Ramanujan Hypergraphs and Automorphic Forms

Winnie Li

Penn State Univ

§1. Spectral theory of regular graphs

X = k-regular connected undirected graph on n vertices

$A = A(X)$ = adjacency matrix of X

$Af(x) = \sum_{x \to y} f(y)$

Eigenvalues of A are the spectrum of X

$$k = \lambda_1 > \lambda_2 \geq \cdots \geq \lambda_n \geq -k.$$

$\lambda^+(X)$ = largest eigenvalue $< k$

$\lambda^-(X)$ = smallest eigenvalue $> -k$

Let $\{X_j\}$ be an infinite family of k-reg. graphs such that $|X_j| \to \infty$ as $j \to \infty$.

Study the behavior of $\lambda^{\pm}(X_j)$.

Thm (Alon - Boppana) $\liminf_{|X_j|\to\infty} \lambda^+(X_j) \geq 2\sqrt{k-$

- Lubotzky - Phillips - Sarnak :

  lower bound of $\lambda^+(X)^{2n}$

- Serre : $\forall \varepsilon>0, \quad \dfrac{\#\{\lambda : \lambda \geq (2-\varepsilon)\sqrt{k-1}\}}{|X|} \geq c$

  $c = c(\varepsilon, k)$

- Nilli: $\lambda^+(X) > 2\sqrt{k-1} - \dfrac{2\sqrt{k-1}-1}{d},$

  where $\operatorname{diam}(X) \geq 2d+2$

Thm (L.) Assume $\{X_j\}$ satisfy

(0) the length of shortest odd cycle in X_j tends to ∞ as $j\to\infty$.

Then

$$\limsup_{|X_j|\to\infty} \lambda^-(X_j) \leq -2\sqrt{k-1}.$$

Universal cover of k-reg graphs is

$\mathcal{T}_k =$ k-reg tree $\leftrightarrow \mathfrak{h}$

A $\leftrightarrow$ Laplacian

Spectrum $= [-2\sqrt{k-1}, 2\sqrt{k-1}]$ $\leftrightarrow [\frac{1}{4}, \infty)$

normalized measure on $[-2, 2]$

$$\mu_k = \frac{k}{\left(\sqrt{k-1} + \frac{1}{\sqrt{k-1}}\right)^2 - x} \mu_{ST}$$

with

$$\mu_{ST} = \frac{1}{\pi}\sqrt{1 - \frac{x^2}{4}}\, dx$$

If $k-1 = q =$ prime power, then

$$\mathcal{T}_k = PGL_2(F) / PGL_2(\mathcal{O}_F)$$

and $A = T =$ Hecke operator

<u>Def.</u> A k-reg. graph X is called Ramanujan if $|\lambda^{\pm}(X)| \leq 2\sqrt{k-1}$.

<u>Remark.</u> X Ramanujan $\Leftrightarrow$ $Z_X(s)$ satisfies R.H

§2. Spectral theory of hypergraphs

An n-hypergraph consists of a set of vertices and a set of hyperedges with each hyperedge being a simplex on n vertices so that X is an (n-1)-dim'l simplicial complex.

Assume

(P) (type) Each vertex x has a type $\tau(x) \in \mathbb{Z}/n\mathbb{Z}$ so that no two vertices of the same type are adjacent.

Define n-1 adjacency operators $A_i = A_i(X)$ for $i = 1, \cdots, n-1$ by

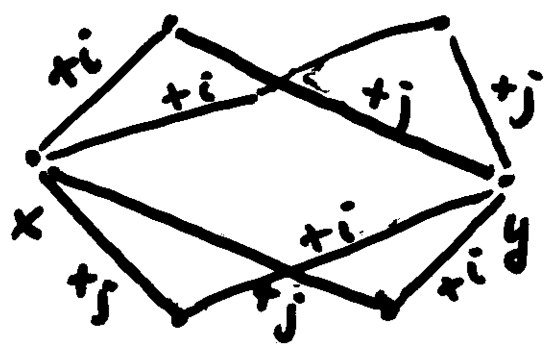
$$A_i f(x) = \sum_{\substack{y \text{ adj. to } x \\ \tau(y) - \tau(x) = i}} f(y).$$

(t) (transpose) A_i and A_{n-i} are transpose of each other.

(C) (commutativity) The operators A_i, $i=1,\cdots,n-$ commute.

