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"Random walks on trees and Ramanujan graphs"

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Γ a connected k-regular graph on n vertices

$$\mathrm{Sp}\,\mathrm{Adj}(\Gamma) = \{ k = \lambda_1 > \lambda_2 \geq \ldots \geq \lambda_n \}$$

Γ is Ramanujan if $\lambda_2(\Gamma) \leq 2\sqrt{k-1}$

$$\qquad\qquad\qquad\qquad\qquad \| $$

$$\qquad\qquad\qquad\qquad \rho(\tilde{\Gamma}) \text{ spectral radius}$$

$\tilde{\Gamma} = T_k$ the infinite k-regular tree

$k \geq 3$. Existence problem

$\exists$? an infinite family of k-regular Ramanujan graphs

⟨ Does T_k have infinitely many Ramanujan quotients? ⟩

LPS, Morgenstern: Yes if $k = p^\alpha + 1$

explicit construction

Γ a connected graph on n vertices

$\mathrm{Sp}\,\mathrm{Adj}(\Gamma) = \{\lambda_1 > \lambda_2 \geq \dots \geq \lambda_n\}$

Γ is Ramanujan if $\lambda_2(\Gamma) \leq \rho(\tilde{\Gamma})$

- $\tilde{\Gamma}_1 = \tilde{\Gamma}_2 \Rightarrow \lambda_1(\Gamma_1) = \lambda_1(\Gamma_2)$

- Greenberg's generalization of Alon-Boppana's Thm

X an infinite, locally finite, connected graph
$\varepsilon > 0$
$\exists\, C = C(X, \varepsilon) > 0$ s.t.
$\forall$ finite graph Y covered by X

$$|\{\lambda \in \mathrm{Sp}\,\mathrm{Adj}(Y) : |\lambda| \geq \rho(X) - \varepsilon\}| \geq C \cdot |Y|$$

Q. Given Γ, how to check whether it is Ramanujan?

An infinite tree T is uniform iff it covers a finite graph (equivalently, ∞ many finite graphs)

Q. Let T be a uniform tree. Does T cover infinitely many Ramanujan graphs?

Lubotzky, N. '98

There are infinitely many uniform trees which cover NO Ramanujan graph

- Give a sufficient condition for a finite graph to be covered by a tree with no Ramanujan quotient

- Construct ∞ many such graphs with distinct universal covers

$\Gamma' \to \Gamma \quad \Rightarrow$

$$\mathrm{Sp\,Adj}(\Gamma') \supset \mathrm{Sp\,Adj}(\Gamma)$$

"old eigenvalues"

$\parallel$

old eigenvalues $\amalg$ new eigenvalues

A finite graph Γ is minimal if

$$\Gamma = \tilde{\Gamma} / \mathrm{Aut}(\tilde{\Gamma})$$

Γ minimal $\Rightarrow$ any graph covered by $\tilde{\Gamma}$ covers Γ.

<u>Thm</u> Let Γ be a minimal finite graph. Denote by k the maximal vertex degree in Γ.

1) Suppose $\exists\ x_0 \in V(\Gamma)$ st $\Gamma \setminus \{x_0\} = \Gamma_1 \amalg \Gamma_2$

2) Suppose the mean degrees of Γ_1 and Γ_2 are $> 2\sqrt{k-1}$

1) + 2) $\Rightarrow$ $\tilde{\Gamma}$ covers no Ramanujan graph

Bass-Tits Algorithm

Γ a finite graph. Define a sequence of equivalence relations on Vert(Γ):

R_0 : all vertices are R_0-equivalent

$x \underset{R_1}{\sim} y \iff \deg x = \deg y$

$x \underset{R_2}{\sim} y \iff x \underset{R_1}{\sim} y$ and # neighbours of x of degree n = # neighbours of y of deg n $\forall n$

$x \underset{R_n}{\sim} y \iff x \underset{R_{n-1}}{\sim} y$ and $\forall$ R_{n-1}-class C # neighbours of x in C = # neighbours of y in C

$R_n = R$ for all n large enough

Thm (Bass-Tits) Let Γ^* be the quotient graph Γ/R. If $\Gamma = \Gamma^*$ then Γ is minimal

The prob. of a graph in $C_n(G)$ being ~~to~~ ν-weakly Ram. with $\nu = \sqrt{\lambda_0 \rho} + \alpha(n)$ goes to 1 as $n \to \infty$ for some function $\alpha(n)$ with $\alpha(n) \to 0$ as $n \to \infty$

