

Institute for Pure and Applied Mathematics  
University of California, Los Angeles presents

# Multiscale Geometry and Analysis in High Dimensions

September 7 - December 17, 2004

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(University of Geneva)

"Random walks on trees  
and Ramanujan graphs"

IPAM, February 12, 2004

Workshop 1: September 20-24: Multiscale Geometry in Image Processing and Coding  
Workshop 2: October 18-22: Multiscale Geometry in Scientific Computing  
(including a satellite workshop on Search and Matching in High Dimensions)  
Workshop 3: October 25-29: Multiscale Structures in the Analysis of High-Dimensional Data  
Workshop 4: November 8-12: Multiscale Geometric Methods in Astronomical Data Analysis  
Workshop 5: November 15-19: Mathematical Analysis and Multiscale Geometric Analysis

Participation:  
This program will involve a community of senior and junior researchers. The intent is for long-term participants to have an opportunity to learn about multiscale geometry and analysis from the perspectives of many different fields--mathematical, science and engineering--and to meet a diverse group of people and have an opportunity to form new collaborations.  
Full and partial support for long-term participants is available, and those interested are encouraged to fill out an online application at <http://www.ipam.ucla.edu/programs/mgs2004>. Support for individual workshops will also be available, and may be applied for through the online application for each workshop. We are especially interested in applicants who are interested in becoming core participants and participating in the entire program (September 7 - December 17, 2004), but give consideration to applications for shorter periods. Funding for participants is available at all academic levels, through recent Ph.D. graduates, students, and researchers in the early stages of their careers and especially encouraged to apply. Encouraging the careers of women and minority mathematicians and scientists is an important component of IPAM's mission and we welcome their applications.

Please visit our website at  
<http://www.ipam.ucla.edu/programs/mgs2004>  
or email questions to [mgs2004@ipam.ucla.edu](mailto:mgs2004@ipam.ucla.edu)

$\Gamma$  a connected  $k$ -regular graph  
on  $n$  vertices

$$\text{Sp Adj}(\Gamma) = \{ k = \lambda_1 > \lambda_2 \geq \dots \geq \lambda_n \}$$

$\Gamma$  is Ramanujan if  $\lambda_2(\Gamma) \leq 2\sqrt{k-1}$

$\tilde{\Gamma} = T_k$  the infinite  $k$ -regular tree

$\rho(\tilde{\Gamma})$  spectral radius

$k \geq 3$ . Existence problem

$\exists?$  an infinite family of  $k$ -regular Ramanujan graphs

< Does  $T_k$  have infinitely many Ramanujan quotients?

LPS, Morgenstern : Yes if  $k = p^a + 1$   
explicit construction

$\Gamma$  a connected graph on  $n$  vertices

$$\text{Sp Adj}(\Gamma) = \{ \lambda_1 > \lambda_2 \geq \dots \geq \lambda_n \}$$

$\Gamma$  is Ramanujan if  $\lambda_2(\Gamma) \leq \rho(\tilde{\Gamma})$

•  $\tilde{\Gamma}_1 = \tilde{\Gamma}_2 \Rightarrow \lambda_1(\Gamma_1) = \lambda_1(\Gamma_2)$

• Greenberg's generalization of Alon-Boppana's  
Thm

$X$  an infinite, locally finite, connected graph

$$\epsilon > 0$$

$$\exists c = c(X, \epsilon) > 0 \text{ s.t.}$$

$\forall$  finite graph  $Y$  covered by  $X$

$$|\{ \lambda \in \text{Sp Adj}(Y) : |\lambda| \geq \rho(X) - \epsilon \}| \geq c \cdot |Y|$$

Q. Given  $\Gamma$ , how to check whether it  
is Ramanujan?

An infinite tree  $T$  is uniform iff it covers a finite graph (equivalently,  $\infty$  many finite graphs)

Q. Let  $T$  be a uniform tree. Does  $T$  cover infinitely many Ramanujan graphs?

Lubotzky, N.  
198

There are infinitely many uniform trees which cover NO Ramanujan graph

- Give a sufficient condition for a finite graph to be covered by a tree with no Ramanujan quotient
- Construct  $\infty$  many such graphs with distinct universal covers

$$\Gamma' \rightarrow \Gamma \Rightarrow$$

$$\text{Sp Adj}(\Gamma') \supset \text{Sp Adj}(\Gamma)$$

"old eigenvalues"

||

old eigenvalues  $\perp$  new eigenvalues

A finite graph  $\Gamma$  is minimal if

$$\Gamma = \tilde{\Gamma} / \text{Aut}(\tilde{\Gamma})$$

$\Gamma$  minimal  $\Rightarrow$  any graph covered by  $\tilde{\Gamma}$  covers  $\Gamma$ .

