Mumford's 2-dimensional complex and generalizations

Ron Livné, Hebrew University of Jerusalem

presented by Shahar Mozes
Hebrew University of Jerusalem

Automorphic Forms, Group Theory and Graph Expansion

IPAM, February 11, 2004

There are several examples of complexes built up from groups:

- The Cayley graph of a group G and a set of generators S.
- The spherical building of a reductive group over a finite field, of dimension $\operatorname{rank} G - 1$.
- The affine building of a reductive group over a local field, of dimension $\operatorname{rank} G$, and its quotients by discrete groups.

Here we give a different example. Set

$$G = \mathsf{PGL}_2(\mathbb{F}_q) = \mathsf{GL}_2(\mathbb{F}_q)/\text{center}$$

It acts on its spherical complex (of dim $= 0$)

$$P = \mathbb{P}^1(\mathbb{F}_q) = \mathbb{F}_q \cup \{\infty\}$$

$$\text{by} \quad \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} z = \frac{\alpha z + \beta}{\gamma z + \delta}.$$

This is boring: P has more symmetries than G! However, G acts simply triply transitively: for any distinct a, b, c in P there exists a unique

$$g = g_{a,b,c} \in G$$

mapping $\infty, 0, 1$ to a, b, c.

When $\{a, b\} \cap \{c, d\} = \emptyset$, the cross-ratio

$$\lambda(a, b, c, d) = \frac{(c-a)(d-b)}{(c-b)(d-a)} \quad (\text{where } \tfrac{\infty}{\infty} = 1)$$

is defined and is G-invariant.

The complex $X(q,d)$

The 2-skeleton $\Sigma^2 P$ of the simplex on P is the simplicial complex with vertices P, edges the unordered pairs $\overline{ab}$, and triangles the unordered triples $\overline{abc}$ of distinct vertices. This is still boring, since all of Per_P acts, not just G.

For $d|q-1$ we build $X(q,d)$ by replacing each edge of $\Sigma^2 P$ by d ones, while leaving the vertices and the triangles unchanged. *The triangles $\overline{abc}$ and $\overline{abd}$ have the same edge in $X(q,d)$ above $\overline{ab}$ iff $\lambda(a,b,c,d)$ is a dth power in $\mathbb{F}_q$.* This is a G-equivariant equivalence relation, independent of the order on $\{a,b\}$, since

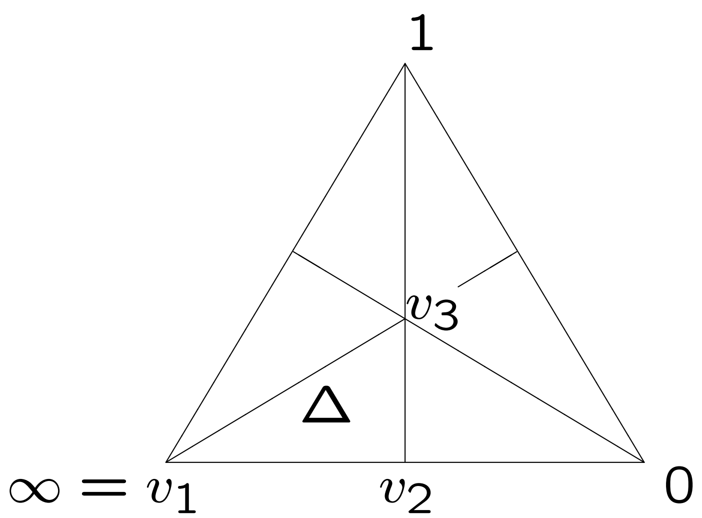
$\lambda(a,b,c,c) = 1$ (reflexivity)
$\lambda(a,b,d,c) = \lambda(a,b,c,d)^{-1}$ (symmetry)
$\lambda(a,b,c,d)\lambda(a,b,d,e) = \lambda(a,b,c,e)$ (transitivity)
$\lambda(b,a,c,d) = \lambda(a,b,c,d)^{-1}$ (order invariance)
$\lambda(ga,gb,gc,gd) = \lambda(a,b,c,d)$ (G-equivariance).

