Natural measures on random fractals

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Based on works with:
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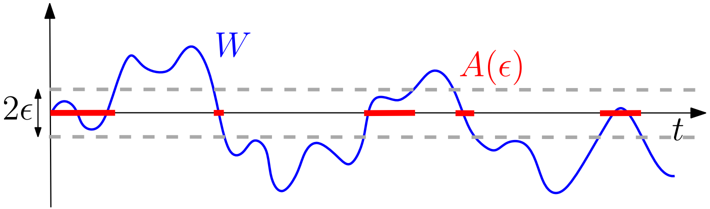
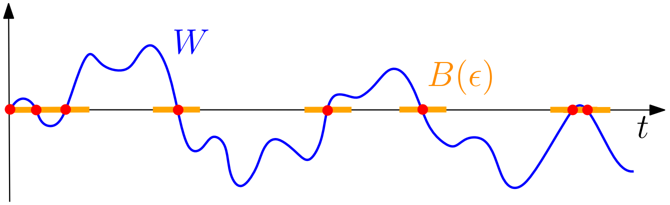
Brownian local time: density of occupation measure at 0



$$L(t) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \operatorname{Leb}(A(\epsilon) \cap [0, t])$$

Brownian local time: 1/2-Minkowski content



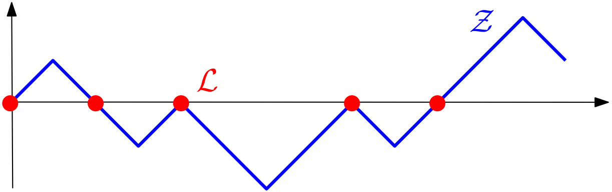
$$L(t) = \lim_{\epsilon \to 0} \frac{c}{\epsilon^{1/2}} \operatorname{Leb}(B(\epsilon) \cap [0,t])$$

- Let W be a Brownian motion with local time L at 0.
- Let $\mathcal{Z}$ be a simple random walk with local time $\mathcal{L}$ at 0.

Theorem 1 (Révész'81)

$$\left(\frac{1}{\sqrt{n}}\mathcal{Z}(n\cdot), \frac{c}{\sqrt{n}}\mathcal{L}(n\cdot)\right) \Rightarrow (W, L).$$

See also Csáki-Révész'83, Borodin'89, Bass-Koshnevisan'93.



Brownian local time: axiomatic characterization

For W a Brownian motion let $t \mapsto L(t, W)$ be

1. continuous,
2. increasing,
3. non-negative,
4. adapted,
5. constant outside the zero set of W, and
6. additive $(L(s+t, W) = L(t, W) + L(s, W(t+\cdot)))$.

Then L is a deterministic multiple of the local time of W.

See e.g. McKean-Tanaka'61.

Minkowski content ⟷ Limit of counting measure

Axiomatic characterization

Natural measures on fractals

Minkowski content

Limit of counting measure

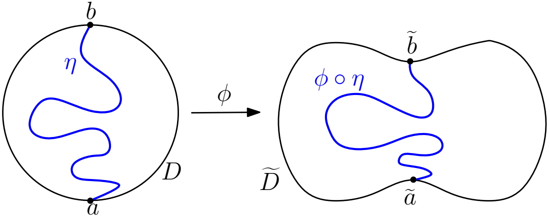
Axiomatic characterization

What natural measures are supported on other fractal sets?

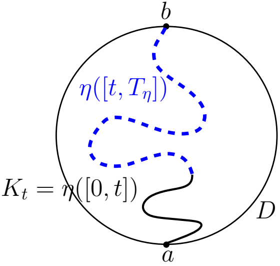
1. Schramm-Loewner evolutions
2. SLE_6 pivotal points and Brownian cut points
3. Fractals in Liouville quantum gravity (LQG) environment

An SLE_κ η is a random curve **modulo time reparametrization** satisfying

- **Conformal invariance:** $\phi \circ \eta$ is an SLE_κ in $(\widetilde{D}, \widetilde{a}, \widetilde{b})$.
- **Domain Markov property:** $\eta|_{[t,T_\eta]}$ is an SLE_κ in $(D \setminus K_t, \eta(t), b)$.



