psi_{\mathbb{D}}(1,0) = 1, \qquad \Psi_{\mathbb{H}}(0,1) = 1.$$

- For chordal case, $\Psi_D(z,w) = H_D(z,w)^b$ where H_D is the boundary Poisson kernel. Not true in radial case if $\kappa \neq 2$.

Radial restriction property

- γ radial SLE_κ from 1 to 0 in $\mathbb{D}$. $\gamma_t = \gamma[0,t]$.
- $U = \mathbb{D} \setminus K \subset \mathbb{D}$ simply connected, $0 \in U$, $\operatorname{dist}(1,K) > 0$.
- Let $D_t = \mathbb{D} \setminus \gamma_t, U_t = U \setminus \gamma_t$,

$$\Psi_t = \frac{\Psi_{U_t}(\gamma(t),0)}{\Psi_{D_t}(\gamma(t),0)}.$$

- SLE in U is SLE in $\mathbb{D}$ "weighted locally" by Ψ_t.
- More precisely find local martingale $M_t = A_t\,\Psi_t$ where A_t is differentiable, and use Girsanov theorem.
- The drift (logarithmic derivative of Ψ_t) is the same as that obtained for SLE_κ in U.

- Calculation shows that

$$A_t = 1\{\gamma_t \subset U\} \exp\left\{\frac{\mathbf{c}}{2}\, m_D(\gamma_t, K)\right\}.$$

- If $\kappa \le 4$, can let $t \to \infty$,

$$\Psi_0 = \mathbb{E}[\Psi_\infty] = \mathbb{E}\left[1\{\gamma \subset U\} \exp\left\{\frac{\mathbf{c}}{2}\, m_D(\gamma, K)\right\}\right].$$

- Proof uses continuity of radial SLE_κ. Also need to consider separately loops of zero winding number about 0 and those of nonzero winding number.

General strategy $\kappa \leq 4$

- Using the λ-SAW as a model, can define multiple SLE and SLE in nonsimply connected domains using the restriction method.
- This is defined as a positive measure (not necessarily finite) on curves, not necessarily finite. When finite, can normalize to make it a probability measure.
- This is well defined but might not be exactly the same limit as a different discrete model.
- The partition function is immediately defined. If the partition function is sufficiently smooth, one can use Girsanov to find the behavior of tilted curves.
- These probability measures are the objects that come under the name of $SLE(\kappa, \rho)$ and generalizations. More canonical to define the process in terms of the partition function rather than the noncanonical parameter ρ.

- Although this direct approach requires $\kappa \leq 4$, many of the contructions seem to make sense for $\kappa < 8$.
- Imaginary geometry has been useful in handling this regime.
- Perhaps the generalized λ-SAW will be a useful heuristic here.

Multiple radial SLE to the same point (with V. Healey)

- Natural measure on n-tuple of nonintersecting ($\kappa \leq 4$) or noncrossing ($4 < \kappa < 8$) curves in the disk from distinct points on the unit circle to the origin.
- Want to grow curves at the “same time”.
- Consider $\kappa \leq 4$ for which we will use the λ-SAW viewpoint.
  - Independent SLEs are weighted by loops that hit them
  - Compensate by tilting SLE by loops that hit more than one path.
  - Must use a limit process since the total measure of such loops is infinite (since all paths end at origin)

- Express points z_t^j on unit circle as $z_t^j = \exp\{2i\theta_t^j\}$. Factor of 2 is convenient for several reasons
    - Makes easy comparisons to angles in upper half plane that range from 0 to π.
    - $|z_t^j - z_t^k| = 2\,|\sin(\theta_t^j - \theta_t^k)|$.
- Want to find quantities that are independent of the choice of parametrization (don't have to grow curves at same "rate").