$\Updownarrow$

(U) (uniformity) Given any pair of vertices x, y of X, for any $i, j \in \mathbb{Z}/n\mathbb{Z}$ such that $\tau(y) = \tau(x) + i + j$, the number of common nbhrs of x and y of type $\tau(x)+i$ is the same as those of type $\tau(x)+j$



(c) $\Rightarrow$

(d) (diagonalizability) The operators A_i are simultaneously diagonalizable.

An n-hypergraph is called $(q+1)$-regular if

(R) (regularity) For $i = 1, \cdots, n-1$, each vertex x has exactly

$$q_{n,i} := \frac{(q^n - 1) \cdot \cdots \cdot (q-1)}{(q^i - 1) \cdots (q-1)(q^{n-i}-1) \cdots (q-1)}$$

nbhrs of type $\tau(x) + i$.

$\therefore$ a $(q+1)$-regular 2-hypergraph is a $(q+1)$-regular bipartite graph.

F = nonarch. local field with q elements in its residue field (eg. $F = \mathbb{F}_q((T))$)

$\mathcal{O}_F$ = ring of integers in F ($\mathcal{O}_F = \mathbb{F}_q[[T]]$)

π = a uniformizer ($\pi = T$)

$\mathcal{B}_{n,F} := PGL_n(F)/PGL_n(\mathcal{O}_F)$ Bruhat-Tits building

Claim: $\mathcal{B}_{n,F}$ is a $(q+1)$-regular n-hypergraph

a vertex $x = g\,GL_n(\mathcal{O}_F)\,Z(F)$ represents the equiv. class of rank n lattices L_x over $\mathcal{O}_F$ with basis the columns of g.

$\det g = \pi^m \cdot$ unit, m is well-def. mod n.

$\tau(x) = m \mod n$.

Two vertices x, y of $\mathcal{B}_{n,F}$ are adjacent iff x and y can be represented by lattices L_x and L_y such that

$$L_x \supsetneq L_y \supsetneq \pi L_x \quad (\supsetneq \pi L_y).$$

$$[L_x : L_y] = q^i \iff \tau(y) = \tau(x) + i$$

Vertices $x_1, \cdots, x_n$ form a hyperedge iff $x_1, \cdots, x_n$ can be represented by lattices $L_1, \cdots, L_n$ so that

$$L_1 \supsetneq L_2 \supsetneq \cdots \supsetneq L_n \supsetneq \pi L_1 .$$

This verifies (P).

For $i = 1, \cdots, n-1$, A_i = Hecke operator T_i

$$T_i \leftrightarrow PGL_n(\mathcal{O}_F) \begin{pmatrix} \pi & & & & & \\ & \ddots & & & & \\ & & \pi & & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix} PGL_n(\mathcal{O}_F)$$

(with i entries equal to π)

$$= \coprod_{\alpha \in S_i} \alpha \, PGL_n(\mathcal{O}_F)$$

$$(T_i f)([g]) = \sum_{\alpha \in S_i} f([g\alpha]).$$

Fix the following choice of representatives in S_i:

$$\alpha = \begin{pmatrix} 1 & & & & & \\ & \pi & & * & * & * \\ & & \pi & * & * & * \\ & & & 1 & & \\ & 0 & & & 1 & \\ & & & & & 1 \\ & & & & & & \pi \end{pmatrix} = (a_{j\ell})$$

1) upper-triangular, entries in $\mathcal{O}_F$.

2) exactly i diagonal entries equal to π, remaining ones equal to 1

3) For $j < \ell$, $a_{j\ell} = \begin{cases} * \text{ in } \mathcal{O}_F/\pi\mathcal{O}_F & \text{if } a_{jj} = \pi \text{ and } a_{\ell\ell} = 1 \\ 0 & \text{otherwise} \end{cases}$

Write $a_{jj} = \pi^{w_j}$; call

$\underline{w} = (w_1, \ldots, w_n) \in \{0,1\}^n$ the type of α,

and $\{j : w_j = 1\}$ the support of α or $\underline{w}$.

Call α basic matrix of type $\underline{w}$.

$I_i = \{ \underline{w} : |\operatorname{supp} \underline{w}| = i \}$, $\# I_i = \binom{n}{i}$.

For each $\underline{w} \in I_i$, there are

$$q^{e(\underline{w})} = q^{in - i(i-1)/2 - w_1 - 2w_2 - \cdots - nw_n}$$

basic matrices in S_i of type $\underline{w}$.

