LAPLACIANS, HOMOLOGY and HYPERGRAPH MATCHING

ROY MESHULAM

- EIGENVALUES & HOMOLOGY OF FLAG COMPLEXES
- VECTOR DOMINATION OF GRAPHS & CONNECTIVITY OF THE INDEPENDENCE COMPLEX
- HYPERGRAPH MATCHING VIA HOMOLOGY

JOINT WORK WITH

RON AHARONI & ELI BERGER

GRAPH $G=(V,E)$, $|V|=n$

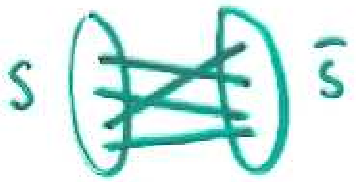
LAPLACIAN OF G

$$L_G(u,v) = \begin{cases} \deg u & u=v \\ -1 & uv \in E \\ 0 & \text{else} \end{cases}$$

EIGENVALUES OF L_G :

$$0 = \lambda_1(G) \leq \lambda_2(G) \leq \cdots \leq \lambda_n(G)$$

SPECTRAL GAP $= \lambda_2(G)$



$$e(S,\bar{S}) = |E \cap S \times \bar{S}|$$

ALON-MILMAN, TANNER:

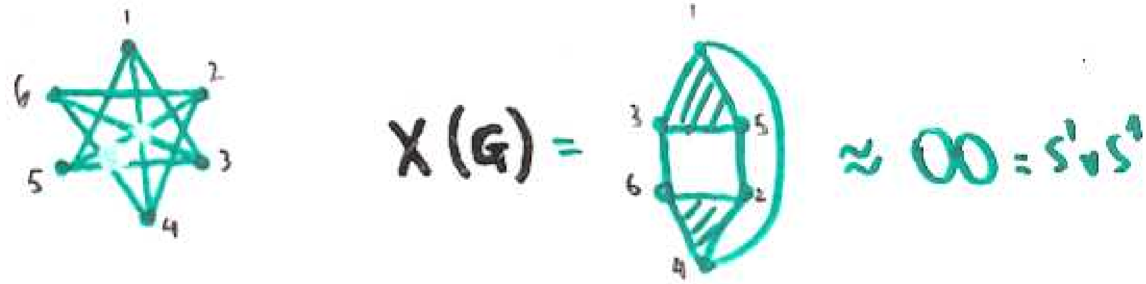
$$e(S,\bar{S}) \geq \frac{|S|\,|\bar{S}|}{n} \lambda_2(G)$$

$X(G)$ FLAG COMPLEX OF G:

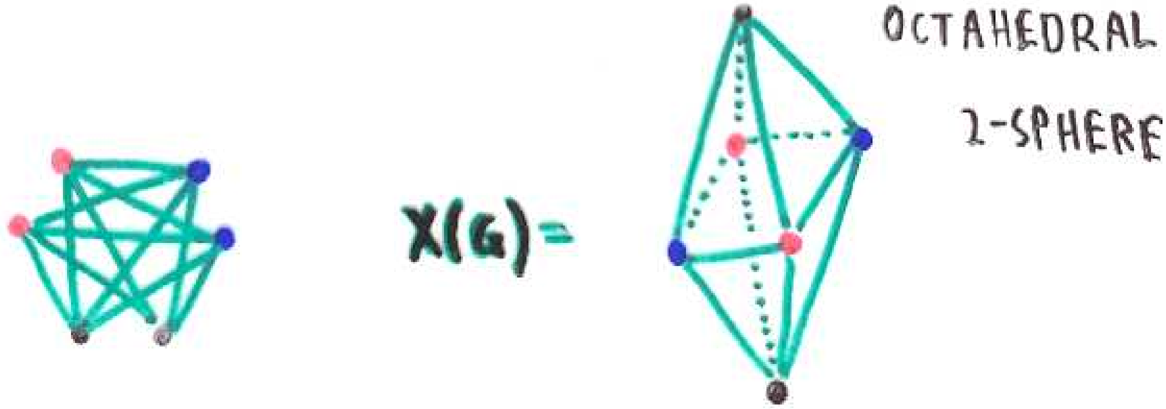
VERTICES $= V$

SIMPLICES $= \sigma \subset V \quad G[\sigma]$ CLIQUE

$G =$



$G =$



$\lambda_2(G) > 0 \iff X(G)$ 0-CONNECTED

HOW DOES $\lambda_2(G)$ RELATES TO THE TOPOLOGY OF $X(G)$?

X SIMPLICIAL COMPLEX ON V, $|V| = n$

$X(k)$ = ORDERED k-SIMPLICES OF X

$\sigma = [v_0, \dots, v_k]$ $\qquad$ $\sigma_i = [v_0, \dots, \hat{v}_i, \dots, v_k]$

$C^k(X)$ = SKEW-SYMM. $\phi : X(k) \to \mathbb{R}$

$$\to C^{k-1}(X) \underset{d_{k-1}^*}{\overset{d_{k-1}}{\rightleftarrows}} C^k(X) \underset{d_k^*}{\overset{d_k}{\rightleftarrows}} C^{k+1}(X) \to$$

$$d_k \phi(\sigma) = \sum_{i=0}^{k+1} (-1)^i \phi(\sigma_i)$$

k-th LAPLACIAN:

$$\Delta_k = d_{k-1} d_{k-1}^* + d_k^* d_k : C^k(X) \to C^k(X)$$

k-th COHOMOLOGY:

$$H^k(X) = \frac{\ker d_k}{\operatorname{Im} d_{k-1}} \cong \ker \Delta_k$$

X  $\qquad H^1 \cong \mathbb{R}^2$, $H^2 \cong \mathbb{R}$

X SIMPLICIAL COMPLEX ON V, $|V| = n$

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X  $\quad H^1 \cong \mathbb{R}^2, \quad H^2 \cong \mathbb{R}$

$\mu_k(X)$ = MINIMAL EIGENVALUE OF Δ_k

$\mu_0(X) = \lambda_2$(1-skeleton of X)

THM (ABM): LET $X = X(G)$ BE THE FLAG COMPLEX OF G. THEN $\forall k \geq 0$

$$k\,\mu_k(X) \geq (k+1)\,\mu_{k-1}(X) - n$$

IN PARTICULAR:

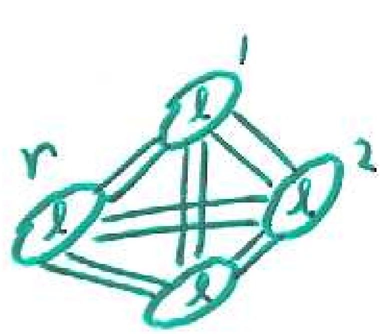
$$\lambda_2(G) > \frac{kn}{k+1} \implies \mu_i > 0 \quad \forall\, i \leq k$$

$$\implies \tilde{H}_i(X) = 0 \quad \forall\, i \leq k$$

EXAMPLE: $n = r\ell$

$T_r(n)$ = COMPLETE r-partite

$$X(T_r(n)) \approx \underbrace{S^{r-1} \vee \cdots \vee S^{r-1}}_{(\ell-1)^r}$$



$$\mu_k(T_r(n)) = \ell(r-k-1) \quad \forall\ 0 \leq k \leq r-1$$

$$\lambda_2(T_r(n)) = \ell(r-1) = \frac{r-1}{r} \cdot n$$

MAIN POINT IN PROOF:

FOR $\tau \in X(i)$ LET $\deg \tau = |\{\sigma \in X(i+1) : \sigma \supset \tau\}|$

FOR $\phi \in C^k(X)$, $u \in V$ DEFINE $\phi_u \in C^{k-1}(X)$:

$$\phi_u(\tau) = \begin{cases} \phi(u\tau) & \tau \in lk(u) \\ 0 & \text{else} \end{cases}$$