- For any loop ℓ, let

$$s^j(\ell) = \min\{t : \gamma^j(t) \in \ell\}, \quad s(\ell) = \min_j s^j(\ell).$$

  Depends on parameterization of γ^j.
- Interested in loops that intersect γ^j but for which $s^j(\ell) > s(\ell)$ (overcounted).
- Let

$$\mathcal{L}_T = \exp\left\{\sum_{j=1}^{n} m(L_T^j)\right\}.$$

  Here I_T is the indicator function for each $j \neq k$, $\gamma_T^j \cap \gamma^k \neq \emptyset$ and L_T^j is the set of loops with $s(\ell) < s^j(\ell), s(\ell) < T$.
- Goal: determine $\mathbb{E}[I_T \, \mathcal{L}_T^{\mathbf{c}/2}]$ (which depends on the parametrization) and the conditional distribution on the paths (which should not depend on parametrization) when tilted by $\mathcal{L}_T^{\mathbf{c}/2}$.

- $t < T$, write
$$\mathcal{L}_T = \hat{\mathcal{L}}_t \; \frac{\mathcal{L}_t}{\hat{\mathcal{L}}_t} \; \mathcal{L}_{T,t},$$
where these represent the contributions for loops satisfying
  - $s(\ell) < s^j(\ell) \le t$ (depends only on $\boldsymbol{\gamma}_t$)
  - $s(\ell) \le t, s^j(\ell) > t$ (depends on $\boldsymbol{\gamma}$; not T)
  - $t < s(\ell) \le T$ (depends on $\boldsymbol{\gamma}_T \setminus \boldsymbol{\gamma}_t$)
- Restriction property shows that
$$R_t := \mathbb{E}\left[I_t \, \mathcal{L}_t^{\mathbf{c}/2} \mid \gamma_t\right] = \hat{I}_t \, \hat{\mathcal{L}}_t^{\mathbf{c}/2} \, \Psi_t = \hat{I}_t \, \hat{\mathcal{L}}_t^{\mathbf{c}/2} \prod_{j=1}^{n} \Psi_t^j,$$
where $\hat{I}_t$ is the indicator function of $\{\gamma_t^j \cap \gamma_t^k = \emptyset, \, j \neq k\}$ and
$$\Psi_t^j = \frac{\Psi_{\mathbb{D}\setminus\gamma_t^j}(\gamma^j(t), 0)}{\Psi_{\mathbb{D}\setminus\boldsymbol{\gamma}_t}(\gamma^j(t), 0)}.$$

- Moreover, it we tilt by R_t, the the conditional distribution of the remainder of the path $\boldsymbol{\gamma} \setminus \boldsymbol{\gamma}_t$ is that of independent SLE_κ paths from $\gamma^j(t)$ to the origin in $\mathbb{D} \setminus \boldsymbol{\gamma}_t$.
- Let $g_t : \mathbb{D} \setminus \boldsymbol{\gamma}_t \to \mathbb{D}$ be the conformal transformation with $g_t(0) = 0, g_t'(0) > 0$ and write

$$z_t^j = g_t(\gamma^j(t)) = \exp\{2i\theta_t^j\}.$$

- Let

$$q(T, \boldsymbol{\theta}) = \mathbb{E}^{\boldsymbol{\theta}} \left[I_T \, \mathcal{L}_T \right].$$

  Then,

$$\mathbb{E}^{\boldsymbol{\theta}} \left[I_T \, \mathcal{L}_T \mid \gamma_t \right] = R_t \, q(T - t, \boldsymbol{\theta}_t).$$

- $$M_t := R_t \, q(T - t, \boldsymbol{\theta}_t)$$

  is a $\mathbb{P}$-martingale

- R_t is *not* a martingale. Find the C^1 function A_t such that

$$N_t = R_t\, A_t$$

is a $\mathbb{P}$-martingale. A_t has the form

$$A_t = \exp\left\{\int_0^t G(\boldsymbol{\theta}_s)\, ds\right\}$$

for some G.