The $\#$ of nbhrs of x of type $\tau(x) + i$

$$= |S_i| = \sum_{\underline{w} \in I_i} q^{e(\underline{w})} = q_{n,i} .$$

This proves (R).

(C) follows from (U) (easy)

$\mathcal{B}_{n,F}$ is contractible $\Rightarrow$ $\mathcal{B}_{n,F}$ is the universal cover of certain $(q+1)$-reg. n-hypergraphs.

Every vertex in $\mathcal{B}_{n,F}$ is represented by a finite product of basic matrices.

Another way to see regularity (R)

$L_1/\pi L_1 \cong \mathbb{F}_q^n$

A hyperedge corresponds to

$$\begin{array}{ccccccc} L_1/\pi L_1 & \supsetneq & L_2/\pi L_1 & \supsetneq \cdots \supsetneq & L_n/\pi L_1 & \supsetneq & \{0\}, \\ \wr\| & & \wr\| & & \wr\| & & \\ \mathbb{F}_q^n & & \mathbb{F}_q^{n-1} & & \mathbb{F}_q & & \end{array}$$

that is, a filtration of the n-dim'l vector space $\mathbb{F}_q^n$ into nested subspaces with codim 1 in the previous space.

# of nbhrs of $[L_1]$ with type difference

$= \#$ of subspaces of $\mathbb{F}_q^n$ of codim i

$$= \frac{\#\text{ of } l.\text{ indep. vectors } (v_1, \cdots, v_{n-i}) \text{ in } \mathbb{F}_q^n}{\#\text{ of } l.\text{ indep. vectors } (v_1, \cdots, v_{n-i}) \text{ in } \mathbb{F}_q^{n-i}}$$

$$= \frac{(q^n-1)(q^n-q)\cdots(q^n-q^{n-i-1})}{(q^{n-i}-1)(q^{n-i}-q)\cdots(q^{n-i}-q^{n-i-1})} = q_{n,i}$$

Spectrum of A_i on $\mathcal{B}_{n,F}$

Let

$$\sigma_i(z_1, \cdots, z_n) = \sum_{\underline{w} = (w_1, \cdots, w_n) \in I_i} z_1^{w_1} \cdots z_n^{w_n}$$

= ith elementary sym. poly in $z_1, \cdots$

Let

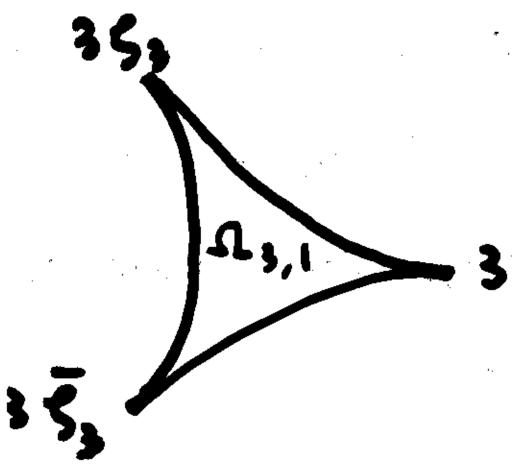
$$\Omega_{n,i} = \{ \sigma_i(z_1, \cdots, z_n) : z_1, \cdots, z_n \in S^1, \ z_1 \cdots z_n = 1 \}$$

The spectrum of A_i is $q^{i(n-i)/2} \, \Omega_{n,i}$.

Cartwright & Steger : the boundary of $\Omega_{n,i}$ is the curve traced by

$$\sigma_i(e^{\sqrt{-1}\theta}, \cdots, e^{\sqrt{-1}\theta}, e^{-\sqrt{-1}(n-1)\theta}), \quad 0 \le \theta \le 2\pi.$$

Eg. $n = 3$. $\Omega_{3,1} = \Omega_{3,2}$



A $(q+1)$-regular n-hypergraph X is said to have radius d if d is the largest integer m such that X contains a ball of radius m which is isomorphic to a ball of radius m in $\mathcal{B}_{n,F}$.

<u>Thm</u> Let $\{X_j\}$ be a family of finite $(q+1)$-regular n-hypergraphs such that the radius of X_j tends to ∞ as $j \to \infty$. Then for $1 \leq i \leq n$ the closure of the collection of eigenvalues of $A_i(X_j)$, $j \geq 1$, contains $q^{i(n-i)/2} \Omega_{n,i}$.