$C_n(G)$ = probab. space of degree n covers of G

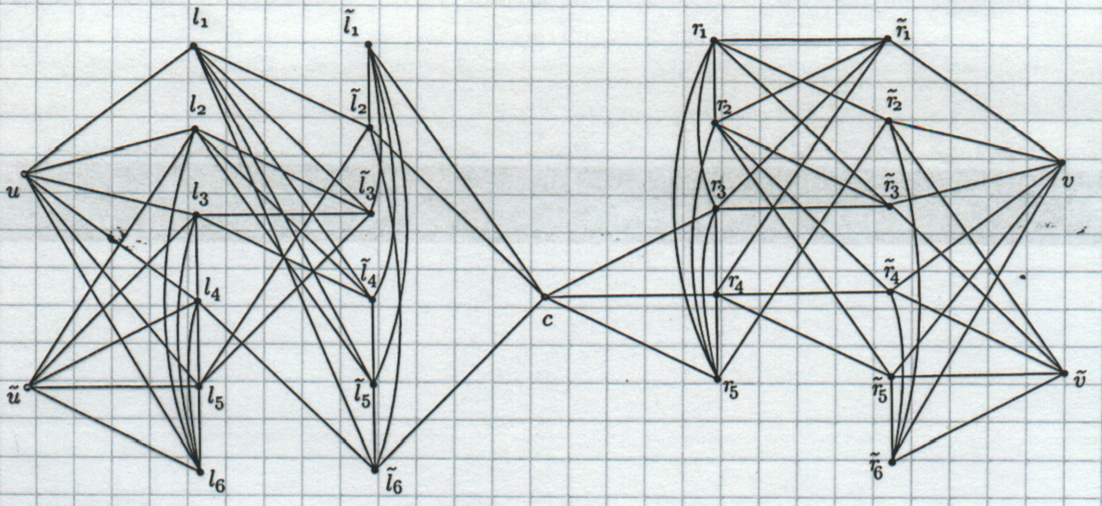
(random graph has vertex $V \times \{1, \dots, n\}$; $\forall e \in G$ choose an arbitrary orientation $e = (u, v)$, pick uniformly a random permutation σ_e of $\{1, \dots, n\}$

$\Rightarrow (u, i) \sim (v, \sigma_e(i)) \ \forall i$

What if there is no minimal G?

Generalize to pregraphs

Γ



X

Figure 2

$\deg x = 5$ or 6

$x \in V(\Gamma)$

$\tilde{\Gamma}$ has no Ramanujan quotient

Γ is minimal $\Rightarrow$ $\mathrm{Aut}(\tilde{\Gamma})$ is discrete

Friedman '00

Let T be a uniform tree.
"Most" finite quotients of T are relatively weakly Ramanujan

Let Γ be a finite graph.
For a graph Γ', an m-degree cover of Γ,

$$\Pr\{\text{new eigenvalues} \in [-x, x]\} \xrightarrow[m\to\infty]{} 1$$

with $x = \sqrt{\lambda_1(\Gamma)}\,\rho(F) + \alpha(m)$

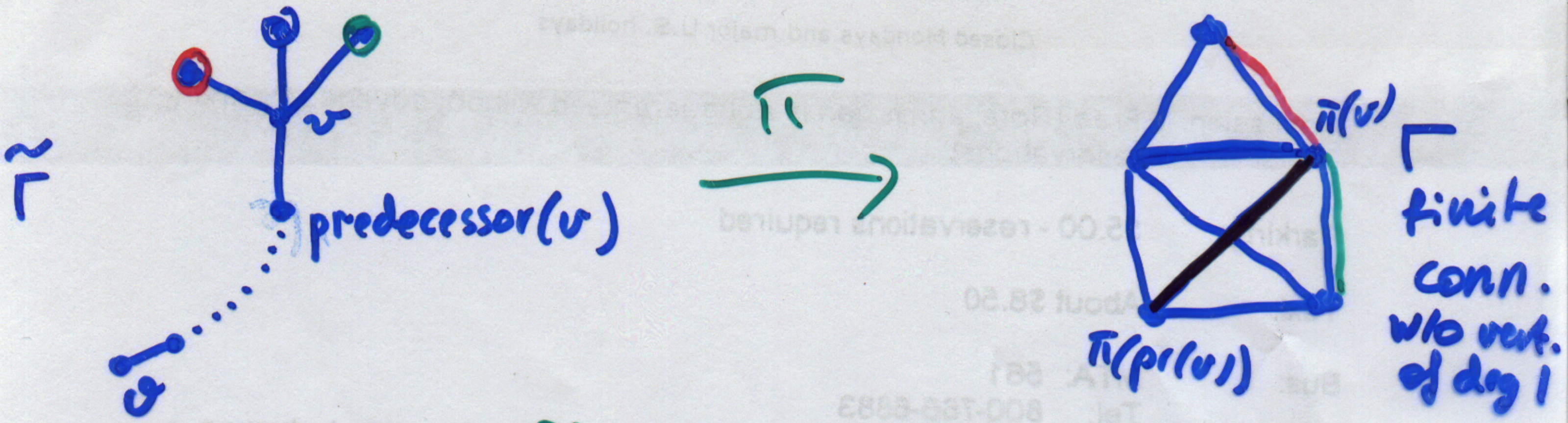
with $\alpha(m) \xrightarrow[m\to\infty]{} 0$

Relative Alon-Boppana's Thm:
Let Γ be a finite graph. For any m-degree cover Γ' of Γ, there exists a new eigenvalue λ with $|\lambda| \geq \rho(F) - \omega(m)$

and $\omega(m) \xrightarrow[m\to\infty]{} 0$

Q. Let T be a uniform tree. Does it cover a tower of finite graphs s.t. infinitely many of them are relatively Ramanujan?