- If we tilt by N_t using Girsanov, we get a new measure $\mathbb{P}_*$ under which the paths are locally independent SLE_κ. At each time t, the n paths move like independent SLE_κ in the slit domain $\mathbb{D} \setminus \boldsymbol{\gamma}_t$ (In original measure, γ^j moves like SLE_κ in $\mathbb{D} \setminus \gamma_t^j$.)
- Write

$$M_t = N_t\, A_t^{-1}\, q(T-t, \boldsymbol{\theta}_t)$$

Since M_t, N_t are $\mathbb{P}$-martingales, $\tilde{M}_t := A_t^{-1}\, q(T-t, \boldsymbol{\theta}_t)$ is a $\mathbb{P}_*$-martingale.

- Feynman-Kac gives

$$\begin{aligned}\tilde{M}_t &= \mathbb{E}_*\left[\exp\left\{-\int_0^T G(\theta_s)\,ds\right\} \mid \boldsymbol{\gamma}_t\right] \\ &= \exp\left\{-\int_0^t G(\theta_s)\,ds\right\} q(T-t,\boldsymbol{\theta}_t).\end{aligned}$$

$$\tilde{M}_0 = q(\boldsymbol{\theta},T) = \mathbb{E}_*^{\boldsymbol{\theta}}[\tilde{M}_T] = \mathbb{E}_*^{\boldsymbol{\theta}}\left[\exp\left\{-\int_0^T G(\theta_s)\,ds\right\}\right].$$

- Hope: find a function $F(\boldsymbol{\theta})$ and ξ such that

$$Z_t := e^{\xi t}\,F(\boldsymbol{\theta}_t)\,\exp\left\{-\int_0^t G(\theta_s)\,ds\right\} = e^{\xi t}\,F(\boldsymbol{\theta}_t)\,A_t^{-1}$$

is a $\mathbb{P}_*$-martingale.

  - In this case we will give F explicitly.
  - More generally, write PDE that F must satisfy and hope to find solution.

- Let $\tilde{\mathbb{P}}$ denote probabilities obtained by tilting $\mathbb{P}_*$ by Z_t.
-

$$\begin{aligned} q(t, \boldsymbol{\theta}) &= \mathbb{E}_*^{\boldsymbol{\theta}}[A_t^{-1}] \\ &= e^{-\xi t}\, \mathbb{E}_*^{\boldsymbol{\theta}} \left[Z_t\, F(\boldsymbol{\theta}_t)^{-1} \right] \\ &= e^{-\xi t}\, F(\boldsymbol{\theta}_0)\, \tilde{\mathbb{E}}^{\boldsymbol{\theta}}[F(\boldsymbol{\theta}_t)^{-1}] \end{aligned}$$

- We now find the invariant probability distribution f for the paths under the measure $\tilde{\mathbb{P}}$, and if we have an exponential rate of convergence to equilbrium,

$$\tilde{\mathbb{E}}^{\boldsymbol{\theta}}[F(\boldsymbol{\theta}_t)^{-1}] = c_* + O(e^{-ut}), \quad c_* = \int F(\boldsymbol{\theta})^{-1}\, f(\boldsymbol{\theta})\, d\boldsymbol{\theta}$$

- Therefore,

$$q(t, \boldsymbol{\theta}) = c_*\, e^{-\xi t}\, F(\boldsymbol{\theta}_0)\, [1 + O(e^{-ut})].$$

- Finally, if $t < T$, then exponential rate of convergence to equilibirum shows that the correlation between $F(\theta_T)^{-1}$ and $\boldsymbol{\gamma}_t$ is $O(e^{-u(T-t)})$.
- This implies that if we tilt the paths in $\mathbb{P}$ by $\mathcal{L}_T$, then, up to a $O(e^{-u(T-t)})$ correction, the restriction to γ_t is $\tilde{\mathbb{P}}$.

Return to radial SLE_κ, $\kappa \leq 4$.

