Ramanujan Graphs (Complexes) and Algebraic Geometry

I. The Explicit Construction from Division Algebras and Shimura Varieties

II. The general Ramanujan graphs and curves and abelian varieties over finite fields and rings

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** I) is reflecting years of joint work with Ron Livné!

The Explicit Construction:

$B(q\infty)$ = rational quaternion division algebra ramified at q, ∞

$\cup\!|$

$\mathcal{M}$ = maximal order, $\tilde{\mathcal{M}} = \mathcal{M}[\frac{1}{p}]$

$\Gamma_0 = \tilde{\mathcal{M}}^\times / \pm 1 \subseteq PGL_2(\mathbb{Q}_p)$

$Gr(q) = \Gamma_0 \backslash \Delta_p$ regularity $k = p+1$

$\#$ vertices $= h(B(q\infty))$

adjacency matrix $A(Gr(q)) = A_q$

Laplacian $\Box_0$ has eigenvalues $\{p+1-\lambda\}$, λ = eigenvalue of T_p on $S_2(\Gamma_0(q))$

$\Gamma_+ = \{\gamma \in \tilde{\mathcal{M}}^\times \mid Norm(\gamma)$ has even p-adic val$\}$

$$\begin{array}{c} Gr^+(q) = \Gamma_+ \backslash \Delta_p \\ \downarrow \text{ étale double cover} \\ Gr(q) \end{array}$$

adjacency matrix of $Gr^+(q) = \begin{pmatrix} 0 & A_q \\ A_q & 0 \end{pmatrix}$; bipartite

Laplacian $\Box_0$ has (nontrivial) eigenvalues

$\{p+1-\lambda, p+1+\lambda\}$, λ = eigenvalue of T_p on $S_2(\Gamma_0(q))$

Sightings of these constructions

① J-Livné analyze the graphs $\Gamma_0 \backslash \Delta_p$ and $\Gamma_+ \backslash \Delta_p$ as main tool in determining whether Shimura curves have local points over $\mathbb{Q}_p$: $V_{B(pq)}(\mathbb{Q}_p) \overset{??}{=} \phi$.

② Ribet studies congruences between newforms and old-forms:

$$\left.\begin{array}{l} f \in S_2(\Gamma_0(q)) \\ g \in S_2(\Gamma_0(pq))^{p\text{-new}} \end{array}\right. \quad f \equiv g \bmod \ell.$$

Result: Such a congruence exists $\Leftrightarrow$

$\ell \mid \prod (p+1-\lambda)(p+1+\lambda)$, λ = eigenvalue of T_p on $S_2(\Gamma_0(q))$

(ℓ not Eisenstein)

<u>Example:</u> $q=11$, $p=3$

		2	5	
$f \in S_2(\Gamma_0(11))$		-2	$+1$	$\cdots$
$g \in S_2(\Gamma_0(33))$		1	-2	$\cdots$

T_3 acts on $S_2(\Gamma_0(11))$ as $-1 = \lambda$

$$(1+p-\lambda)(1+p+\lambda) = (1+3-(-1))(1+3+(-1)) = 5 \cdot 3$$

5 is Eisenstein for level 11 $\Rightarrow$ There should be such a congruence only mod 3. ✓✓

First Appearance of the explict construction of these graphs:

Brandt 1943

If $h = h(B(q\infty))$, define matrices $B(n)$ for $n \geq 1$ such that

- size $h \times h$
- symmetric, integral
- rows and columns sum to $p+1$ for $n = p$ prime.

$I_1, \dots, I_h$ = representatives of ideal classes

$$B(n)_{i,j} = \left\{ \alpha \in I_j^{-1} I_i \;\middle|\; \frac{\mathrm{Norm}(\alpha)}{\mathrm{Norm}(I_j^{-1} I_i)} = n \right\}.$$

Equivalently, for $n = p$ prime

$E_1, \dots, E_h$ = supersingular elliptic curves in char q

$B(p)_{i,j}$ = # of isogenies of degree p from E_i to E_j.

Eichler's Trace Formula: (1950's)

(non)-trivial eigenvalues of $\mathcal{B}(p)$ =

eigenvalues of T_p on $S_2(\Gamma_0(q))$

Eichler-Shimura: $T_p = F + {}^tF$ on $X_0(q)$

Riemann hypothesis known for curves $\Rightarrow$

By end of 1950's we have explicit matrices $B(q)$ of size $h \times h$ with

- integral, symmetric, rows + columns add to $q+1$
- nontrivial eigenvalues $\leq 2\sqrt{p}$.