Conformal invariance



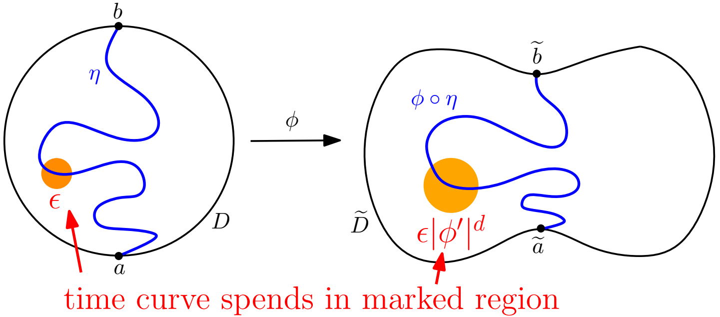
Domain Markov property

An SLE_κ η with its **natural parametrization** is a random curve satisfying

- **Conformal invariance:** $\phi \circ \eta$ is an SLE_κ in $(\widetilde{D}, \widetilde{a}, \widetilde{b})$ such that $(\phi \circ \eta)([0,t])$ is traced in time

$$\int_0^t |\phi'(\eta(s))|^d \, ds, \qquad d = \left(1 + \frac{\kappa}{8}\right) \wedge 2.$$

- **Domain Markov property**



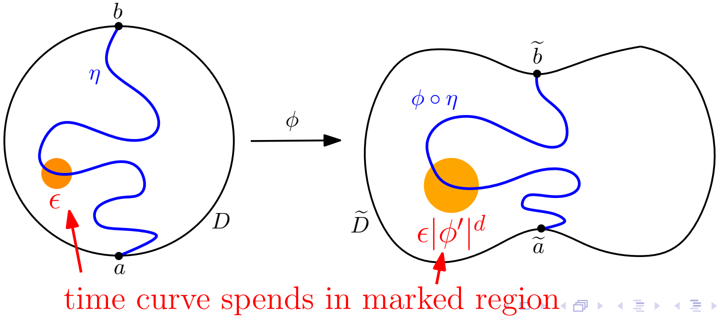
The natural parametrization of SLE

Lawler-Sheffield'09:

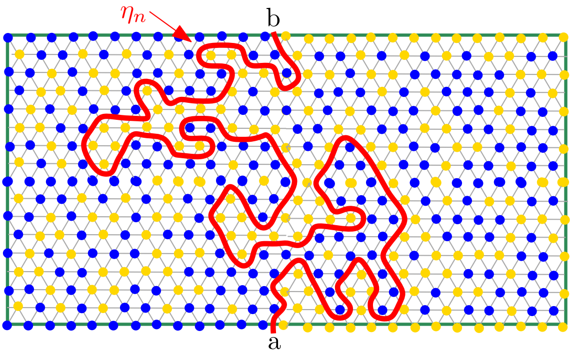
- introduced the natural parametrization
- uniqueness for all $\kappa \in (0,8)$ (under assumption of finite expectation)
- existence for $\kappa < 5.021...$

Lawler-Zhou'13, Lawler-Rezaei'15:

- For all $\kappa \in (0,8)$ the natural parametrization exists and given by $1+\frac{\kappa}{8}$-Minkowski content.



Percolation interface $\Rightarrow$ SLE$_6$

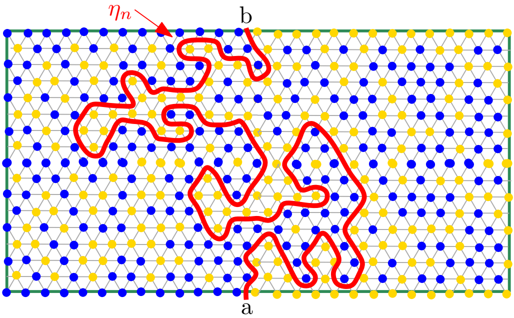


Let η be an SLE$_6$ in (D, a, b).

Theorem 2 (Smirnov'01)

When $n \to \infty$, $\eta_n \Rightarrow \eta$ as a curve modulo reparametrization of time.

Percolation interface $\Rightarrow$ SLE$_6$ in natural parametrization



each face traversed in time $n^{-7/4+o(1)}$

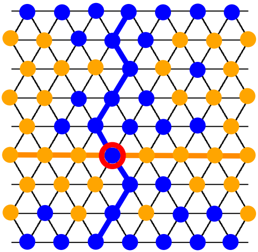
Let η be an SLE$_6$ in (D, a, b) with its natural parametrization.

Theorem 3 (H.-Li-Sun'18)

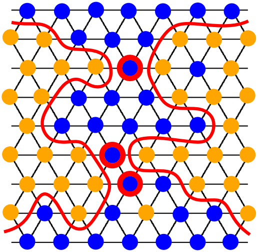
When $n \to \infty$, $\eta_n \Rightarrow \eta$ for the uniform topology.