<u>Sketch of proof.</u>

Given $z_1, \cdots, z_n \in S^1$ with $z_1 \cdots z_n = 1$, write

$$\lambda_i = q^{i(n-i)/2} \sigma_i(z_1, \cdots, z_n).$$

ETS if X is a finite $(q+1)$-reg. n-hypergraph of radius $d+2$, then there is a fcn f on X such that for $1 \le i \le n-1$, $\langle A_i f - \lambda_i f, A_i f - \lambda_i f \rangle / \langle f, f \rangle \to 0$ as $d \to \infty$.

$\exists\ B_u(d+2)$ in $X \cong B(d+2)$ in $\mathcal{B}_{n,F}$.

$f = 0$ outside $B_u(d+1)$. w.m.a. $B_u(d+2) = B(d+2)$ centered at $[id]$.

Order elts in $\bigcup_{i=1}^{n-1} I_i$ so that

$\underline{w} = (w_1, \dots, w_n) > \underline{w}' = (w_1', \dots, w_n')$

if $\exists\ j$ s.t. $w_j > w_j'$ and $w_m = w_m'$ for $m > j$.

Among all $\underline{w} \in I_i$,

$$\underline{w}(i) = (\underbrace{1, \cdots, 1}_{i}, 0, \cdots, 0)$$

is the smallest in order and $q^{e(\underline{w}(i))} = q^{i(n-i)}$ is the largest. Call $\underline{w}(i)$ minimal.

<u>Lemma 1</u> $[g] \in \mathcal{B}_{n,F}$ of distance m to $[id]$ can be represented (in a unique way) by $M({}^1\underline{w}) \cdots M({}^m\underline{w})$ with ${}^1\underline{w} \geqslant {}^2\underline{w} \geqslant \cdots \geqslant {}^m\underline{w}$. Here $M(\underline{w})$ is a basic matrix of type $\underline{w}$.

Say $[g]$ of minimal type if each ${}^j\underline{w}$ is min'l. In this case, the support seq. of $[g]$ satisfies

$$\mathrm{Supp}\,{}^1\underline{w} \geq \mathrm{Supp}\,{}^2\underline{w} \geq \cdots \geq \mathrm{supp}\,{}^m\underline{w}$$

$$M(d) = \{ [g] \in B(d) \text{ of min'l type} \}$$
$\cup$
$$M^0(d) = \{ [g] \in M(d) \text{ whose supp seq contains}$$

$$\{1, 2, \cdots, n-1\} \supset \{1, \cdots, n-2\} \supset \cdots \supset \{1, 2\} \supset \{1\}$$

as a subseq $\}$, interior of $M(d)$

Set

$$a_j = q^{-j(n-j)/2} z_1 \cdots z_j \quad \text{for } j = 1, \cdots, n-1.$$

Given a type $\underline{w} = (w_1, \cdots, w_n)$, define

$$a(\underline{w}) = a_1^{w_1 - w_2} \, a_2^{w_2 - w_3} \cdots a_{n-1}^{w_{n-1} - w_n}.$$

Define $f = 0$ outside $M(d+1)$, and for

$[g] = [M({}^1\underline{w}) \cdots M({}^m\underline{w})] \in M(d+1)$ minimal

of dist m to $[Id]$,

$$f([g]) = a({}^1\underline{w}) \cdots a({}^m\underline{w}).$$

Lemma 2. For $[g] \in M^0(d)$, all nbhrs of $[g]$ lie in $M(d+1)$. Further, for $1 \le i \le n-1$,

$$(A_i f)([g]) = f([g]) \sum_{\underline{w} \in I_i} q^{e(\underline{w})} a(\underline{w}) = \lambda_i f([g]).$$

Therefore $A_i f - \lambda_i f$ is zero except on

(i) the boundary pts $M(d+1) \setminus M^0(d)$,

(ii) vertices in $B(d+2) \setminus M(d+1)$ which have some nbhrs in $M(d+1)$.

Finally, compute

$$\langle f, f \rangle = O(d^{n-1})$$

$$\langle A_i f - \lambda_i f, \; A_i f - \lambda_i f \rangle = O(d^{n-2}). \quad \text{Q.E.D.}$$

<u>Def</u>. A finite $(q+1)$-reg. n-hypergraph X is called Ramanujan if, for $1 \le i \le n-1$, all eigenval of $A_i(X)$ other than $g_{n,i}\, \zeta_n^m$, $1 \le m \le n$, fall in the region $q^{i(n-i)/2}\, \Omega_{n,i}$.