Eichler's mass formula shows

$h \to \infty$ as $q \to \infty$.

** There are compelling reasons to put Brandt matrices back in the game.

The fundamental context for the division algebra construction = p-adic uniformization of Shimura varieties

$$B(pq) = \text{quaternion division algebra ramified at } p \text{ and } q$$

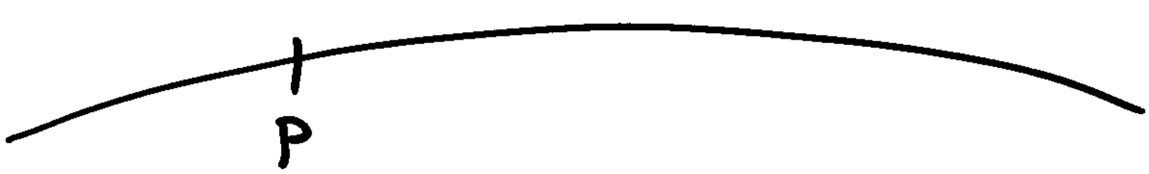
$$\cup\!|$$

$$\mathfrak{M} = \text{maximal order}$$

$$\Gamma_1 = \mathfrak{M}_1/\pm 1 \hookrightarrow PSL_2(\mathbb{R})$$

$$\Gamma_1\backslash\mathbb{H} = V_{B(pq)}(\mathbb{C})$$

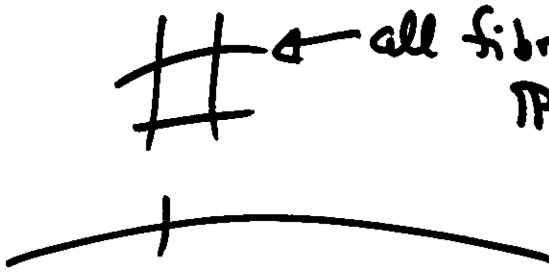
<u>Shimura</u>: canonical model $V_{B(pq)}/\mathbb{Q}$



<u>Drinfeld</u>: $V_{B(pq)}/\mathbb{Z}_p \cong \left(\Gamma_+\backslash\hat{\Omega}_p\right)^{\chi}$ $\quad \chi: \mathrm{Frob}_p \mapsto w_p$.

$$V_{B(pq)}/w_p \Big/ {}_{\Gamma}\mathbb{Z}_p = \Gamma_0 \backslash \mathfrak{h}_p .$$

$V_{B(pq)}$



all fibres $\mathbb{P}^1$'s

p

dual graph of special fiber:
vertices = components
edges = crossings

dual graph 

Mumford uniformization:

$$\text{dual graph of } V_{B(pq)/w_p} \times \overline{\mathbb{F}}_p = \Gamma_0 \backslash \Delta_p = Gr(q)$$

$$\text{dual graph of } V_{B(pq)} \times \overline{\mathbb{F}}_p = \Gamma_+ \backslash \Delta_p = Gr^+(q)$$

Hecke correspondences T_ℓ on $V_{B(pq)}$ act on special fiber $\rightsquigarrow$ Hecke action on vertices, edges of $Gr^+(q) \equiv Gr$:

$$\mathbb{Z}^{Ed} = C^1(Gr) \underset{\delta}{\overset{d}{\rightleftarrows}} C^0(Gr) = \mathbb{Z}^{Ver} \cong \mathbb{Z}^{2h(B(q\infty))}$$

canonical basis $\hat{v}_1, \ldots, \hat{v}_{2h}$

T_ℓ acts as $\begin{pmatrix} B(\ell) & 0 \\ 0 & B(\ell) \end{pmatrix}$.

$$C^0(Gr) \otimes \mathbb{C} = S_2(B(q\infty)) \oplus \text{Eisenstein}$$
$$\cong M_2(\Gamma_0(q))^2$$

*** The graph gives a canonical $\mathbb{Z}$-structure to spaces of modular forms for the definite algebra $B(q\infty)$. The graph gives a canonical basis. Hecke operators act via Brandt matrices.