- Garban-Pete-Schramm'13: Counting measure on the percolation interface has a scaling limit.
- Lawler-Viklund'17: Loop-erased random walk $\Rightarrow$ SLE$_2$ in natural parametrization

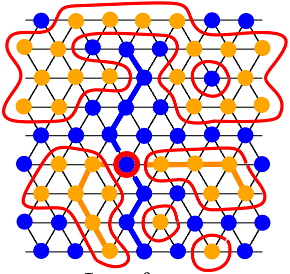
Percolation pivotal points



4-arm event



Pair of interfaces

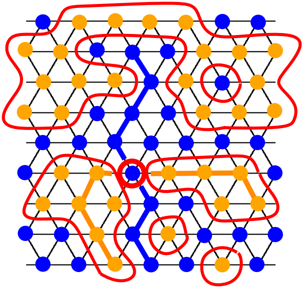


Interface intersection points

Percolation pivotal points

Theorem 4 (H.-Li-Sun'18)

- *The ϵ-important pivotal points (double points) of SLE_6 and CLE_6 have a.s. non-trivial and finite $3/4$-Minkowski content ν.*
- *If ν_n is counting measure on discrete pivotal points, then $\nu_n \Rightarrow \nu$ as $n \to \infty$. The convergence is joint with convergence to CLE_6.*



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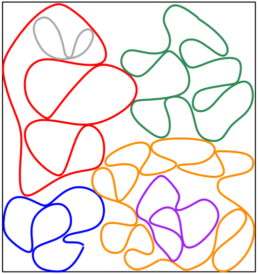


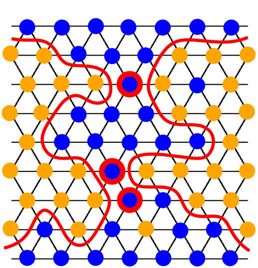
Conformal loop ensemble CLE_6

Brownian cut points

Theorem 5 (H.-Lawler-Li-Sun'18)

- *$(W_t)_{t\in\mathbb{R}}$ a 2d Brownian excursion; $A \subset \mathbb{R}^2$ is the set of cut-points.*
- *Then A has a.s. locally finite and non-trivial 3/4-Minkowski content.*

We also prove the analogous 3d result (but cut point dimension unknown).



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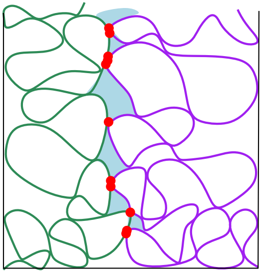
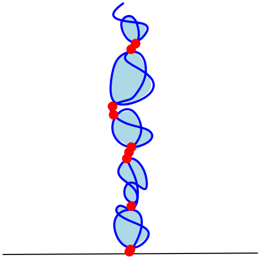


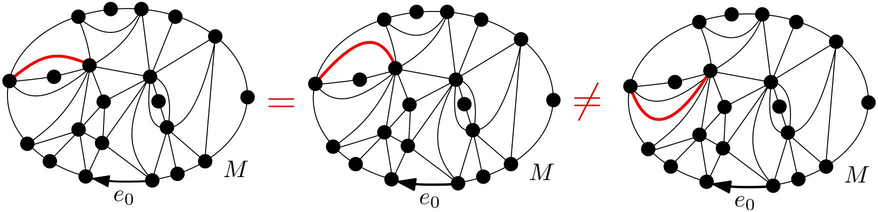
Pair of SLE$_6$



Brownian excursion

Random planar maps (RPM)

- **Planar map**: graph on the sphere, modulo continuous deformations.



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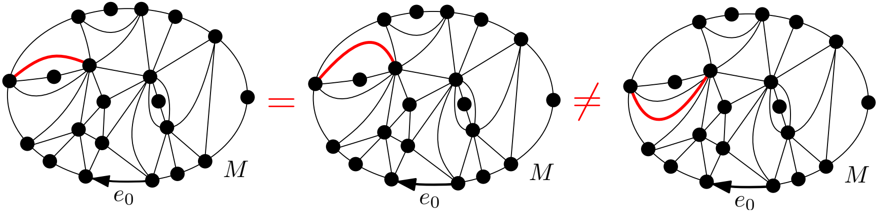
- **Planar map**: graph on the sphere, modulo continuous deformations.
- **Triangulation of a disk**: planar map where all the faces have three edges, except one distinguished face (the exterior face) with arbitrary degree and simple boundary.



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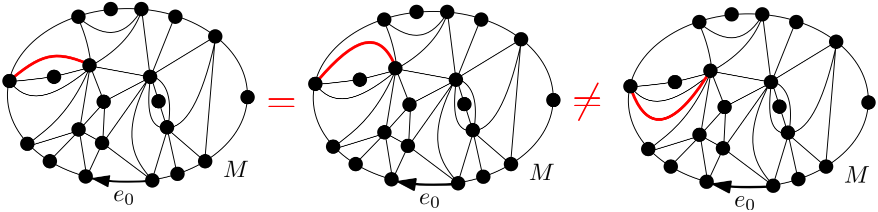
- Let M be a **uniform** triangulation with n vertices and boundary length m.