"Ramanujan complexes of type $\tilde{A}_n$"

Cartwright, Solé and Zuk

§3. Explicit constructions of R-graphs

(I) $K = \mathbb{Q}$ or function field (eg. $\mathbb{F}(t)$)

∞ = place of K (eg. $\frac{1}{t}$)

$v \neq \infty$ another place (eg. p if $K = \mathbb{Q}$
t if $K = \mathbb{F}(t)$)

H = quaternion alg. over K, unram. at v
and ram. at ∞

$D = H^\times / \text{center}$

$$X_{\mathcal{K}} = D(K) \backslash D(\mathbb{A}_K) / D(K_\infty) D(\mathcal{O}_v) \mathcal{K}$$

where $\mathcal{K}$ is a cpt open subgp of $\prod_{w \neq \infty, v} D(\mathcal{O}_w)$

strong approx. theory $\Rightarrow$

$$X_{\mathcal{K}} = \Gamma_{\mathcal{K}} \backslash D(K_v) / D(\mathcal{O}_v). \quad \Gamma_{\mathcal{K}} = D(K) \cap \mathcal{K}$$

$$= \Gamma_{\mathcal{K}} \backslash \underbrace{PGL_2(K_v) / PGL_2(\mathcal{O}_v)}_{\substack{\| \\ \mathcal{J}_{q+1}}}$$

(eg $q = p$; $|\mathbb{F}|$)

$A(D, \mathcal{K})$ = functions on $X_{\mathcal{K}}$

= auto. forms of $D(\mathbb{A}_K)$

= { consts, + maybe alternating const. } $\oplus A(D, \mathcal{K})$

$\updownarrow$ JL

certain cusp forms for $Gl_2(\mathbb{A}_K)$

RC holds $\Rightarrow$ nontrivial eigenvalues λ of $X_{\mathcal{K}}$ satisfy $|\lambda| \leq 2\sqrt{q} = 2\sqrt{k-1}$

(by Eichler - Shimura for $K = \mathbb{Q}$
Drinfeld for K = fcn field)

Lubotzky - Phillips - Sarnak / Margulis
$K = \mathbb{Q}$, H = Hamiltonian quat., $\rightsquigarrow \{X_{\mathcal{K}}\}$

Mestre - Oesterlé $K = \mathbb{Q}$, H_ℓ = quat. ram. at ∞ & $\ell \neq p$.

$\mathcal{K}$ = max'l cpt. $\rightsquigarrow \{X_\ell\}$

Morgenstern $K = \mathbb{F}(t)$, $q = |\mathbb{F}|$ odd $\rightsquigarrow \{X_{\mathcal{K}}\}$

$H = K \oplus Ki \oplus Kj \oplus Kij$, $i^2 = \delta$, $j^2 = t-1$
ram. at ∞ & $t-1$ $\qquad ij = -ji$

§4. Explicit construction of Ramanujan n-hypergraphs

$n \geq 3$. K = fcn field with a finite field of cons

Fix a place v, another place $\infty \neq v$.

q = cardinality of the residue field at v.

H = division alg def. over K of dim n^2.

tot. ram. at ∞ and unram. at v.

$D = H^\times /$ center

$$X_{\mathcal{K}} = D(K) \backslash D(\mathbb{A}_K) / D(K_\infty) D(\mathcal{O}_v) \mathcal{K}$$

$$= \Gamma_{\mathcal{K}} \backslash D(K_v) / D(\mathcal{O}_v)$$

$$= \Gamma_{\mathcal{K}} \backslash PGL_n(K_v) / PGL_n(\mathcal{O}_v)$$

$$= \Gamma_{\mathcal{K}} \backslash \mathcal{B}_{n, K_v}$$

finite $(q+1)$-reg. n-hypergraph

$A(D,\mathcal{H}) =$ fcns on $X_{\mathcal{H}}$

$= \{$consts, twists by $\zeta_n^{m}\} \oplus A(D,\mathcal{H})^{+}$

$\updownarrow$ "JL"

certain auto. forms for $GL_n(\mathbb{A}_K)$
($\Rightarrow$ cuspidal)

Lafforgue : RC true for GL_n

This would imply $X_{\mathcal{H}}$ Ramanujan.