Then Let  $\Gamma$  be a minimal finite graph. Denote by  $k$  the maximal vertex degree in  $\Gamma$ .

1) Suppose  $\exists x_0 \in V(\Gamma)$  st  $\Gamma \setminus \{x_0\} = \Gamma_1 \perp \Gamma_2$

2) Suppose the mean degrees of  $\Gamma_1$  and  $\Gamma_2$  are  $> 2\sqrt{k-1}$

1) + 2)  $\Rightarrow \tilde{\Gamma}$  covers no Ramanujan graphs

# Bass-Tits Algorithm

$\Gamma$  a finite graph. Define a sequence of equivalence relations on  $\text{Vert}(\Gamma)$ :

$R_0$ : all vertices are  $R_0$ -equivalent

$$x \sim_{R_1} y \iff \deg x = \deg y$$

$$x \sim_{R_2} y \iff x \sim_{R_1} y \text{ and } \# \text{ neighbours of } x \text{ of degree } n = \# \text{ neighbours of } y \text{ of degree } n \forall n$$

$$x \sim_{R_n} y \iff x \sim_{R_{n-1}} y \text{ and } \forall R_{n-1}\text{-class } C \# \text{ neighbours of } x \text{ in } C = \# \text{ neighbours of } y \text{ in } C$$

$R_n = R$  for all  $n$  large enough

Then (Bass-Tits) let  $\Gamma^*$  be the quotient graph  $\Gamma/R$ . If  $\Gamma = \Gamma^*$  then  $\Gamma$  is minimal

The prob. of a graph in  $C_n(G)$  being  $d$ -weably ran. goes to 1 as  $n \rightarrow \infty$  for some function  $\alpha(n)$  with  $\alpha(n) \rightarrow 0$  as  $n \rightarrow \infty$ .  
 with  $d = \sqrt{\log p} + \alpha(n)$

$C_n(G)$  = probab. space of degree  $n$  covers of  $G$

(random graph has vertex  $v \in \{1, \dots, n\}$ ;  $\forall e \in E$  choose an arbitrary orientation  $e = (u, v)$ , pick uniformly a random permutation  $\sigma$  of  $\{1, \dots, n\}$   
 $\Rightarrow (u, i) \sim (v, \sigma(i)) \quad \forall i$

What if there is no minimal  $G$ ?  
 Generalize to pregraphs.

