RAMANUJAN TYPE GRAPHS AND BIGRAPHS

Cristina M. Ballantine

College of the Holy Cross

cballant@holycross.edu

February 2004

IPAM - Los Angeles

Problem: Construct a graph:

- well connected
- reasonable number of edges

Definition: An (n, k, c)-expander is a k-regular graph $X_{n,k}$ on n vertices: $\forall A \subset V$ with $|A| \leq \frac{n}{2}$ satisfies $|\partial A| \geq c|A|$. (c is called the expansion coefficient.)

δ - adjacency matrix of X

$\lambda(X)$ - eigenvalue of δ with second largest abs. value.

Proposition: A finite k-regular graph $X_{n,k}$ on n vertices is an (n, k, c)-expander with $2c = 1 - \frac{\lambda}{k}$.

good expanders $\leftrightarrow$ λ small.

Alon-Boppana: $\lim_{n\to\infty} \lambda(X_{n,k}) \geq 2\sqrt{k-1}$

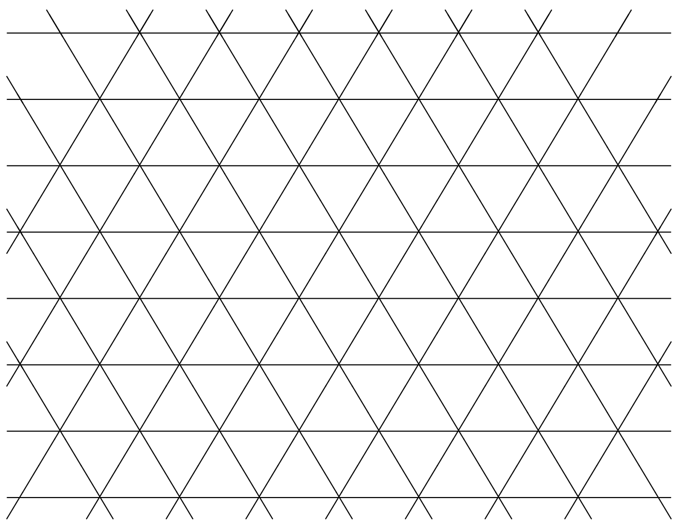
Definition (Lubotzky, Phillips, Sarnak): $X_{n,k}$ is called a Ramanujan graph if $|\lambda| \leq 2\sqrt{k-1}$.

$G = GL_3(\mathbb{Q}_p)$

$\mathcal{B}$ - Bruhat-Tits building associated with G - obtained by "gluing together" the apartments corresponding to different tori. G acts on $\mathcal{B}$.

Apartment $\mathcal{A}(G,T)$: canonical affine space under $X_* \otimes \mathbb{R}$.

Apartment of $SL_3(\mathbb{Q}_p)$: 2-dim'l simplicial complex. Maximal simplices are triangles.



Structure: facets, walls, half-apartments, chambers; special vertices.

Building: "ramify" along every edge. Each edge belongs to $p + 1$ triangles. All vertices are special.

$\mathcal{B} =$ Building of $SL_3(\mathbb{Q}_p)\times$affine line.

$SL_3(\mathbb{Q}_p)$ semisimple and simply connected $\rightarrow$ maximal compact subgroups are the stabilizers of the vertices of $\mathcal{B}$.

$\mathcal{B}$ is a labellable hypergraph with labelling $\{0, 1, 2\}$.

The underlying graph $\underline{\mathcal{B}}$ is $2(p^2 + p + 1)$-regular.

$G = GL_3(\mathbb{Q}_p)$:

- max'l compact subgps: $K = GL_3(\mathbb{Z}_p)$ and its conj.

- G acts transitively on vertices.

- vertices $\leftrightarrow G/K$

- $V = C_c(G/K)$

Hecke algebra $\mathcal{H}_p = C_c(K\backslash G/K)$.