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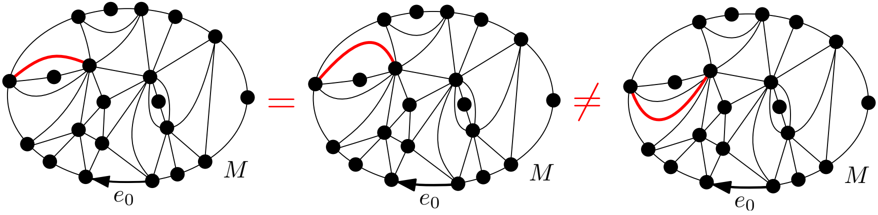
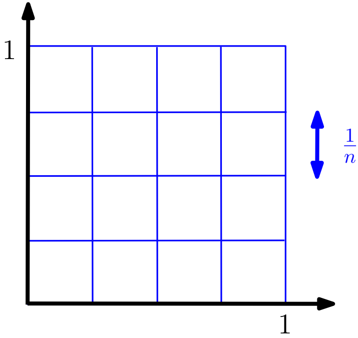
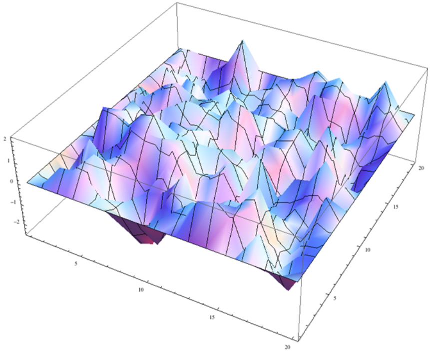
- Let M be a **uniform** triangulation with n vertices and boundary length m.
- What is the scaling limit of M?



The Gaussian free field (GFF)

- The **discrete Gaussian free field** (GFF) $h_n : \frac{1}{n}\mathbb{Z}^2 \cap [0,1]^2 \to \mathbb{R}$ is a random function.





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- $h_n(z)$ is a normal random variable such that

$$\mathbb{E}[h_n(z)] = 0, \qquad \mathrm{Var}(h_n(z)) \approx \log n, \qquad \mathrm{Cov}(h_n(z), h_n(w)) \approx \log|z-w|^{-1}.$$

North of LA

Yosemite

Himalaya

$\longrightarrow$ increasing n

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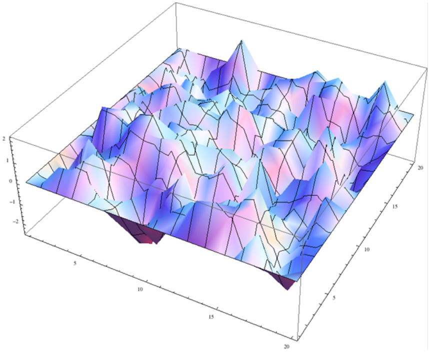
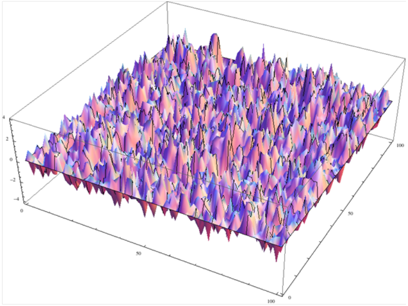


$n = 20$



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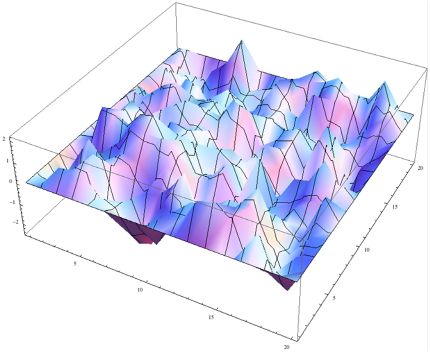
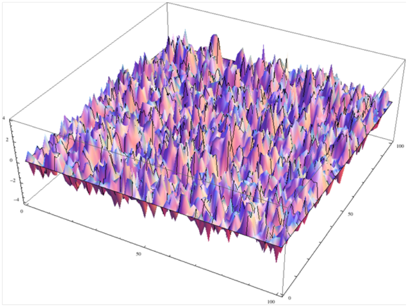
- The **Gaussian free field** h is the limit of h_n when $n \to \infty$.



$n = 20$



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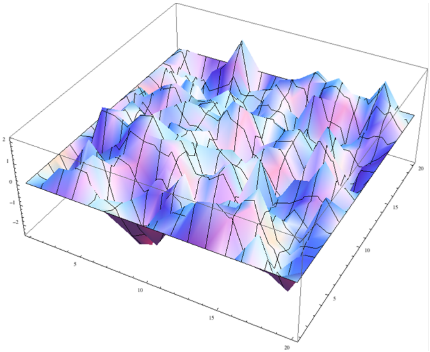
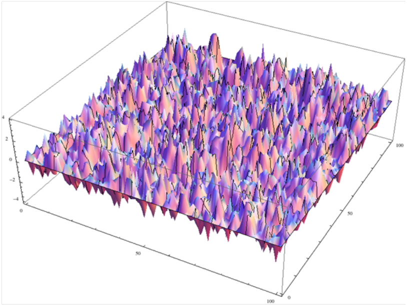
- The GFF is a **random distribution (generalized function)**.