$$\begin{tikzcd}
\overset{\langle , \rangle_1}{C^1(Gr)} \arrow[loop below] \arrow[r, dashed, bend right, "\delta"'] & \overset{\langle , \rangle_0}{C^0(Gr)} \arrow[l, "d"'] \arrow[loop below]
\end{tikzcd}$$

$\square_1 = d\delta$ $\qquad\qquad$ $\square_0 = \delta d$

$\operatorname{Ker} \square_1 = \mathbb{H}^1$ $\qquad\qquad$ $\operatorname{Ker} \square_0 = \mathbb{H}^0$

Over $\mathbb{Q}$:

$$C^0(Gr) = \mathbb{H}^0 \oplus (\mathbb{H}^0)^\perp = \mathbb{H}^0 \oplus \delta C^1(Gr) = \mathbb{H}^0 \oplus \square_0 C^0(Gr)$$
$$C^1(Gr) = \mathbb{H}^1 \oplus (\mathbb{H}^1)^\perp = \mathbb{H}^1 \oplus d\, C^0(Gr) = \mathbb{H}^1 \oplus \square_1 C^1(Gr)$$

But not true over $\mathbb{Z}$!

$$\Phi_0 = (\mathbb{H}^0)^\perp / \square_0 C^0(Gr), \qquad \Phi_1 = H^1(Gr, \mathbb{Z}) / \mathbb{H}^1$$

$$\bar{\delta} : \Phi_1 \xrightarrow{\approx} \Phi_0$$

$$\Phi_0 = \left(\mathbb{Z}^{\mathrm{ver}}\right)_0 \Big/ \begin{bmatrix} p+1 & -B(p) \\ -B(p) & p+1 \end{bmatrix} \mathbb{Z}^{\mathrm{ver}}.$$

$$\shortparallel$$

$$\square_0$$

essentially order of $\Phi_0 = \prod (p+1-\lambda)(p+1+\lambda)$.

λ eigenvalue of T_p on $S_2(\Gamma_0(q))$

** $\Phi := \Phi_1 \approx \Phi_0$ detects congruences between newforms and old forms.

II. Relations with Algebraic Varieties: Experimental and Speculative

$q \geq 1$ be an integer

q-Weil number: algebraic integer α s.t. $\|\alpha\|_\infty = q^{1/2}$ $\forall \infty$

q-Ramanujan number: totally real algebraic integer β s.t. $|\beta| \leq 2\sqrt{q}$

Root of $x^2 - \beta x + q$ are q-Weil numbers $\Leftrightarrow$ β is a q-Ramanujan number

q-Weil polynomial $P_W(x)$: $\in \mathbb{Z}[x]$; roots are all q-Weil numbers s.t. ω is a root $\Leftrightarrow$ q/ω is a distinct root

q-Ramanujan polynomial $P_R(x)$: $\in \mathbb{Z}[x]$; roots are all q-Ramanujan numbers