- Let $a = 2/\kappa$.
- Let $z_t = \exp\{2i\theta_t\}$. Use radial Loewner equation

$$\dot{g}_t(z) = 2a\, g_t(z)\, \frac{z_t + g_t(z)}{z_t - g_t(z)}.$$

  Parametrized so that $g_t'(0) = e^{2at}$. If $|w| = 1$ and $w_t = g_t(w) = \exp\{2i\psi_t\}$, then,

$$\dot{\psi}_t = a\, \cot[\psi_t - \theta_t].$$

- If θ_t is a standard Brownian motion, then this gives radial SLE_κ.
- Choose this parametrization so that the variance parameter of θ_t is one. Note that z_t does Brownian motion on the circle with variance parameter 4.

SLE_κ paths with common parametrization

- $z_t^1 = \exp\{2i\theta_t^1\}, \quad \ldots, \quad z_t^n = \exp\{2i\theta_t^1\}.$
- Loewner equation

$$\dot{g}_t(z) = 2a\, g_t(z) \sum_{j=1}^{n} \frac{z_t^j + g_t(z)}{z_t^j - g_t(z)}$$

  Parametrized so that $g_t'(0) = e^{2ant}$.
- We will have θ_t satisfy SDE of the form

$$d\theta_t^j = \phi^j(\boldsymbol{\theta}_t)\, dt + dB_t^j,$$

  where $B_t^1, \ldots, B_t^n$ are standard Brownian motions.
- Might want variance parameter $1/n$ with Loewner equation

$$\dot{g}_t(z) = \frac{2a}{n}\, g_t(z) \sum_{j=1}^{n} \frac{z_t^j + g_t(z)}{z_t^j - g_t(z)}$$

  so that $g_t'(0) = e^{2at}$ but we will not today.

Independent SLE

- Independent SLE corresponding to independent θ_t^j.
- They are only independent modulo reparametrization. The common parameterization looks at all the paths.

Multiple radial Bessel process (Dyson Brownian motion on the circle)

- Let

$$F(\boldsymbol{\theta}) = \prod_{1 \le j < k \le n} |\sin \theta^k - \sin \theta^j| = c_n \prod_{1 \le j < k \le n} |z^k - z^j|.$$

$$F_a(\boldsymbol{\theta}) = [F(\boldsymbol{\theta})]^a$$

$$\psi(\boldsymbol{\theta}) = \sum_{j=1}^{n} \sum_{k \ne j} \csc^2(\theta^j - \theta^k) = 2 \sum_{1 \le j < k \le n} \csc^2(\theta^j - \theta^k)$$

- The (n)-radial Bessel process with parameter a on the torus $[0, \pi)^n$ is Brownian motion weighted locally by F_a.

- To be more precise,

$$M_{t,a} := F_a(\boldsymbol{\theta}_t) \exp\left\{ \frac{a^2 n(n^2-1)}{6} t \right\} \exp\left\{ \frac{a-a^2}{2} \int_0^t \psi(\boldsymbol{\theta}_s) ds \right\}$$

  is a local martingale satisfying

$$dM_{t,a} = M_{t,a} \sum_{j=1}^{n} \left(\sum_{k \neq j} a \cot(\theta_t^j - \theta_t^k) \right) d\theta_t^j .$$

- If $\mathbb{P}_a$ denotes the probability measure obtained by tilting by $M_{t,a}$, then

$$d\theta_t^j = a \sum_{k \neq j} \cot(\theta_t^j - \theta_t^k)\, dt + dW_t^j ,$$

  where $W_t^1, \ldots, W_t^n$ are independent standard Brownian motions with respect to $\mathbb{P}_a$.