- $$k(\Delta_k \phi, \phi) =$$

$$\sum_u (\Delta_{k-1}\phi_u, \phi_u) - \sum_{\sigma \in X(k)} \Big(\sum_{\tau \in \sigma(k-1)} \deg \tau - k \deg \sigma\Big) \phi(\sigma)^2$$

- $\forall \sigma \in X(k)$

$$\sum_{\tau \in \sigma(k-1)} \deg \tau - k \deg \sigma \leq n$$

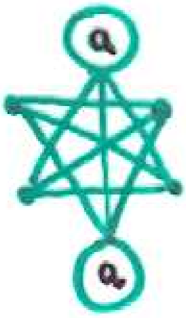
$$\eta(X) = 1 + \min\{ i : \tilde{H}_i(X) \neq 0 \} =$$

1+ HOMOLOGICAL CONN. OF X

- $$\eta(X(G)) \geq \left\lceil \frac{n}{n - \lambda_2(G)} \right\rceil$$

$\overline{C_{3k}} =$  $X(\overline{C_{3k}}) \simeq \infty = S^{k-1} \vee S^{k-1}$

$$\left\lceil \frac{3k}{3k - \lambda_2(\overline{C_{3k}})} \right\rceil \approx \frac{3k}{4} \qquad \eta = k+1$$

$\overline{C^a_{3k}} =$ 

$X(\overline{C^a_{3k}}) \simeq X(\overline{C_{3k}})$

$$\max_a \left\lceil \frac{(2+a)k}{(2+a)k - \lambda_2(\overline{C^a_{3k}})} \right\rceil = k+1$$

$A_1, \dots, A_m$ SETS

SYSTEM OF DISTINCT REPRESENTATIVES (SDR):

DISTINCT $x_1 \in A_1, \dots, x_m \in A_m$

HALL'S THM: $\left|\bigcup_{i \in I} A_i\right| \geq |I| \quad \forall I \subset [m]$

$\Rightarrow \exists$ SDR

$\mathcal{F} \subset 2^V$ HYPERGRAPH.

$M \subset \mathcal{F}$ MATCHING IF $F \cap F' = \emptyset \quad \forall F \neq F' \in M$

$\mathcal{F}_1, \dots, \mathcal{F}_m$ HYPERGRAPHS ON V

SYSTEM OF DISJOINT REPRESENTATIVES (SDR):

MATCHING $F_1 \in \mathcal{F}_1, \dots, F_m \in \mathcal{F}_m$

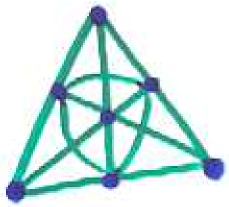
HYPERGRAPH MATCHING PROBLEM:

WHEN DOES $(\mathcal{F}_1, \dots, \mathcal{F}_m)$

HAVE AN SDR ?

$\nu(\mathcal{F})$ MATCHING NUMBER OF $\mathcal{F}$:

$\max |M| :\ M \subset \mathcal{F}$ MATCHING

$\mathcal{F} =$  $\nu(\mathcal{F}) = 1$

THM (AHARONI-HAXELL)

$\mathcal{F}_1, \dots, \mathcal{F}_m$ r-UNIFORM $\mathcal{F}_i \subset \binom{V}{r}$.

$$\nu\Big(\bigcup_{i \in I} \mathcal{F}_i\Big) \geq (|I|-1)r+1 \quad \forall I \subset [m]$$

$\Rightarrow \exists$ SDR

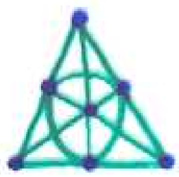
FRACTIONAL MATCHING OF $\mathcal{F}$:

$f : \mathcal{F} \to \mathbb{R}$, $f \geq 0$

$$\sum_{F \ni v} f(F) \leq 1 \qquad \forall v \in V$$

$\nu^*(\mathcal{F})$ FRACTIONAL MATCHING NUMBER OF $\mathcal{F}$:

$$\max \sum_{F \in \mathcal{F}} f(F) : f \text{ FRAC. MATCHING}$$

$\mathcal{F} =$ 

$f \equiv 1/3$ FRAC. MATCH.