Liouville quantum gravity

- If $h : [0,1]^2 \to \mathbb{R}$ smooth and $\gamma \in (0,2)$, then $e^{\gamma h}(dx^2 + dy^2)$ defines the metric tensor of a Riemannian manifold.

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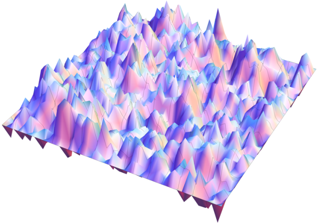
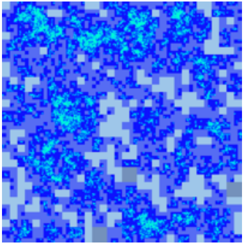


GFF



LQG

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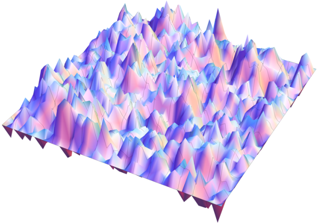
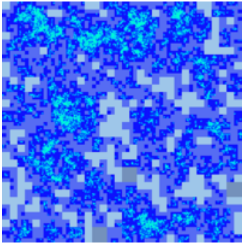
- Definition does not make rigorous sense since h is a distribution.



GFF



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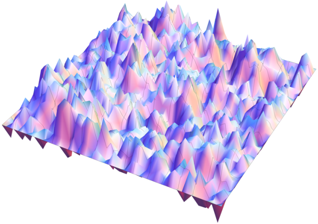
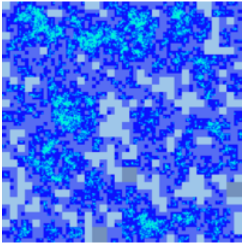
- γ-Liouville quantum gravity (LQG): h is the **Gaussian free field** (GFF).
- Definition does not make rigorous sense since h is a distribution.
- **Area measure** $\mu = e^{\gamma h} dxdy$ rigorously defined by regularizing h

$$\mu(U) = \lim_{\epsilon \to 0} \epsilon^{\gamma^2/2} \int_U e^{\gamma h_\epsilon} dxdy, \qquad h_\epsilon \text{ regularized verison of } h, \quad U \subset \mathbb{C}.$$

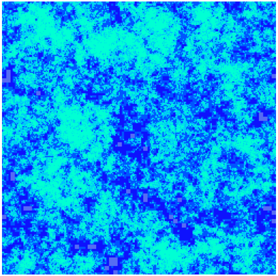


GFF

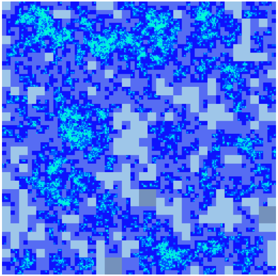


LQG

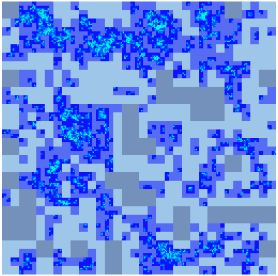
Illustration of LQG area measure



$\gamma = 1$



$\gamma = 1.5$



$\gamma = 1.75$

Area measure of random surface $e^{\gamma h} dxdy$, by J. Miller

Random planar maps converge to LQG

Two models for random surfaces:

- Random planar maps (RPM)
- Liouville quantum gravity (LQG)

What does it mean for a RPM to converge?

Random planar maps converge to LQG

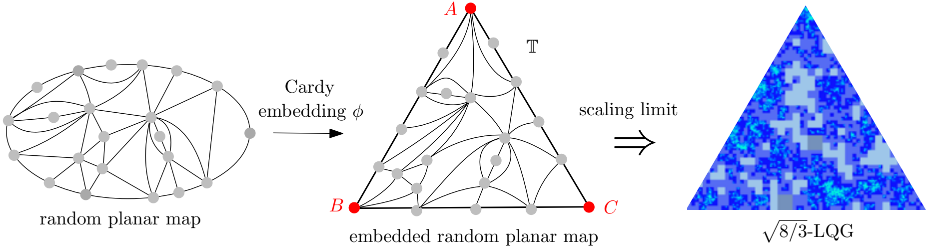
Two models for random surfaces:

- Random planar maps (RPM)
- Liouville quantum gravity (LQG)

What does it mean for a RPM to converge?

- Metric structure (Le Gall'13, Miermont'13, ...)
- Conformal structure (H.-Sun'19)
- Statistical physics decorations (Duplantier-Miller-Sheffield'14, ...)