- $\mathcal{H}_p$ acts on V by convolution on the right.

$$(f \star \varphi)(x) = \int_{G(\mathbb{Q}_p)} \varphi(y) f(xy)\, dy, \quad \varphi \in \mathcal{H}_p, \quad f \in V.$$

- φ_i characteristic function of Kt_iK, $i = 0, 1, 2, 3$,

$$t_i = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & p \end{pmatrix} \}i$$

are the generators of $\mathcal{H}_p$.

Theorem: φ_1, φ_2 are the partial Laplacians for $\mathcal{B}$:

$$(\varphi_i(f))(x^{(0)}) = \sum_{x^{(i)} \sim x^{(0)}} f\left(x^{(i)}\right), \ i = 1, 2.$$

- φ_1, φ_2 commute and $\delta = \varphi_1 + \varphi_2 \ \longrightarrow$

estimate spectrum of φ_1, φ_2.

1 dim'l representations of $\mathcal{H}_p \leftrightarrow$ irreducible unramified representations of G.

$GL_n(\mathbb{Q}_p)$ - Tate: Fourier transform for φ_i (Satake) $\rightarrow$ for each φ_i: eigenvalues determined by the unramified representations.

G' is a $\mathbb{Q}$-form of $U(3)$:

$G'(\mathbb{Q}_p) \cong GL_3(\mathbb{Q}_p)$ & $G'(\mathbb{R})$ compact.

Rogawski's classification of automorphic representations of $U(3) \rightarrow$ unramified representations of $GL_3(\mathbb{Q}_p)$ $\rightarrow$ (Tate's Theorem) eigenvalues of φ_i.

Γ_p discrete co-compact subgroup of $G'(\mathbb{Q}_p)$:
Γ_p acts on $G'(\mathbb{Q}_p)$ without fixed points.

$\mathbb{B}$ - quotient of the building of $G'(\mathbb{Q}_p)$ by Γ_p.

Main Theorem: $\mathbb{B}$ is a Ramanujan type building quotient with

- $|\lambda| \leq 6p$ if G' arises from a division algebra

- $|\lambda| \leq 2p(p^{1/2} + 1 + p^{-1/2})$ if G' arises from an unramified algebra

Idea of proof:

Tate:
$\chi = (\chi_1, \ldots, \chi_n)$ n-tuple of unram. characters of $\mathbb{Q}_p^*$.

φ_i has eigenvalue $p^{\frac{1}{2}i(n-i)}\sigma_i(\chi_1(p), \ldots, \chi_n(\pi))$.

Rogawski:

Packet structure on $\mathbf{G} = U(3)$
$\updownarrow$
Packets on the endoscopic group $\mathbf{H}$

Global packet $\Pi = \otimes \Pi_v$ on $\mathbf{G}$: $\pi = \otimes \pi_v$ with $\pi_v \in \Pi_v$ for all v and π_v unramified for a.a. v.

$\Pi_a(\mathbf{G}) = \{\Pi(\xi) : \xi \in \Pi(\mathbf{H}), \dim(\xi) = 1\}$

$\Pi_e(\mathbf{G}) = \{\Pi(\varrho) : \varrho \in \Pi_\circ(\mathbf{H}), \varrho \neq \varrho(\theta) \text{ for } \theta \text{ s.r.}\}$

$\Pi_s(\mathbf{G}) = \{\Pi : \nexists \varrho \in \Pi(\mathbf{H}) \text{ with } \Pi_v = \Pi(\varrho_v) \text{ a.a. } v\}$

$\{$global packets on $\mathbf{G}'\}$
$\qquad\updownarrow$e.v.p.
$\{$global packets on $\mathbf{G}\}$

Eigenvalue package (e.v.p.): $t = t_S = \{t_v : v \notin S\}$
t_v homomorphism of $\mathcal{H}_v$ into $\mathbb{C}$.

e.v.p.: can work with either $\mathbf{G}$ or $\mathbf{G}'$.