We have a G-equivariant map $X(q,2) \to \Sigma^2 P$.

A presentation of $\pi_1(X(q,d),\infty)$

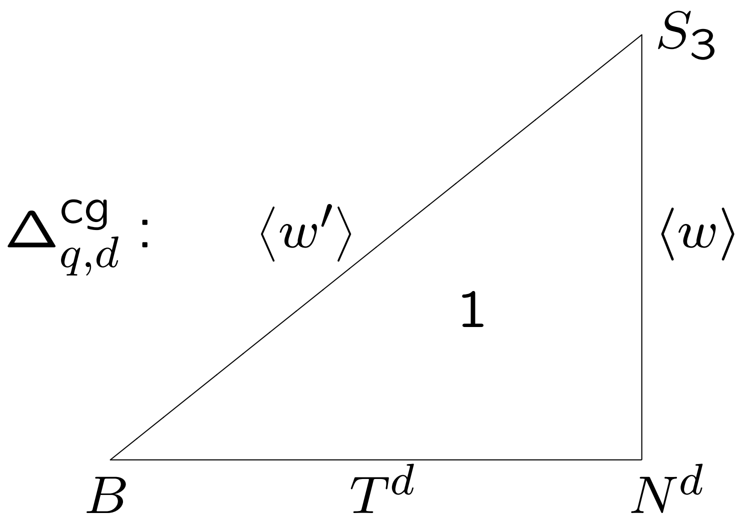
In the barycentric subdivision of $X(q,d)$ let Δ be the triangle with vertices

- $v_1 = \infty$;
- $v_2 =$ the barycenter of $\overline{\infty 0}$;
- $v_3 =$ the barycenter of $\overline{\infty 0 1}$.



Δ is a fundamental domain for the action of G on $\Sigma^2 P$ by the simple triple transitivity of G on P.

The stabilizer of Δ in G is trivial. The stabilizers S_i of v_i and S_{ij} of $\overline{v_i v_j}$ are



with B the upper triangular matrices, T^d the dth powers of the diagonal T, $w \in G$ the involution $z \mapsto 1/z$, w' the involution $z \mapsto 1 - z$, S_3 the stabilizer in G of the set $\{\infty, 0, 1\}$, and N^d the group generated by T^d and w.

The orbifold data Δ, S_i, S_{ij} and the inclusions

$$\iota_{i,j} : S_{ij} \to S_j$$

form a triangle with groups denoted $\Delta^{\mathrm{cg}}_{q,d}$.

The theory of complexes with groups (Bass-Serre for graphs, Stallings for triangles, Haefliger in general) presents the fundamental group $\pi_1(\Delta^{\mathrm{cg}}_{q,d}, \infty)$ of $\Delta^{\mathrm{cg}}_{q,d}$ as the free product

$$S_1 * S_2 * S_3 \tag{1}$$

divided by the relations

$$\iota_{i,j} x = \iota_{j,i} x$$

for all $x \in S_{ij}$, where

$$\iota_{i,j} : S_{ij} \to S_j$$

are the inclusions. Its crucial property is the existence of an exact sequence

$$0 \to \pi_1(X(q,d), \infty) \to \pi_1(\Delta^{\mathrm{cg}}_{q,d}, \infty) \to G \to 1,$$

where the map to G is the natural one.

Since $S_2 = N^d$ is generated by $\iota_{1,2}(T^d)$ and by $\iota_{3,2}(w)$, we can remove it from the generators, adding the relation $wtw = t^{-1}$ for any $t \in T^d$.

Since $S_3 = \left\langle w, w' | w^2 = (w')^2 = (ww')^3 = 1 \right\rangle$, and since w' comes from B, the presentation for $\pi_1(\Delta^{\text{cg}}_{q,d}, \infty)$ becomes the free product

$$B * \langle w \rangle$$

modulo the relations

- $wtw = t^{-1}$ for all $t \in T^d$;
- $(ww')^3 = 1$ (viewing w' in B).