$\nu^*(\mathcal{F}) = \frac{7}{3}$

THM (ABM)

$\mathcal{F}_1, \dots, \mathcal{F}_m$ r-UNIFORM $\mathcal{F}_i \subset \binom{V}{r}$

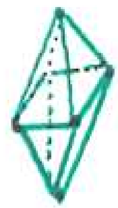
$$\nu^*\Big(\bigcup_{i \in I} \mathcal{F}_i\Big) > (|I|-1)r \qquad \forall I \subset [m]$$

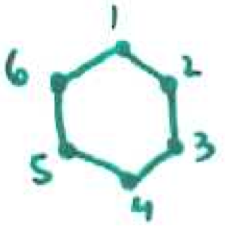
$\Rightarrow \exists$ SDR

$I(G)$ = INDEPENDENCE COMPLEX OF $G = X(\bar{G})$

VERTICES = V

SIMPLICES = IND. SUBSETS $\sigma \subset V$

$M_k =$  $I(M_k) =$  $\cong S^{k-1}$

$C_6 =$  $I(C_6) =$  $\approx \infty$

C_n = n-cycle $\quad I(C_n) \simeq \begin{cases} S^{k-1} & n = 3k \pm 1 \\ S^{k-1} \vee S^{k-1} & n = 3k \end{cases}$

THM (M): $\quad \tilde{H}_{k-1}(I(G)) \neq 0 \Rightarrow$

$f_i(I(G))$ = # i-simplices of $I(G)$

$\geq f_i(I(M_k)) = \binom{k}{i+1} \cdot 2^{i+1}$

e.g. $\quad f_0(I(G)) = |V| \geq 2k$

$\eta(X) = \min\{ i : \tilde{H}_i(X) \neq 0 \} + 1$ =

HOMOLOGICAL CONN. OF X + 2

- HALL TYPE THEOREMS FOLLOW FROM GOOD LOWER BOUNDS ON $\eta(I(G))$

- G IS "HARD TO DOMINATE" $\Rightarrow$ $\eta(I(G))$ IS LARGE

EXAMPLE:

$\tilde{\gamma}(G)$ = STRONG DOMINATION NUM. OF G =

$\min |S| \; : \; N(S) = V$

$\tilde{\gamma}($  $) = 2$

THM (M): $\eta(I(G)) \geq \dfrac{\tilde{\gamma}(G)}{2}$

VECTOR REPRESENTATION OF $G=(V,E)$:

$P: V \to \ell_2$ SUCH THAT

$$P(u)\cdot P(v) \geq \begin{cases} 1 & uv \in E \\ 0 & \text{else} \end{cases}$$

VALUE OF P :

$$|P| = \min \sum_{v \in V} \alpha(v)$$

(*) $\alpha(v) \geq 0$; $\sum_{v \in V} \alpha(v) P(v) \cdot P(u) \geq 1 \quad \forall u \in V$

VECTOR DOMINATION NUM. OF G =

$$\Gamma(G) = \max \{ |P| : \text{ALL REP. } P \text{ OF } G \}$$

$S \subset V$ STRONGLY DOMINATING SET

$\Rightarrow$ $\alpha = 1_S$ SATISFIES (*)

