IPAM Workshop: Automorphic Forms, Group Theory, and Graph Expansion (February 9-13, 2004)

See http://www.ipam.ucla.edu/schedule.aspx?pc=agg2004
for the list of workshop talks and some slides from the talks.

PROBLEM 1 (Avi Wigderson)

Prove or disprove:

(A) For all finite groups G and irreducible representations $\rho \in \widehat{G}$,

$$Pr_{x,y,z \in G}\Big\{|\rho(x) + \rho(y) + \rho(z)| \leq 2.999\Big\} \geq \frac{1}{2}$$

(B) S_n can be generated as expanders with n^ϵ generators.
Note: (A) implies $\epsilon \leq \frac{1}{2}$.

(C) If G has a constant set of expanding generators, then for all d,

$$\Big|\{\rho \in \widehat{G} : \dim \rho = d\}\Big| \leq \exp(d)$$

PROBLEM 2 (Alex Lubotzky)

(A) Find the correct definition of Ramanujan complex. The one currently used for buildings of type $A_n^{\sim}$ is based on the colors of the vertices but this does not take the higher-dimensional structure into explicit consideration. A different natural definition is to bound the eigenvalues of the higher-dimensional Laplacians and a third way is to require that every irreducible subrepresentation of $L^2(\Gamma\backslash G)$ with Iwahori-fixed vectors is tempered. What are the relations between these definitions?

(B) For a family of finite simple groups, study the diameter of their Cayley graphs with respect to random (resp., worst case) generators.

(C) Do we know any family of finite simple groups $\{G_i\}$ for which

$$\mathrm{diam}(\mathrm{Cay}(G_i, S_i)) = O(\log |G_i|)$$

with respect to any choice of generators S_i of g_i.

PROBLEM 3 (Salil Vadhan) Constructing explicit Expanders with near-optimal vertex expansion.

Fix a degree D and constants A, α. A family of expanders is an infinite family of D-regular graphs G such that for all $S \subset \text{Vertices}(G)$ such that $|S| \leq \alpha N$ ($N =$ number of vertices of G):

$$|\text{Neighbors}(S)| \geq AN$$

Bipartite version: $D =$ left-degree, consider only $S \subset \text{Left-Vertices}(G)$

Goal: maximize A as a function of D

Known:

- Random Graphs: $A = D - O(1)$
- Ramanujan Graphs: $A = D/2$ (but cannot do better [Kahale])
- Bipartite Graphs: explicit construction [Capalbo-Reingold-Vadhan-Wigderson, 02]: for all $\epsilon > 0$, $\exists\ D, \alpha$ and an explicit infinite family with $A = (1-\epsilon)D$.

Problems:

(A) Construct explicit non-bipartite expanders with $A > D/2$ for linear-sized sets. More precisely, show that there exists D, A, α where $A > D/2$ and $\alpha > 0$, and an infinite family $\{G_i\}$ of D-regular graphs with N_i vertices such that for all subsets of vertices S with $|S| \leq \alpha N_i$, we have $|\Gamma(S)| \geq A|S|$). (This is not even known for $D = \log^{O(1)} N$).

(B) Construct explicit expanders (bipartite or not) with $A = D - c$ with fixed c and arbitrarily large D. This is not even known for $D = \log^{O(1)} N$.

(C) Find an algebraic method that implies expansion $A > D/2$ (λ_2 does not suffice).

PROBLEM 4 (Nati Linial)

(A) Fix d. Can you construct two-dimensional complexes X_n with $n \to \infty$ with the property that for all $x \in \mathrm{Vert}(X_n)$, the link $lk(x)$ is a d-regular Ramanujan graph on $n-1$ vertices?

(B) A λ-Steiner triple system is a collection of triples $\{\sigma_i\}_{i \in I}$ such that for all $x, y \in V$, there are exactly d triples σ_i containing x and y. Can you generate such triples at random? Comment: by old work of Wilson, there are $\exp(\Omega(n^2))$ such system – see work of Cameron on a Markov Chain on such systems.

(C) Let $g(n,d)$ be the largest girth among all d-regular graphs with n vertices. Conjecture: there exists $\epsilon_0 > 0$ such that

$$g(n,d) < (2 - \epsilon_0)\frac{\ln n}{\ln(d-1)}$$

Can you decide this problem for Cayley graphs?