Since $X(q,1) = \Sigma^2 P$ we get $\pi_1(\Delta^{\text{cg}}_{q,1}, \infty) = G$. The relations above on $B * \langle w \rangle$ are then precisely those coming from the multiplication table of G relative to the Bruhat decomposition

$$G = B \sqcup NwB$$

where N is the group of translations $z \mapsto z + b$.

The relations defining $\pi_1(\Delta^{\text{cg}}_{q,d}, \infty)$ are thus a natural weakening of the Bruhat relations.

Finiteness and non-finiteness of $\pi_1(X(q,d),\infty)$

GAP computations by A. Besser (suggested by D. Wise) proved that $\pi_1(X(q,2),\infty)$ has order 6 if $43 \geq q \geq 11$ for a prime $q \equiv 3 \pmod 4$.

In all other cases with $d \geq 2$, including $X(3,2)$ and $X(7,2)$, the machine failed to compute the order of $\pi_1(X(q,2),\infty)$, suggesting it might be infinite. ($q > 45$ is too big for the machine.)

We can explain these results completely. Consider first $d = 2$ and $q \equiv 3 \pmod 4$:

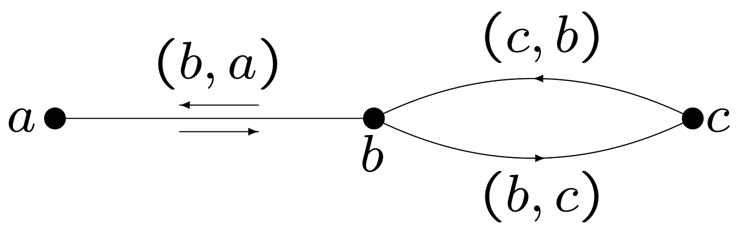
- $X(3,2)$ is a tetrahedron slit along its six edges. Its π_1 is infinite, free of rank 5.
- $X(7,2)$ is a quotient of an A_2 building. Its π_1 is an infinite, S-arithmetic group.

On the other hand we have the following

Theorem 1 *For* $q \equiv 3 \pmod 4$ *a prime power* ≥ 11 *we have*

$$\pi_1(X(q,2),\infty) \simeq \mathbb{Z}/6\mathbb{Z}.$$

Since $\det g_{b,a,c} = -\det g_{a,b,c}$, we orient canonically an edge above $\overline{ab}$ as (a,b) if $\det g_{a,b,c}$ is a square. Let (a,a) be the constant path at a, let $\overline{(a,b)}$ be the path backing along (a,b) from b to a. Let $\gamma^a_{b,c}$ be the loop based at a given by

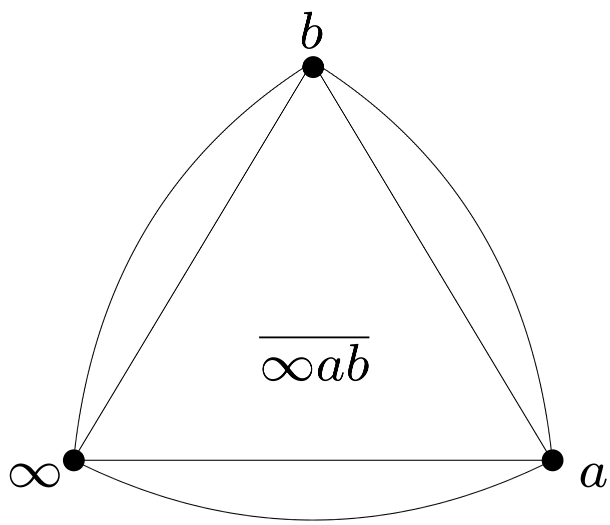


Or $\gamma^a_{b,c} = \overline{(b,a)} * (b,c) * (c,b) * (b,a)$. Let $[\gamma^a_{b,c}]$ be the class of $\gamma^a_{b,c}$ in $\pi_1(X(q,2),a)$. G-equivariance takes the form

$$g\gamma^a_{a,b} = \begin{cases} \gamma^{ga}_{ga,gb} & \text{if } \det g \text{ is a square,} \\ (\gamma^{ga}_{ga,gb})^{-1} & \text{otherwise.} \end{cases} \qquad (2)$$

Step 1. By induction on the length, a loop on $X(q,2)$ based at ∞ is homotopic to a product of γ_{ab}^{∞}'s, so these generate $\pi_1(X(q,d),\infty)$.