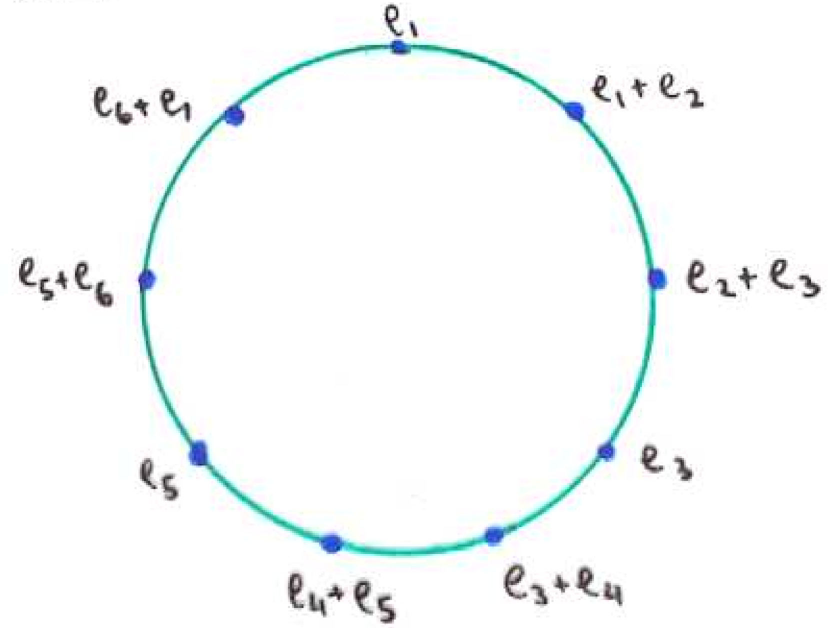
$\Rightarrow$ $\tilde{\gamma}(G) \geq \Gamma(G)$

THM (ABM):

$$\eta(I(G)) \geq \Gamma(G)$$

EXAMPLE:

$G =$



$$(e_1+e_3+e_5)\cdot P(u) \geq 1 \qquad \forall u$$

$$\Gamma(G) = |P| = 3$$

$$I(G) \approx S^2 \Rightarrow \eta(I(G)) = 3$$

MAIN POINTS IN PROOF:

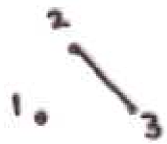
$G = (V, E)$, $V = [n]$

- $$\eta(I(G)) \geq \frac{n}{\lambda_n(G)}$$

$a = (a_1, \dots, a_n)$ $\quad a_i \geq 1 \quad \sum_i a_i = N$

$i \in V \rightsquigarrow$ IND. SET OF SIZE a_i

$G \rightsquigarrow G_a$

$G =$  $\quad G_{(2,3,2)} =$ 

$$I(G) \simeq I(G_a)$$

- $$\eta(I(G)) = \eta(I(G_a)) \geq \frac{N}{\lambda_N(G_a)}$$

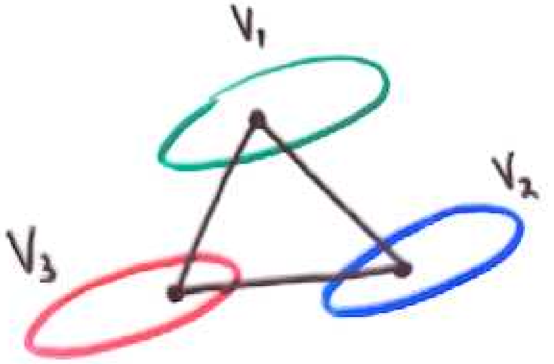
- $$\sup_a \frac{N}{\lambda_N(G_a)} \geq \Gamma(G)$$

X SIMPLICIAL COMPLEX ON V

$V_1 \cup \cdots \cup V_m$ PARTITION OF V

$\sigma \in X$ IS COLORFUL IF

$\forall \; 1 \le i \le m \quad |\sigma \cap V_i| = 1$



$\eta(Y) = \min\{ i : \tilde{H}_i(Y) \neq 0 \} + 1$

THM (AHARONI-HAXELL, M)