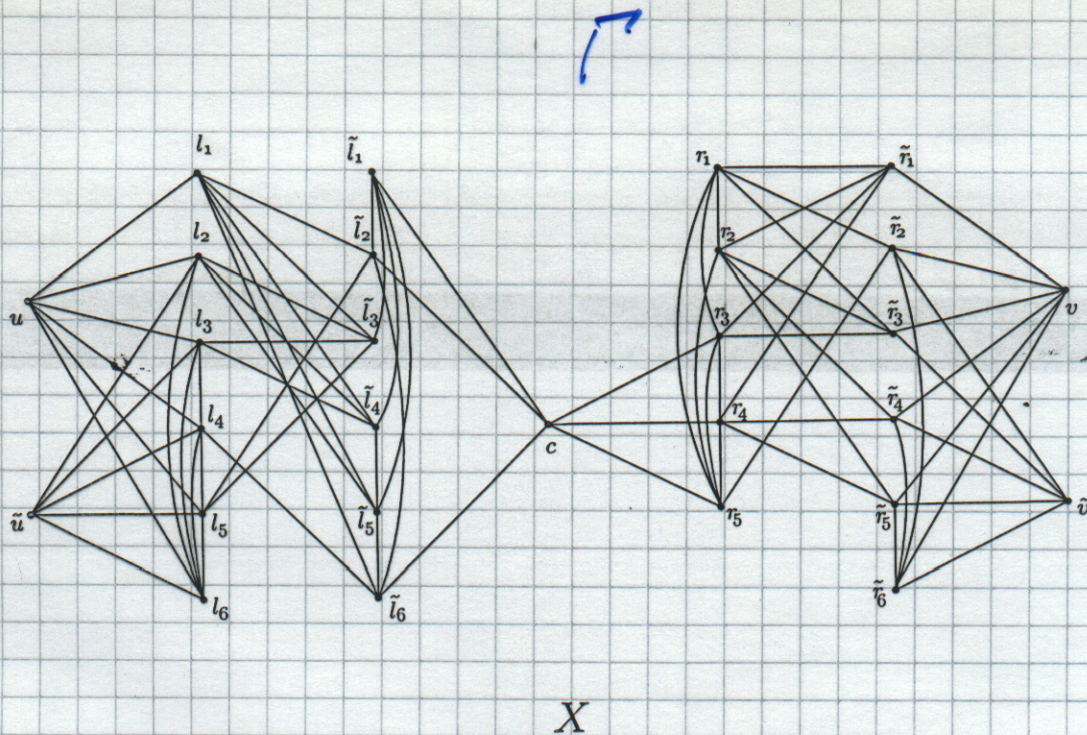


Figure 2

$$\deg x = 5 \text{ or } 6$$

$$x \in V(\Gamma)$$

$\tilde{\Gamma}$  has no Ramanujan quotient  
 $\Gamma$  is minimal  $\Rightarrow \text{Aut}(\tilde{\Gamma})$  is discrete



Friedman  
'00

Let  $T$  be a uniform tree. ⑥  
"Most" finite quotients of  $T$   
are relatively weakly Ramanujan

Let  $\Gamma$  be a finite graph.

For a graph  $\Gamma'$ , an  $m$ -degree cover of  $\Gamma$ ,

$\Pr \{ \text{new eigenvalues } \in [-\varepsilon, \varepsilon] \} \xrightarrow{m \rightarrow \infty} 1$

with  $\varepsilon = \sqrt{\lambda_1(\Gamma) \rho(\Gamma) + o(m)}$

with  $o(m) \xrightarrow{m \rightarrow \infty} 0$

Relative Alon-Koppana's Theorem:

Let  $\Gamma$  be a finite graph. For any  
 $m$ -degree cover  $\Gamma'$  of  $\Gamma$ , there

exists a new eigenvalue  $\lambda$

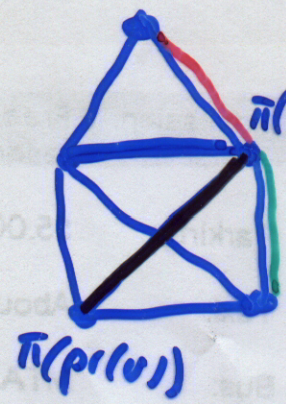
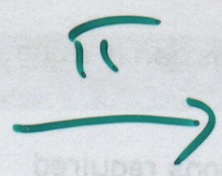
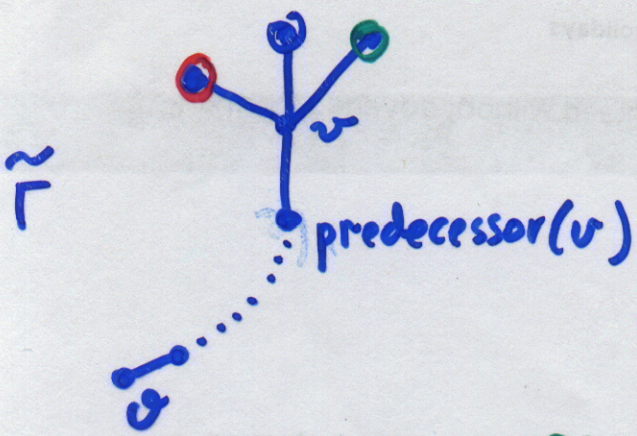
with  $|\lambda| \geq \rho(\Gamma) - \omega(m)$

and  $\omega(m) \xrightarrow{m \rightarrow \infty} 0$

Q. Let  $T$  be a uniform tree. Does it cover  
a tower of finite graphs s.t. infinitely many  
of them are relatively Ramanujan?

