Y finite regular graph of valency $q+1$, $q \in \mathbb{N}$

X universal cover, $\Gamma = \pi_1(Y)$, $G = Aut(X)$

∂X = geometric boundary
= {rays}/parallelity

∞ fixed pt in ∂X

P = stabilizer of ∞ in G

x vertex $\Rightarrow$ $\exists!$ ray

$$\left(\underset{\overset{\shortparallel}{x}}{v_0(x)}, v_1(x), \ldots \right) \in \infty$$

x, y are in the same <u>horosphere</u> if $\exists_N : v_N(x) = v_N(y)$

$S \subset P$ subgroup that preserves horospheres

$\Rightarrow$ P/S cyclic of infinite order

let a_0 be the generator that "moves away from ∞"

$$\mathrm{Hom}(P/S, \mathbb{C}^\times) \cong \mathbb{C}^\times$$
$$\alpha \longmapsto \alpha(a_0)$$

For $\lambda \in \mathbb{C}^\times$ and a $\mathbb{C}[P/S]$- module V let

$$V^\lambda = \left\{ v \in V : (a_0 - \lambda)^n v = 0 \text{ for some } n \in \mathbb{N} \right\}$$

= generalized λ-eigenspace

Theorem Let $\lambda \in \mathbb{C}^\times$ and let

$$Z_Y(u) = \prod_{c \text{ prim}} \left(1 - u^{\ell(c)}\right)^{-1}$$

be the Ihara zeta function of Y. Then

$$\operatorname{ord}_{u=\lambda} Z_Y(u) = -\dim H_0(S, L^2(\Gamma\backslash G))^\lambda$$

Let π be a representation of G

$\pi^\infty = \{ v \in V_\pi : \ \mathrm{stab}(v) \text{ is open} \}$

π is <u>smooth</u> if $\pi = \pi^\infty$

π <u>admissible</u> if $\dim \pi^K < \infty \quad \forall K \subset G$ compact, open.

$\mathrm{Adm}(G)$ = category of admissible rep. of G

$\mathrm{Adm}^\infty(G)$ = " smooth admissible rep. of G

$$F : \mathrm{Adm}(G) \longrightarrow \mathrm{Adm}^\infty(G)$$
$$\pi \longmapsto \pi^\infty$$

For $V \in \mathrm{Adm}^\infty$ let $\tilde{V} = (V^*)^\infty$ "admissible dual"

$$V^{-\infty} = \mathrm{Hom}_G(\tilde{V}, C(G))$$

= "maximal completion"

$$R : \mathrm{Adm}^\infty \longrightarrow \mathrm{Adm}$$
$$V \longmapsto V^{-\infty}$$

Then
- $FR = Id$
- R is right adjoint to F, i.e.

$$\mathrm{Hom}_G(FW, V) \cong \mathrm{Hom}_{G,\mathrm{cont}}(W, RV)$$

<u>Theorem</u> (Patterson Conjecture)

If $\lambda \neq \pm\sqrt{q}$, then

$$\operatorname{ord}_{u=\lambda} Z_{Y}(u) = -\dim H^0\left(\Gamma, I^{-\infty}_{1/\lambda q}\right)$$

If $\lambda = \pm\sqrt{q}$, then the same holds with I replaced with a certain self-extension $\hat{I}$ of I.

Here $I_\lambda = \operatorname{Ind}_P^G(\lambda)$.

This follows from the previous theorem a

<u>Theorem</u> (Gelfand Duality)

$$H^0(\Gamma, V^{-\infty}) \cong \operatorname{Hom}_G\left(C^\infty(\Gamma\backslash G), V\right)$$

$0 \neq \infty$ new boundary point.

$\Rightarrow \exists!$ line $\ell = (\ldots, v_{-1}, v_0, v_1, \ldots)$ from 0 to ∞

$L \subset P$ subgroup preserving ℓ.

$M = L \cap S$ = pointwise stabilizer of ℓ. (compact)

exact sequence

$$1 \to M \to L \to \mathbb{Z} \to 1$$

with a section $s : \mathbb{Z} \to L$.