Conformally embedded uniform triangulation $\Rightarrow \sqrt{8/3}$-LQG



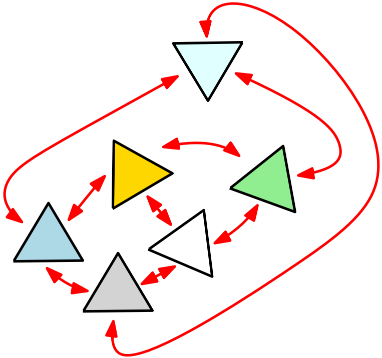
- Uniform triangulation with n vertices and boundary length $\Theta(n^{1/2})$.
- **Cardy embedding:** uses properties of percolation on the RPM.
- Let μ_n be renormalized counting measure on the vertices in $\mathbb{T}$.
- Let μ be $\sqrt{8/3}$-LQG area measure in $\mathbb{T}$.

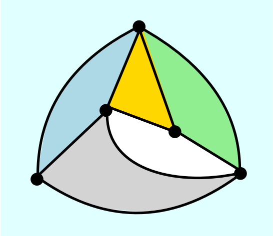
Theorem 6 (H.-Sun'19)

In the above setting, $\mu_n \Rightarrow \mu$.

Discrete conformal embeddings

- Circle packing
- Riemann uniformization
- Tutte embedding
- Cardy embedding

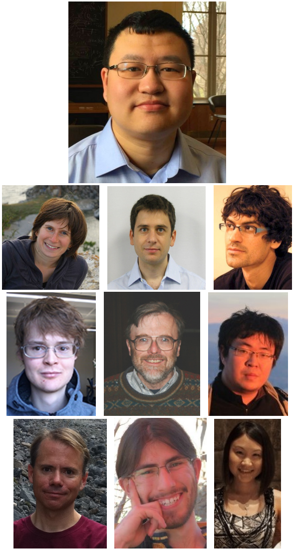




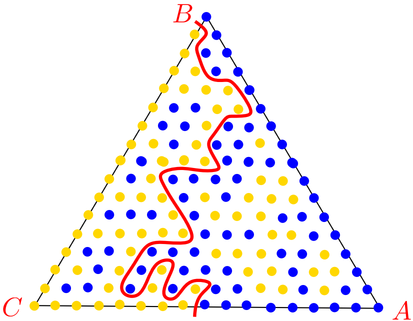
Conformally embedded RPM converge to $\sqrt{8/3}$-LQG

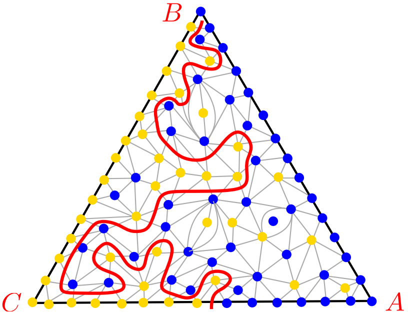
The proof is based on multiple works, including, but not limited to (*=in preparation):

- Percolation on triangulations: a bijective path to Liouville quantum gravity (Bernardi-H.-Sun)
- Minkowski content of Brownian cut points (Lawler-Li-H.-Sun)
- Natural parametrization of percolation interface and pivotal points (Li-H.-Sun)
- Uniform triangulations with simple boundary converge to the Brownian disk (Albenque-Sun-Wen)*
- Joint scaling limit of site percolation on random triangulations in the metric and peanosphere sense (Gwynne-H.-Sun)*
- Liouville dynamical percolation (Garban-H.-Sepúlveda-Sun)*
- Convergence of uniform triangulations under the Cardy embedding (H.-Sun)



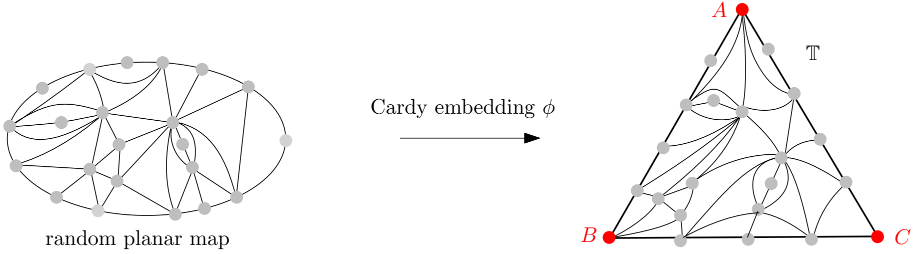
Percolation on uniform triangulations $\Rightarrow$ SLE$_6$





- Smirnov'01: The percolation interface for critical percolation on the triangular lattice converges to a Schramm-Loewner evolution (SLE$_6$).
- H.-Sun'19: The percolation interface for critical percolation on a Cardy embedded uniform triangulation converges to SLE$_6$ in a quenched sense.