# Method of computation of $p(T)$

$T$  a uniform tree



finite conn. w/0 vert. of deg 1

Vertices in  $\tilde{\Gamma}$  = paths of finite length from a basepoint in  $\Gamma$

Future of a vertex  $v$  in  $\tilde{\Gamma}$  is determined by an oriented edge  $(\pi(pr(v)), \tilde{u}(v))$  in  $\Gamma$

$$K := \# \text{ of oriented edges in } \Gamma$$

$$= 2 \cdot \# \text{ of edges in } T = \sum_{v \in \text{Vert}(T)} \deg v$$

1, ..., K types of vertices in  $T (= \tilde{\Gamma})$   
"cone types"

Irreducibility condition is satisfied:

(\*)  $\forall v \in \text{Vert } T$ , Cone  $(v)$  contains vertices of types 1, ..., K.

T, cone types 1, ..., K.

$$\text{Adj}(T) = (a(v, w))_{v, w \in \text{Vert}(T)}$$

$a^{(n)}(v, w)$  = # of paths of length n from v to w

$$\rho(T) = \limsup_{n \rightarrow \infty} (a^{(n)}(v, w))^{1/n}$$

(indep. of v, w)

$$G(v, w | z) = \sum_{n \in \mathbb{N}} a^{(n)}(v, w) z^n$$

Green function of T

$\rho^{-1}(T)$  = radius of convergence of  $G(o, o | z)$

↙  
almost

①

First of all <sup>many</sup> thanks ~~for~~ to the organizers for the invitation and even more for such an interesting conference.

My talk will touch upon still one more aspect of the Ramanujan business, namely, <sup>JH talk</sup> about the ~~notion of being Ramanujan~~ Ramanujan property for arbitrary, not necessarily regular graphs.

To start with, for those who are good friends with all previous speakers and consequently slept through all previous talks, I remind the definition of R-g.

$\Gamma$  a ~~graph~~ <sup>connected</sup>  $k$ -regular graph on  $n$  vert.  
 $A(\Gamma) = \text{adj. op.}$   $\text{Sp } A(\Gamma) = \{ \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \}$

$\Gamma$  is Ramanujan if  $\lambda_2(\Gamma) \leq 2\sqrt{k-1}$ .

Two important points:  $\lambda_1 = k$  for every  $k$ -reg. graph  
PF ev of mult 1

$\lambda_2(\Gamma)$  compared with  $\rho(F)$   
 $\tilde{\Gamma} = \Gamma_k =$  infinite <sup>the</sup> tree  
for any  $k$ -reg. ~~graph~~  $\Gamma$ .

Main problem about Ramanujan graphs is existence problem

$k \geq 2$ .  $\exists?$  an infinite family of  $k$ -reg. Ramanujan graphs?

LPS Mors. Yes for  $k = p^2 + 1 \rightarrow$  explicit construction.

$k=2$   $\exists?$  an infinite family of  $k$ -reg. Ramanujan graphs?

$j = 1, \dots, K$

$F_j(z) = \sum_{n \in \mathbb{N}} \varphi_j^{(n)} z^n$  where

$\varphi_j^{(n)}$  = # of paths (of length  $n$ ) starting at a vertex of type  $j$  and arriving for the first time to predecessor ( $v$ ) after  $n$  steps

Lemma. Under irreducibility condition  $(*)$ ,  $\{F_j(z)\}_1^K, G(0,0|z)$  have the same radius of conver.

Lemma.  $\{F_j(z)\}_1^K$  satisfy the following system of polynomial equations

$i = 1, \dots, K$   
 $w_i = z + z \sum_{j=1}^K s_{ij} w_i w_j =: P_i(z, w_1, \dots, w_K)$

$s_{ij}$  = # of successors of type  $j$  of a vertex of type  $i$

$$J(z) = \left( \frac{\partial P_i(z, w_1, \dots, w_k)}{\partial w_j} \right)_{i,j=1}^K$$

Jacobian of the system above

$\lambda_{PF}(z)$  continuous increasing function,  $\lambda_{PF}(0) = 0$

$\bar{\rho}^{-1}(T) = \min \{ z > 0 : \lambda_{PF}(z) = 1 \}$   
 radius of conv. of  $F_j(z) \forall j$

$$\begin{cases} w_i = z + z \sum_{j=1}^K s_{ij} w_i w_j & i=1, \dots, K \\ \det ( J(z) - \lambda(z) \cdot Id ) = 0 \end{cases}$$

elimination of variables  $\rightarrow$

$\exists!$  polynomial with integer coeff

$P(z, w)$  s.t.

$P(z, \lambda(z)) = 0$ , and

$\bar{\rho}^{-1}(T)$  is the smallest positive solution of the polynomial eq. in 1 variable.  
 $P(z, 1) = 0$