The induction step replaces a loop starting by going from ∞ to b in two steps through a, by a product of $\gamma_{*,*}^{\infty}$'s followed by going from ∞ to b in one step. The picture shows this is possible.



Step 2. Four distinct points a, b, c and d in P span the 2-skeleton $\Sigma = \Sigma^2\overline{abcd}$ of a tetrahedron in $\Sigma^2 P$. In $X(q,2)$ some of the edges are slit (the others have an extra edge connected to their endpoints). We have the following

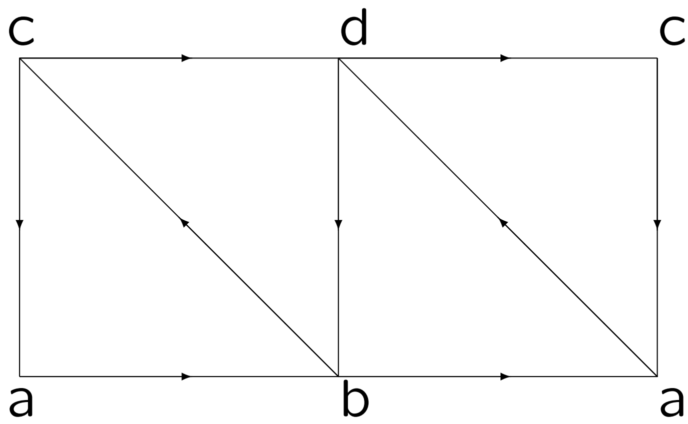
Lemma 2 *1. The edge $\overline{ab}$ is slit in $X(q,2)$ iff its opposite edge $\overline{cd}$ is.*
2. The number of pairs of slit edges is 3 or 1. (It is 0 or 2 if $q \equiv 1 \pmod 4$.)
3. $\Sigma^2\overline{abcd}$ is slit precisely along $\overline{ab}$ and $\overline{cd}$ iff

$$-\lambda(a,b,c,d) \quad \textit{and} \quad \lambda(a,c,b,d)$$

are both squares in $\mathbb{F}_q^\times$.

The first two assertions follow from
1. $\lambda(c,d,a,b) = \lambda(a,b,c,d)$
2. $\lambda(a,b,c,d)\lambda(a,c,d,b)\lambda(a,d,b,c) = -1$
respectively. Together they imply Part 3.

A tetrahedron with 3 slit edge-pairs is four triangles meeting at the vertices; for 2 slits, join two squares at their corresponding vertices and divide each square into two triangles.



The picture shows $\Sigma^2\overline{abcd}$ slit along the $\overline{ab}$ and $\overline{cd}$ edge-pair, with the left and right (a, c) sides glued so as to form a ring. We see that in this case we have $[\gamma^a_{a,b}] = [\gamma^a_{c,d}]$.

Lemma 3 $[\gamma^\infty_{\infty,0}] = [\gamma^\infty_{\infty,1}]$.

Assuming this we will show the crucial claim: The $[\gamma^\infty_{a,b}]$'s are all equal in $\pi_1(X(q, 2), \infty)$.

Sketch: For general a, b, c, d the Lemma gives, by G-equivariance, $[\gamma^a_{a,b}] = [\gamma^a_{a,c}]$. Adding a tail to ∞ gives $[\gamma^\infty_{a,b}] = [\gamma^\infty_{a,c}]$. One directly checks $[\gamma^a_{c,a}] = [\gamma^a_{a,c}]$. Thus $[\gamma^\infty_{a,b}] = [\gamma^\infty_{a,\infty}] = [\gamma^\infty_{\infty,a}] = [\gamma^\infty_{\infty,c}] = [\gamma^\infty_{c,\infty}] = [\gamma^\infty_{c,d}]$, proving the claim.