Method of computation of $\rho(T)$
T a uniform tree



Vertices in $\tilde{\Gamma}$ = paths of finite length from a basepoint in Γ

Future of a vertex v in $\tilde{\Gamma}$ is determined by an oriented edge $(\pi(pr(v)), \pi(v))$ in Γ

$K := \#$ of oriented edges in Γ

$= 2 \cdot \#$ of edges in $\Gamma = \sum_{v \in Vert(\Gamma)} \deg v$

$1, \dots, K$ types of vertices in T $(= \tilde{\Gamma})$

"cone types"

Irreducibility condition is satisfied:

(*) $\forall\, v \in Vert\, T$, Cone(v) contains vertices of types $1, \dots, K$.

T, cone types $1, \dots, K$.

$$\mathrm{Adj}(T) = \big(a(v,w)\big)_{v,w \in \mathrm{Vert}(T)}$$

$a^{(n)}(v,w) = \#$ of paths of length n from v to w

$$\boxed{\rho(T) = \limsup_{n \to \infty} \big(a^{(n)}(v,w)\big)^{1/n}}$$

(indep. of v, w)

$$G(v,w \mid z) = \sum_{n \in \mathbb{N}} a^{(n)}(v,w)\, z^n$$

Green function of T

$\rho^{-1}(T) =$ radius of convergence of $G(o,o \mid z)$

almost

(1)

~~Re~~ First of all many thanks ~~for~~ to the organizers for the invitation and even more for such an interesting conference. ~~It~~

My talk will touch upon still one more aspect of the Ramanujan business, namely, I'll talk about the ~~notion of being Ramanu~~ Ramanujan property for arbitrary, not necessarily regular graphs.

To start with, for those who are good friends with all previous speakers and consequently slept through all previous talks, I remind the definition of R.-G.

Γ a ~~finite~~ connected k-regular graph on n vert.

$A(\Gamma)$ = adj. op. $\quad Sp\, A(\Gamma) = \{ k = \lambda_1 > \lambda_2 \geq \dots \geq \lambda_n \}$.

Γ is Ramanujan if $\lambda_2(\Gamma) \leq 2\sqrt{k-1}$.

Two important points: $\lambda_1 = k$ for every k-reg graph

PF ev of mult 1

$\lambda_2(\Gamma)$ ~~is~~ compared with $\rho(\tilde{\Gamma})$

$\tilde{\Gamma} = T_k$ = the infinite k-reg tree

for any k-reg. ~~Set~~ graph Γ.

Main problem about Ram. graph is existence problem

$k \geq 3$. $\exists$? an infinite family of k-reg. Ram graphs?

LPS Morg. Yes for $k = p^2+1$. $\to$ explicit construction

the ($k=3$) ... [illegible] ... ∞-many Ram graphs?

$j = 1, \dots, K$

$$F_j(z) = \sum_{n \in \mathbb{N}} \varphi_j^{(n)} z^n \quad \text{where}$$

$\varphi_j^{(n)}$ = # of paths (of length n) starting at a vertex v of type j and arriving for the first time to predecessor(v) after n steps

Lemma. Under irreducibility condition (*), $\{F_j(z)\}_1^K$, $G(0,0|z)$ have the same radius of converg.

Lemma. $\{F_j(z)\}_1^K$ satisfy the following system of polynomial equations

$i = 1, \dots, K$

$$w_i = \underbrace{z + z \sum_{j=1}^{K} s_{ij} w_i w_j}_{} =: P_i(z, w_1, \dots, w_K)$$

s_{ij} = # of successors of type j of a vertex of type i

$$J(z) = \left(\frac{\partial P_i(z, w_1,\dots,w_K)}{\partial w_j}\right)_{i,j=1}^{K}$$

Jacobian of the system above

$\lambda_{PF}(z)$ continuous increasing function, $\lambda_{PF}(0)=0$

$$\rho^{-1}(T) = \min\{z>0 : \lambda_{PF}(z) = 1\}$$

radius of conv. of $F_j(z)$ $\forall j$

$$\begin{cases} w_i = z + z\sum_{j=1}^{K} s_{ij}\, w_i w_j & i=1,\dots,K \\ \det\left(J(z) - \lambda(z)\cdot Id\right) = 0 \end{cases}$$

elimination of variables

$\exists!$ polynomial with integer coeff $P(z,w)$ s.t. $P(z,\lambda(z))=0$, and

$\rho^{-1}(T)$ is the smallest positive solution of the polynomial eq. in 1 variable $P(z,1)=0$