Rogawski's packets $\leftrightarrow$ Arthur's parametrization

$L_{\mathbb{Q}}$ - hypothetical Langlands group

$$\text{global packets on } G$$
$$\updownarrow$$
$$\psi : L_{\mathbb{Q}} \times SL(2, \mathbb{C}) \longrightarrow {}^L G$$

Consider $S_\psi = \text{Cent}(\text{Im}(\psi), \widehat{G})$.

$$\text{Representations in the discrete spectrum}$$
$$\updownarrow$$
$$\psi \text{ with } S_\psi \text{ finite}$$

Attached representation:

$$\rho : A_{\mathbb{Q}} \times SL(2, \mathbb{C}) \longrightarrow GL(3, \mathbb{C})$$

S_ψ finite $\Longleftrightarrow$ ρ contains no irreducible factors with multiplicity > 1.

Proof: Case by case discussion of the parameters according to the decomposition of ρ.

- different cases $\leftrightarrow$ Π_s, Π_e and Π_a
- Tate's Theorem
- applications of Deligne's Theorem

Definition: A (k,l)-regular bigraph $B_{k,l}$ is a bipartite graph in which all vertices of one color have degree k and all vertices of the other color have degree l.

Trivial eigenvalues of adjacency matrix: $\sqrt{kl}$, $-\sqrt{kl}$

Feng and Li: $\liminf_{n\to\infty} \lambda(B_{n,k,l}) \geq \sqrt{k-1}+\sqrt{l-1}$

Definition (Solé): A (k,l)-regular bigraph $B_{k,l}$ is a Ramanujan bigraph if any nontrivial eigenvalue α of $B_{k,l}$ satisfies

$$|\sqrt{k-1}-\sqrt{l-1}| \leq |\alpha| \leq \sqrt{k-1}+\sqrt{l-1}.$$

Let $E/\mathbb{Q}$ be a separable quadratic extension of global fields.

Let $E_p/\mathbb{Q}_p$ be unramified. Let G be a $\mathbb{Q}$-form of $U(3)$ such that $G(\mathbb{Q}_p)$ is isomorphic to $U_3(\mathbb{Q}_p)$ and $G(\mathbb{R})$ is compact.

T - Bruhat-Tits tree attached to $U_3(\mathbb{Q}_p)$
T is $(p^3+1, p+1)$-regular.

Questions:

1) Combinatorial meaning of the Hecke operators of $U_3(\mathbb{Q}_p)$ relative to the maximal compact subgroups?

2) How does Tate's Theorem for GL_n translate to U_n?

The vertices of T are stabilized by the maximal compact subgroups of $U_3(\mathbb{Q}_p)$. Up to conjugacy:

$$K_1 = U_3(\mathbb{Q}_p) \cap GL_3(\mathbb{Z}_p)$$

$$K_2 = \left\{ \begin{pmatrix} * & * & p^{-1}* \\ * & * & p^{-1}* \\ p* & p* & * \end{pmatrix} : * \in \mathbb{Z}_p \right\}$$

Vertices of $T \leftrightarrow G/K_1 \cup G/K_2$.

Conjugates of K_1 stabilize vertices of degree p^3+1.

Conjugates of K_2 stabilize vertices of degree $p+1$.

Consider the Hecke algebras $\mathcal{H}(G, K_1)$ and $\mathcal{H}(G, K_2)$.

For each $i = 1, 2$, generators of $\mathcal{H}(G, K_i)$:

- characteristic function of K_i
- the characteristic function φ_i of $K_i t K_i$, where

$$t = \begin{pmatrix} p & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & p^{-1} \end{pmatrix}$$

Each Hecke algebra acts on $C_c(G/K_1 \cup G/K_2)$ by right convolution.

If x is stabilized by K_1, $(f \star \varphi_1)(x)$ is the sum of the values of f at $p+1$ of the p^3+1 neighbors of x.

If x is stabilized by K_2, $(f \star \varphi_1)(x)$ is the sum of the values of f at all $p+1$ neighbors of x.

φ_2 does not seem to give an interesting combinatorial object.

Cartwright and Steger: construction of T from the building of $SL_3(E_p)$. Keep vertices of type 0 and collapse adjacent vertices of type 1 and 2 (delete some collapsed pairs).

Is it possible to obtain an estimation of the spectrum of T in a purely combinatorial fashion?
(using estimation of spectrum of SL_3)