(D) (Bilu-Linial Conjecture) For every d-regular graph G, there is a signing of the adjacency matrix that has spectral radius $\leq 2\sqrt{d-1}$.

(E) The following holds (Linial and London): for every measurable set $X \subset \mathbf{R}^2$,

$$\mu(X \cup SX \cup TX) \geq \frac{4}{3}\mu(X)$$

where $S = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. The bound is tight and the proof is elementary (does not use eigenvalues). Does a similar statement hold modulo 1, that is, in $\mathbf{R}^2/\mathbf{Z}^2$? If so, this would be the first calculation of expansion that does not require eigenvalues.

PROBLEM 5 (Akos Seress) For a simple group G, let $\mu(G)$ be the degree of the smallest permutation representation.
Conjecture: $\exists$ an absolute constant C such that for all simple groups G and all generating sets S of G,

$$\mathrm{diam}(\mathrm{Cayley}(G,S)) < \mu(G)^C \tag{1}$$

This is weaker than the standard conjectures:

$$\mathrm{diam}(\mathrm{Cayley}(G,S)) < (\log|G|)^C$$

However, (1) would have an application to the conjecture: for any transitive subgroup $G \subset S_n$, $\mathrm{diam}(\mathrm{Cayley}(G,S)) < n^C$.

PROBLEM 6 (Sebastian Cioaba)

(A) Let G be the infinite graph whose vertices are the points in $\mathbf{R}^2$, where two points x, y are adjacent if $|x-y| = 1$. It is known that the chromatic number of this graph is between 4 and 7 (Chilakamari, Kirau, *Finite analogues of Euclidean space*, J. Comput. Appl. Math. 68 (1996), no. 1-2, 221-238, and Beuda-Perles *Coloring of metric spaces*, Geombinatorics 9 (2000), no. 3, 113-126). What is the chromatic number?

(B) Finite Euclidean graphs. Let $q = p^r$ with p and odd prime. Let G_q be the graph with vertex set $\mathbf{F}_q^2$, where again (x,y) is adjacent to (a,b) if

$$(x-a)^2 + (y-b)^2 = 1$$

Then G_q is the Cayley graph of $\mathbf{F}_q^2$ with generating set $S_p = \{(a,b) \in \mathbf{F}_q^2 : a^2+b^2 = 1\}$. This graph is $p\pm1$ regular (cf. Eric Moorhouse, *On the chromatic number of the Euclidean plane*, http://math.uwyo.edu/ moorhous/pub/chrom.pdf) and the non-trivial eigenvalues satisfy $\lambda \le 2\sqrt{p}$ (cf. Medrano-Myers-Stark-Terras, (*Finite Euclidean graphs over rings*, Bull. ICA 8 (1993), 39-60). Therefore the chromatic number $\chi(G_q)$ satisfies

$$\frac{1}{2}\sqrt{p} + O(1) \le \chi(G_q) \le p + O(1)$$

The left inequality follows from an old result of Hoffman and the right one is a triviality. Find $\chi(G_q)$.

PROBLEM 7 (Anton Dietmar) Is there an Ihara formula for the Ihara zeta-functions of building complexes of higher dimension? For rank one, the Ihara formula says:

$$Z(u)^{-1} = (1-u^2)^{-\chi} \det(1 - Au + u^2 q)$$

where χ is the Euler characteristic and A is the adjacency matrix. In the case $G = SL_3(\mathbf{Q}_p)$, there are two adjacency operators A_1, A_2 with zeta functions Z_1 and Z_2. Setting $Z(u) = Z_1(u)Z_2(u)$, the conjecture is

$$Z(u)^{-1} = (\text{elementary factor})^{-\chi} \det(1 - A_1 u + u^2 A_2 - u^3 q)$$

PROBLEM 8 (Peter Sarnak)

(A) Is the random 3-regular graph Ramanujan? Numerical simulation by T. Novikoff suggest (surprisingly) that it is Ramanujan with probability approximately .52 and more precisely, that the distribution of the second lowest eigenvalue is related to the Tracy-Widom distribution in random matrix theory (Gaussian orthogonal ensemble).

(B) Using the Brandt-Eichler-Shimura theory and the Kerchoff matrix tree theorem, can one show that for the Brandt graphs (described in the talk of Bruce Jordan), the number of spanning trees is essentially equal to the class number of the functions field $X_0(\ell)/\mathbf{F}_p$. Is there an explicit relation between these spanning trees and divisor classes?