* $P_R(x)$ is q-Ramanujan of degree g

$P_W(x) = x^g P_R\left(x + \frac{q}{x}\right)$ is q-Weil of degree $2g$

$C_{g,q}$ = curves of genus g over $\mathbb{F}_q$, $q = p^r$

$A_{g,q}$ = abelian varieties of dim g over $\mathbb{F}_q$, $q = p^r$

$Gr_{k,n}$ = (connected) k-regular graphs on n vertices

$\cup|$

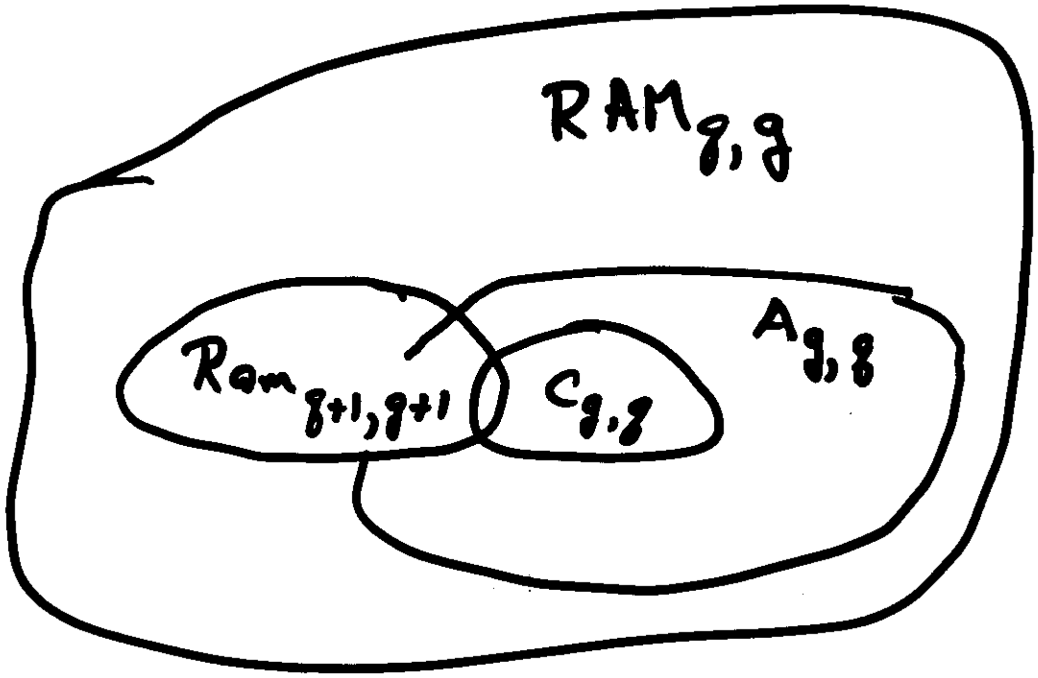
$Ram_{k,n}$ = (connected) k-regular graphs on n vertices which are Ramanujan

$RAM_{q,g}$ = q-Ramanujan polynomials of degree g

$Ram_{k,n} \rightsquigarrow RAM_{k-1,n-1}$ non bipartite

$n-2$ bipartite

$C_{g,q}$, $A_{g,q}$ $\rightsquigarrow RAM_{q,g}$



<u>Recall</u>: T_ℓ for $X_0(p)$ given by $B(\ell)$ on $B(p\infty) \Rightarrow$

$X_0(p)/_{\mathbb{F}_\ell}$ of genus $g \rightsquigarrow Ram_{\ell+1,g+1}$.

What is $C_{g,g} \cap Ram_{g+1,g+1}$?

$A_{g,g} \cap Ram_{g+1,g+1}$?

Example: $P_R(x) = x - 1 \in \mathrm{RAM}_{2,1}$

corresponds to the 2-Weil polynomial

$$P_W(x) = x\left[\left(x + \frac{2}{x}\right) - 1\right]$$
$$= x^2 + 2 - x = x^2 - x + 2$$

Trace of $\mathrm{Frob}_2 = +1 \leftrightarrow E/\mathbb{F}_2$ with 4 points

$$y^2 + y = x + \frac{1}{x}.$$

 $\in \mathrm{Gr}_{3,2}$

Adjacency matrix $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

Char poly $\begin{vmatrix} x-2 & -1 \\ -1 & x-2 \end{vmatrix} = (x-2)^2 - 1$
$$= x^2 - 4x + 3$$
$$= (x-3)(x-1)$$

So in $\mathrm{Ram}_{3,2}$ but associated to a curve in $C_{1,2}$.

One Final Point:

q-Ramanujan polynomials for $q \neq p^r$ can be related to curves:

$x-1$ 6-Ramanujan polynomial

$$P_W(x) = x\left(\left(x+\frac{6}{x}\right)-1\right) = x^2+6-x = x^2-x+6$$

should correspond to $E/\mathbb{F}_6$??

roots: $\frac{1 \pm \sqrt{-23}}{2}$

$E/$ Hilb_K | 3 $\mathbb{Q}(\sqrt{-23}) = K$

ordinary at 2, 3: 2, 3 split in $\mathbb{Q}(\sqrt{-23})$

$(2) = \mathfrak{q}\bar{\mathfrak{q}}$ $(3) = \mathfrak{p}\bar{\mathfrak{p}}$

$[\mathfrak{p}]$ order 3
$[\mathfrak{q}]$ order 3

$R = \mathcal{E}/\mathcal{O}_{\mathrm{Hilb}_K} \bmod \mathfrak{p}\mathfrak{q}$

$E/$ R | $\mathbb{Z}/6\mathbb{Z}$

$\mathrm{Frob}_2 \mathrm{Frob}_3 : E \to E.$