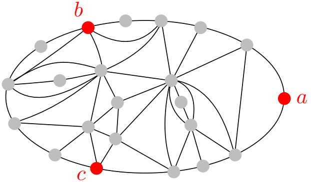
Cardy embedding: percolation-based embedding



random planar map

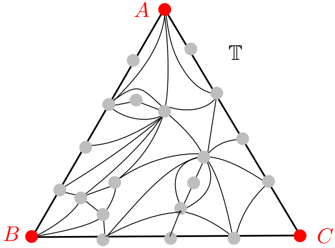
embedded random planar map

Cardy embedding: percolation-based embedding



random planar map

Cardy embedding ϕ



embedded random planar map

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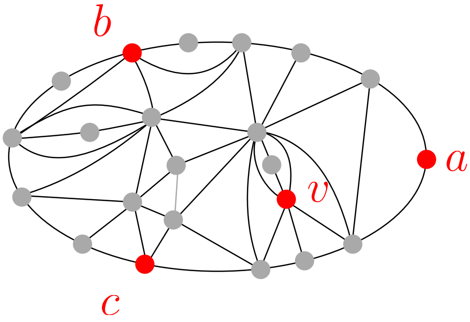

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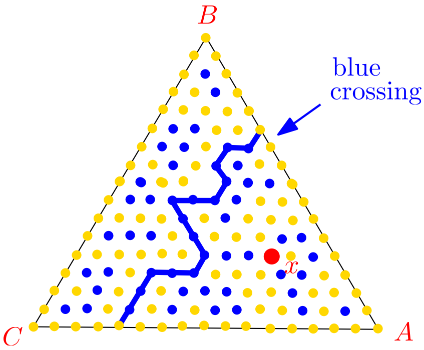

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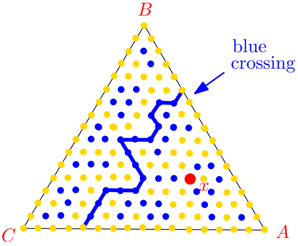
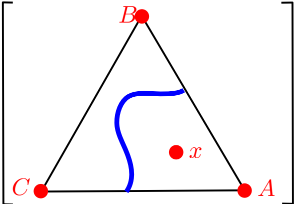


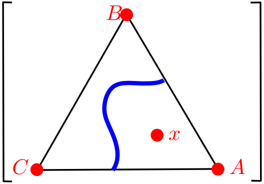
$p_A(x) = \mathbb{P}$

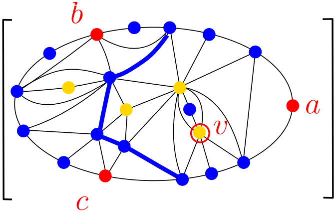


Cardy embedding: percolation-based embedding

- What is the "correct" position of v in $\mathbb{T}$?
- Map $v \in V(M)$ to $x \in \mathbb{T}$ such that

$$(p_A(x), p_B(x), p_C(x)) = (\widehat{p}_a(v), \widehat{p}_b(v), \widehat{p}_c(v)).$$

$p_A(x) = \mathbb{P}$ 

$\widehat{p}_a(v) = \mathbb{P}$ 

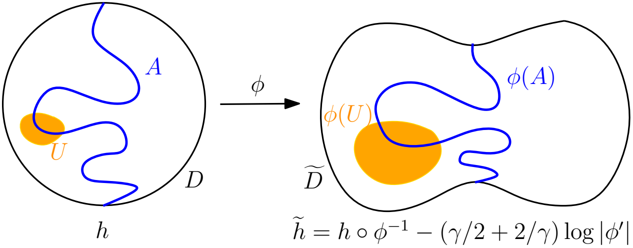
γ-LQG measures on fractals

- $A \subset \mathbb{C}$ fractal with d-Minkowski content $\mathfrak{m}$ for $d \in (0, 2]$.
- γ-LQG measure of A:

$$d\nu_h = \lim_{\epsilon \to 0} \epsilon^{\gamma_d^2/2} e^{\gamma_d h_\epsilon(z)}\, d\mathfrak{m}, \qquad h_\epsilon \text{ regularized verison of } h,$$

where γ_d is chosen such that for any $U \subset D$,

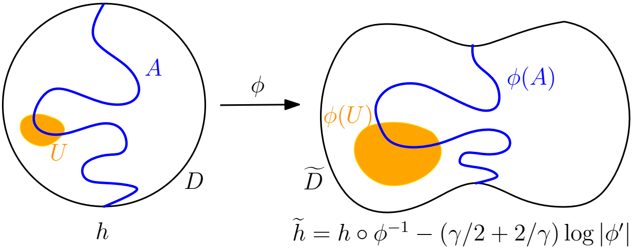
$$\nu_h(U) = \nu_{\widetilde{h}}(\phi(U)).$$



We get the γ-LQG size of

(i) SLE_κ ($\Rightarrow$ quantum natural parametrization) and

(ii) pivotal points.