- $\mathbb{P}_a$ corresponds to $\mathbb{P}_*$, that is, locally independent SLE.
- One can get this by discrete approximations and can show independent of how the curves are grown. The common parametrization is a convenience for nice formulas.
- Using deteministic estimates from the Loewner equation, we get
$$A_t = \exp\left\{ab \int_0^t \psi(\boldsymbol{\theta}_s)\, ds\right\}, \quad b = \frac{3a-1}{2}.$$
- Note that
$$M_{t,2a} = M_{t,a}\, N_t$$
where
$$N_t = F_a(\boldsymbol{\theta}_t)\, \exp\left\{\frac{a^2 n(n^2-1)}{2} t\right\} A_t^{-1}.$$

- $\mathbb{P}_{2a}$ corresponds to $\tilde{\mathbb{P}}$.
- Using the fact that $\mathbb{P}_{2a}$ is obtained from $\mathbb{P}$ (independent Brownian motions) by weighting locally by F_{2a} we can see that the invariant density is

$$f_{2a} = J_{4a}\, F_{4a}, \quad J_{4a} = \int F_{4a}(\boldsymbol{\theta})\, d\boldsymbol{\theta}.$$

$$q(t,\boldsymbol{\theta}) \sim \frac{J_{3a}}{J_{4a}}\, F_{2a}(\boldsymbol{\theta})\, g_t'(0)^{-\xi},$$

$$\xi = \frac{a^2(n^2-1)}{4}.$$

- n-radial SLE_κ corresponds to the invariant density

$$c \prod_{1 \le j < k \le n} |z^k - z^j|^{8/\kappa}$$

- The n-interior scaling exponent is

$$\tilde{b}_n = \xi + \tilde{b} = \frac{4(n^2-1) + (6-\kappa)(\kappa-2)}{8\kappa}.$$

$n = 2$: two-sided radial SLE

- Need only consider $X_t = \theta_t^2 - \theta_t^1$.

  $$dX_t = 2a \, \cot X_t \, dt + dB_t^2 - dB_t^1 .$$

  After a time change this becomes

  $$dX_t = a \, \cot X_t + dB_t.$$

- $\tilde{b}_2 = 2 - d$. $(\kappa \leq 4)$

Two-sided annulus SLE ($\kappa \leq 4$) (with M. Jahangoshahi)

- Annulus SLE_κ connecting points on different boundaries is defined using restriction property comparing to chordal SLE in the covering space (infinite strip).
- Defined as a non-probability measure for paths with each winding number.
- Finiteness and smoothness proved using ideas somewhat similar to above. Here, the partition function is not known precisely (except for some special κ), but one can write the PDE and use PDE results to establish existence of solution.
- The annulus Loewner equation is not used in the definiton but is used in the analysis.

- Given the annulus SLE_κ, two-sided SLE_κ is defined immediately with its partition function.
- Not obvious that the partition function is smooth, but this can be proved using Hormander theorem.
- As annulus approaches full disk, two-sided annulus approaches two-sided radial.
- Both two-sided radial and two-sided annulus used to understand chordal SLE near an interior point (approaching in both directions)

SLE loops (-Field, Zhan)

- The λ-SAW formalism gives a way to introduce loops; most natural to consider unrooted self-avoiding loops.
- The way to construct is to start with a rooted loop measure. This is an infinite measure because there are loops of all size.
- Parametrize by Minkowsi content so that one put a term of the form $1/$length in the definition as in definition of random walk/Brownian motion loop measure.
- Technical hurdle solved by Zhan (using two-sided radial ideas).
- Tricky question how to handle boundary.
- While this is an infinite measure, the measure of "macrosopic loops" (say, loops in an annulus that separate the boundaries) is finite.

Example: $\kappa = 2$, "loop-erased loops"

- Consider $A_n = \{z \in \mathbb{Z}^2 : |z| < n\}$ and let $\mathcal{T}$ be a wired spanning tree.
- For an (unrooted) SAW loop (polgyon), say that it appears in $\mathcal{T}$ if all but one edge appears.
- The probability that η appears in a uniform spanning tree is

$$4^{-|\eta|} \exp\left\{ \sum_{\ell \subset A_n, \ell \cap \eta \neq \emptyset} m(\ell) \right\} \mathcal{E}(\eta, \partial A_n),$$

where $\mathcal{E}$ denotes the (conformal invariant) excursion measure.