To show that $\gamma^{\infty}_{\infty 0} = \gamma^{\infty}_{\infty 1}$, we will find $a, b \in P$, such that a, b, ∞, 0, and 1 are all distinct and both tetrahedra $\overline{\infty 0 ab}$ and $\overline{\infty 1 ab}$ are slit along $\overline{ab}$ and $\overline{\infty 0}$ (respectively along $\overline{ab}$ and $\overline{\infty 1}$). Up to interchanging a and b, this happens iff

a, $a-1$, $a-b$, $1-b$, $-b$ are all squares, and a, b, ∞, 0, and 1 are all distinct.

Take $a = 25/9$; the conditions on a hold (any other such choice for a leads to a similar situation). The other conditions boil down to the existence of u and $k \neq \infty, 0, \pm 1$ in $\mathbb{F}_q$ so that

$$u^2 = k^4 - \frac{14}{25}k^2 + 1.$$

Here $b = -4k^2(k^2-1)^{-2}$. This is an algebraic curve of genus 1, so the Hasse bound gives for its number of points N_q (which includes the two points above $k = \infty$) the inequality

$$N_q \geq q + 1 - 2\sqrt{q}$$

Thus a solution for which k avoids the above 4 values exists if $N_q \geq 9$, which holds for $q \geq 19$. For $q = 11$ we take $a = 5$, $b = -4$. For $q = 7$ the method fails.

Step 3. We now know that $\pi_1(X(q,2),\infty)$ is cyclic with generator $\gamma^{\infty}_{\infty,0}$. In a tetrahedron $\overline{abcd}$ with all edges slit, say $\overline{\infty, 0, 1, -9/16}$, the 1-cycle represented by the sum

$$\gamma^{\infty}_{a,b} + \gamma^{\infty}_{a,c} + \gamma^{\infty}_{a,d} + \gamma^{\infty}_{b,c} + \gamma^{\infty}_{b,d} + \gamma^{\infty}_{c,d} \sim 6\gamma^{\infty}_{\infty,0}$$

bounds, so $\pi_1(X(q,2),\infty)$ is a quotient of $\mathbb{Z}/6\mathbb{Z}$.

Step 4. A study of the homology of the map $X(q,2) \to \Sigma^2 P$ shows that $H_1(X(q,2),\mathbb{Z})$ is the quotient of the group generated by the classes of the $\gamma^{\infty}_{a,b}$'s modulo the relations coming from the tetrahedra. It follows readily that

$$\pi_1(X(q,2),\infty) \simeq H_1(X(q,2),\mathbb{Z}) \simeq \mathbb{Z}/6\mathbb{Z},$$

concluding the proof of Theorem 1.

Remarks: **1.** $X(q, 2)$ is not $K(\pi, 1)$ for q large with respect to d, since it contains octahedra (and tetrahedra for $q \equiv 1 \pmod 4$.) The proof uses the Hasse-Weil bounds as in Theorem 1, but for a curve of higher genus. Is our use of these bounds essential? They also underlie the Ramanujan property.

2. The link of a vertex of $X(q, 2)$ is a graph with dq vertices; the vertices break naturally into d sets of q indexed by $t \in \mathbb{F}_q^\times/(\mathbb{F}_q^\times)^d$, and the vertices of the t'th set connect only with those of the $-t$'th set. In particular, the link is connected iff $d = 2$ and $q \equiv 3 \pmod 4$. One gets a retraction of $X(q, d)$ onto the subgraph spanned by ∞ and 0. Hence $\pi_1(X(q, d), \infty)$ is infinite except in the cases of Theorem 1.

3. The 1-skeleton of $X(q, 2)$ is a Ramanujan $2q$-regular graph on $q + 1$ vertices with eigenvalues $2q - 2, -2, \ldots, -2$. G. Dogon verified by machine that many covers of $X(7, 2)$ are Ramanujan, but results of J. Rogawski imply there are covers of $X(7, 2)$ which are not.