γ-LQG measures on fractals

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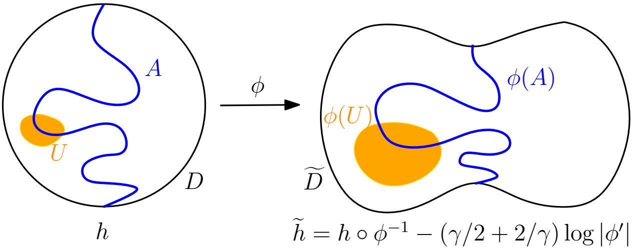
Quantum natural parametrization: unique parametrization of SLE which is

(i) **invariant** under conformal maps and

(ii) **locally** determined by η and h.

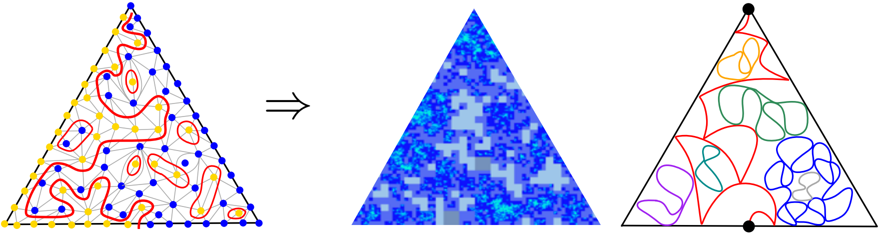


Percolation on RPM $\Rightarrow$ SLE$_6$ on $\sqrt{8/3}$-LQG

- μ_n = counting measure on vertices; $\mu = \sqrt{8/3}$-LQG area measure
- η_n = percolation interface; η = SLE$_6$ w/quantum natural param.
- ν_n = counting measure pivotals; $\nu = \sqrt{8/3}$-LQG pivotal measure

Theorem 7 (H.-Sun'19, Bernardi-H.-Sun'18)

$(\mu_n, \eta_n, \nu_n) \Rightarrow (\mu, \eta, \nu)$



Cardy embedded (M, P) $\quad\Rightarrow\quad$ $\sqrt{8/3}$-LQG $\quad$ SLE$_6$ and CLE$_6$

Quantum natural parametrization

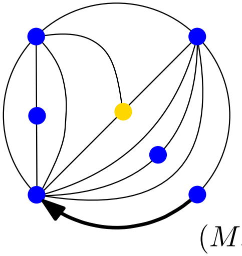
$e^{\gamma h} d\mathfrak{m}$

Limit of counting measure on interface

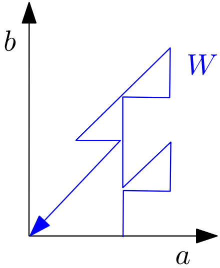
Axiomatic characterization of quantum natural parametrization

Bijection between percolated maps and walks

Bernardi-H.-Sun'18:



$\Leftrightarrow$



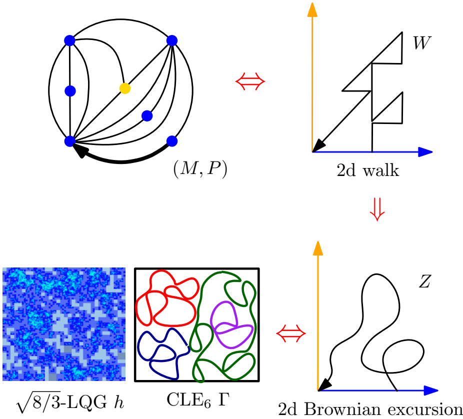
$babcbbabccacc$

Bijection between percolated maps and walks

$$(M,P) \overset{\text{bijection}}{\Leftrightarrow} \text{walk } (W_k)_{k\in[3n]}$$

$$\Downarrow \text{ convergence}$$

$$(h,\Gamma) \overset{\text{“bijection”}}{\Leftrightarrow} \text{Brownian excursion } (Z_t)_{t\in[0,1]}$$



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$\Downarrow$ convergence of walk and observables encoded by walk

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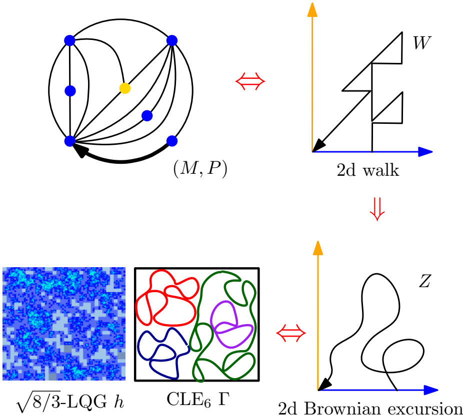

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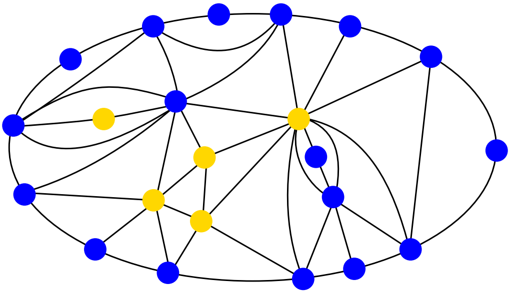

Thanks!