IF $\quad \forall \; I \subset [m]$

$$\eta\Big(X\Big[\bigcup_{i \in I} V_i\Big]\Big) \geq |I|$$

$\Rightarrow \; \exists$ COLORFUL SIMPLEX

$\mathcal{F}$ HYPERGRAPH

$\downarrow$

$G(\mathcal{F})$ INTERSECTION GRAPH

VERTICES = $\mathcal{F}$

EDGES = $\{F, F'\}$ $F \cap F' \neq \emptyset$

- $\mathcal{F}$ r-UNIFORM

$\Rightarrow$ $\Gamma(G(\mathcal{F})) \geq \dfrac{\nu^*(\mathcal{F})}{r}$

THM (ABM): $\mathcal{F}_1, \dots, \mathcal{F}_m$ r-UNIFORM

IF $\forall I \subset [m]$

$$\nu^*\Big(\bigcup_{i \in I} \mathcal{F}_i\Big) > (|I|-1)\, r$$

$\Rightarrow \exists$ SDR.

EQUIVALENTLY:

$I\Big(G\Big(\bigcup_{i=1}^{m} \mathcal{F}_i\Big)\Big)$ CONTAINS A COLORFUL SIMP. WRT THE PART. $\mathcal{F}_1 \cup \dots \cup \mathcal{F}_m$

$$\eta\Big(I\Big(G\Big(\bigcup_{i \in I} \mathcal{F}_i\Big)\Big)\Big) \geq \Gamma\Big(G\Big(\bigcup_{i \in I} \mathcal{F}_i\Big)\Big)$$

$$\geq \frac{\nu^*\Big(\bigcup_{i \in I} \mathcal{F}_i\Big)}{r} > |I|-1$$

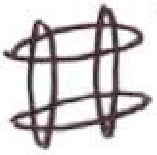
$\blacksquare$

$\mathcal{F}$ HYPERGRAPH

WIDTH $w(\mathcal{F})$ = MINIMAL t

$\exists$ $F_1,\dots,F_t \in \mathcal{F}$ SUCH THAT ANY $F \in \mathcal{F}$

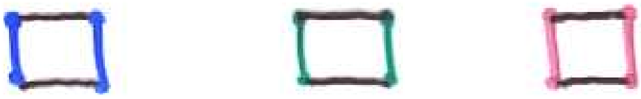
INTERSECTS SOME F_i

$w\Bigg($  $\Bigg) = 2$

THM [HAXELL]: $\mathcal{F}_1,\dots,\mathcal{F}_m$ HYPERGRAPHS

$\forall I \subset [m]$ $\quad w\Big(\bigcup_{i\in I}\mathcal{F}_i\Big) \geq 2|I|-1$

$\Rightarrow$ $\exists$ disjoint $F_1 \in \mathcal{F}_1, \dots, F_m \in \mathcal{F}_m$ SDR



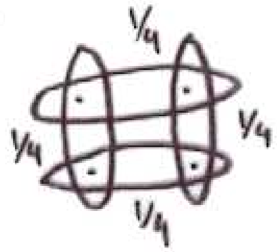
$w(\mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3 \cup \mathcal{F}_4) = 6$ $\nexists$ SDR

$\mathcal{F} = \{F_1, F_2, \dots\}$ HYPERGRAPH

FRACTIONAL WIDTH $w^*(\mathcal{F}) =$

$$\min \sum_i c_i$$

$$c_i \geq 0 \quad , \quad \sum_i c_i |F_i \cap F_j| \geq 1 \quad \forall j$$

$w^*\Big($  $\Big) = 1$

$$\Gamma(G(\mathcal{F})) \geq w^*(\mathcal{F}) \quad \Rightarrow$$

THM (ABM): $\mathcal{F}_1, \dots, \mathcal{F}_m$ HYPERGRAPHS

$$\forall I \quad w^*\Big(\bigcup_{i \in I} \mathcal{F}_i\Big) > |I| - 1$$

$\Rightarrow \exists$ SDR