Mumford's complex

In 1979 Mumford constructed a maximal S-arithmetic group Γ_1, acting on the building $\mathcal{B}$ of $\mathrm{PGL}_3(\mathbb{Q}_2)$, an affine building of type A_2. There is a surjection of Γ_1 on $G_0 = \mathrm{PSL}_2(\mathbb{F}_7)$, yielding an exact sequence

$$1 \to \Gamma_2 \to \Gamma_1 \to G_0 \to 1.$$

Γ_1 is on Kantor, Solomon, and Tits' list for acting "very transitively" on $\mathcal{B}$. Hence G_0 acts "very transitively" on $\Gamma_2 \backslash \mathcal{B}$. One can prove that $\Gamma_2 \backslash \mathcal{B}$ has the same number of vertices, edges, and triangles as $X(7,2)$, and that the transitivity is strong enough to determine completely a G_0-complex of this size. We conclude:

Theorem 4 *$\Gamma_2 \backslash \mathcal{B}$ and $X(7,2)$ are isomorphic G_0-equivariantly. Hence $\pi_1(X(7,2),\infty) \simeq \Gamma_2$, and $\Gamma_1 \subsetneq \pi_1(\Delta^{\mathrm{cg}}_{q,d},\infty)$ is not maximal in* $\mathrm{Aut}\,\mathcal{B}$. *The outer automorphism of Γ_1 reverses the* mod 3 *cyclic order on the vertices of $\mathcal{B}$.*

Let Γ be the inverse image in Γ_1 of a 2-Sylow subgroup S of G_0. Mumford proved that Γ acts freely on $\mathcal{B}$ and transitively on its vertices. Set $Y = \Gamma\backslash\mathcal{B} = S\backslash X(7,2)$. We obtain the following (later verified by G. Dogon on a machine):

Theorem 5 *The complex Y has one vertex $*$. Its edges e_u are the images of (∞, u) for $u \in \mathbb{F}_7$. Representatives in $X(7,2)$ for its triangles σ_i and the boundaries $\partial\sigma_i$ are*

$$\begin{array}{ll} \sigma_1 : (\infty, 0, 1) & \partial\sigma_1 = (e_0, e_6, e_6) \text{ (sic!)} \\ \sigma_2 : (\infty, 0, 2) & \partial\sigma_2 = (e_0, e_3, e_2) \\ \sigma_3 : (\infty, 0, 4) & \partial\sigma_3 = (e_0, e_5, e_4) \\ \sigma_4 : (\infty, 1, 2) & \partial\sigma_4 = (e_1, e_4, e_2) \\ \sigma_5 : (\infty, 1, 3) & \partial\sigma_5 = (e_1, e_5, e_3) \\ \sigma_6 : (\infty, 1, 5) & \partial\sigma_6 = (e_1, e_2, e_5) \\ \sigma_7 : (\infty, 4, 1) & \partial\sigma_7 = (e_4, e_3, e_6). \end{array}$$

Corollary 6 *Let x, y be the classes of e_3 and e_6. Then $\Gamma = \pi_1(Y, *)$ admits the presentation*

$$\left\langle x, y \,|\, y^2 = x^2 y^3 xyx, \;\; xy^2xyx = y^4 xy \right\rangle.$$

Mumford's challenge, to draw Y, still stands.

The work described here was initiated by S. Mozes's expressed hope for a simple description of Mumford's complex. His interest and comments were very encouraging and helpful.

In particular, he suggested to "normalize" the complexes $X(q,d)$ by replacing each vertex by one per component of its link. If -1 is in $(\mathbb{F}_q^\times)^d$ the result $\tilde{X}(q,d)$ has 2 isomorphic components $X(q,d)^\pm$. Otherwise it is connected.

For instance $X(q,2)^+$, for $q \equiv 1 \pmod 4$, is the subcomplex of $\Sigma^2 P$ obtained by gluing to $\Sigma^1 P$ only those triangles $\overline{abc}$ for which $\det g_{a,b,c}$ is a square. We have $\pi_1(X(q,2),\infty) = \mathbb{Z}/2\mathbb{Z}$ for $q > 13$ (again using the Hasse bound).

It still remains to look at the other cases. Also, at other 3-transitive groups (or some similar condition). Using the classification of the finite simple groups these groups are known.