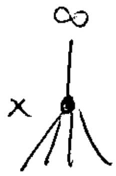
fix section s. let $A = \mathrm{Im}(s)$

Then $L = AM = MA$, $P = MAN$

with N = "unipotent radical" constructed as follows:

R ring with q elements

x vertex



q edges not pointing to ∞

label these edges with R such that

- for $x \in \ell$ the edge pointing to 0 is labelled 0.
- labelling is A-invariant

let $b \in \partial X \setminus \{\infty\} \implies \exists!$ line $(\dots, w_{-1}, w_0, w_1, \dots)$ from ∞ to b

let $\psi(b) \in \mathbb{R}((t))$ defined by

$$\psi(b) = \sum_{j=-\infty}^{\infty} lab(\overline{w_j, w_{j+1}})\, t^j$$

then $\psi : \partial X \setminus \{\infty\} \xrightarrow{\simeq} \mathbb{R}((t))$

$S \hookrightarrow \mathrm{Per}(\partial X \setminus \{\infty\})$

let $N = \{ n \in S : \exists\, n_{\mathbb{R}} \in \mathbb{R}((t)) : n(b) = \psi^{-1}(\psi(b) + n_{\mathbb{R}}) \}$

Then $N \cong (\mathbb{R}((t)), +)$ and $S = \Pi N$, $P = MAN$

$$\boxed{\begin{array}{l} \underline{\text{Proposition}}\ (\text{Bruhat decomposition}) \\ \exists\, w \in G : \quad w^2 = 1, \quad w(v_j) = v_{-j} \\ \qquad\qquad w a w^{-1} = a^{-1}, \quad a \in A \\ \\ G = P \cup PwP \\ PwP = PwN = NwP \end{array}}$$

$g \in G$ is called <u>hyperbolic</u> if g does not fix a point in X.

g hyperbolic $\Rightarrow$ g conjugate to a unique element a_g of

$$A^- = \{ a_0 , a_0^2 , \dots \}$$

g hyperbolic $\Rightarrow$ $\exists!$ line $(\dots V_{-1}(g) , V_0(g) , V_1(g) \dots)$

such that $g V_j(g) = V_{j+l(g)}(g)$ $\quad l(g) \in \mathbb{N}$.

$\Gamma_{hyp} = \Gamma \cap G_{hyp} = \Gamma \setminus \{1\}$.

$\gamma \in \Gamma_{hyp}$ <u>primitive</u> if $\gamma = \sigma^n$, $n \in \mathbb{N}$, $\sigma \in \Gamma \Rightarrow n = 1$.

$\forall \gamma \in \Gamma_{hyp}$ $\exists!$ primitive γ_0 with $\gamma = \gamma_0^k$.

$\mathcal{E}(\Gamma) = \Gamma_{hyp} / \text{conjugation}$

<u>Theorem</u> (Lefschetz Formula)

Let φ be a function on A^- with $\sum_{a \in A^-} |\varphi(a)| \, q^{l(a)} < \infty$

Then

$$\sum_{a \in A^-} \varphi(a) \, q^{l(a)} \, \mathrm{tr}(a \mid H_0(S, L^2(\Gamma \backslash G)))$$

$$= \sum_{[\gamma] \in \mathcal{E}(\Gamma)} l(\gamma_0) \, \varphi(a_\gamma)$$

Now G semisimple p-adic group

$P = LN$ parabolic $A \subset L$ split component

$M = L^{der}$ (derived group)

$A^- \subset A$ negative Weyl chamber

<u>Theorem</u> (General Lefschetz Formula)

Let $\varphi \in C_c^\infty(A^-)$. Then

$$\sum_{p=0}^{\dim M} (-1)^p \int_{A^-} \operatorname{tr}\left(a \mid H_d^p(M, H_0(N, L^2(\Gamma\backslash G))\right) a^{2\rho} \varphi(a)\, da$$

$$= \sum_{[\gamma] \sim_G a_\gamma m_\gamma \in A^- M^{ell}} \chi_r(\Gamma_\gamma)\, \lambda_\gamma\, \varphi(a_\gamma)$$

where $r = \dim A$, $\lambda_\gamma = \operatorname{vol}(A/\Gamma_\gamma)$

$$\chi_r(\Gamma_\gamma) = \sum_q (-1)^{q+r} \binom{q}{r} \dim H^q(\Gamma_\gamma, \mathbb{Q})$$

Γ_γ = centralizer of γ in Γ