JL correspondence is proved for $n=3$ by Jacque[s],
Piatetski-Shapiro & Shalika,
not settled for $n \geq 4$. composite, for $n=$ prime Kazhda[n]

Clozel suggests : use Laumon - Rapoport - Stuhler
π : infinite-dim'l irred. auto. rep'n of a
division alg H' over K

If π_w is Steinberg at some place w where
H' is unram. then RC holds for π.

Take H to be tot. ram. at ∞ and another place $\infty' \neq v, \infty$ such that $H(K_\infty)$ and $H(K_{\infty'})$ are opposite, and H also ram. at a finite nonempty set S of places. Let H' be another div. alg over K of deg n^2 ram. at S.

Let $D' = (H')^\times/$ center. (Go to next page)

Use trace formula for D, D' to show that

∞-dim'l irred. auto. repns σ occurring in $\mathcal{A}(D, \mathcal{K})^+$ lift to auto. repns σ' of $D'(\mathbb{A}_K)$.

Since σ at ∞ is trivial, σ' at ∞ is Steinb

$\therefore$ R.C. holds for σ'.

Thus $X_{\mathcal{K}}$ is Ramanujan.

<u>Thm</u> I. For $n \geq 3$ and prime power q, there exists an infinite family of finite $(q+1)$-reg. Ramanujan n-hypergraphs.

	∞	∞'	$w \neq \infty, \infty'$
D	D_∞	$D_{\infty'}$	D_w
			$\wr\!\shortparallel$
D'	$PGL_n(K_\infty)$	$PGL_n(K_{\infty'})$	D'_w

Badulescu:

Local JL map $\underset{\infty'}{\widehat{D}_{\infty}} \longrightarrow \underset{\infty'}{\widehat{PGL_n(K_\infty)}}$ exists

which sends $\begin{matrix}\text{trivial} \\ \text{rep'n}\end{matrix} \longmapsto \begin{matrix}\text{Steinberg} \\ \text{rep'n}\end{matrix}$

Using trace formula, one can show

<u>Thm</u> Given an infinite-dim'l adm. irred. auto. rep'n $\pi = \bigotimes_w' \pi_w$ of $D(\mathbb{A}_K)$, there exists an inf.-dim'l adm. irred. auto. rep'n $\pi' = \bigotimes_w' \pi'_w$ of $D'(\mathbb{A}_K)$ such that at $w = \infty, \infty'$, $\pi'_w = JL(\pi_w)$, and at $w \neq \infty, \infty'$, $\pi_w \cong \pi'_w$.

Explicit constructions with explicit generator

For Ramanujan graphs.

Lubotzky - Phillips - Sarnak / Morgenstern

Cayley graphs on $PGL_2(\mathbb{F}_q)$ or $PSL_2(\mathbb{F}_q)$

generators are from elements in $H(\mathbb{Z})$ of norm p / function field analogue

Terras graphs. norm graphs are quotients

Morgenstern graphs

For Ramanujan hypergraphs. similar constru.

Alireza Sarveniazi over $\mathbb{F}_q(t)$

He took division alg. ram. at only two places of dim d^2, obtained hypergraphs on $PGL_d(\mathbb{F}_{q^{dn}})$ and $PSL_d(\mathbb{F}_{q^{dn}})$ for $n \geq 1$.

(They are Ramanujan for $d=3$, prime)

Lubotzky, Samuels, Vishne have similar constructions.

"norm hypergraphs".

Let H be the division alg. over $\mathbb{F}_q(t)$ of dim d^2 and center $\mathbb{F}_q(t)$ given by

$$\mathbb{F}_{q^d}(t) + \mathbb{F}_{q^d}(t)\tau + \cdots + \mathbb{F}_{q^d}(t)\tau^{d-1},$$

where $\tau^d = t$ and $\tau\alpha = \alpha^q\tau$ for $\alpha \in \mathbb{F}_{q^d}$.

Then H is ram. at ∞ with invariant $-\frac{1}{d}$

t $\frac{1}{d}$

and unram. elsewhere.

Let $N_{d,a} = \{\alpha \in \mathbb{F}_{q^d} : N_{\mathbb{F}_{q^d}/\mathbb{F}_q}(\alpha) = a\}$.

Facts

- Up to multiplication by elements in $N_{d,1}$, $1-\alpha\tau$, $\alpha \in N_{d,1}$, are the elements in H of reduced norm $1-t$

- Regard a poly $\sum a_i \tau^i$ over $\mathbb{F}_{q^d}$ as an $\mathbb{F}_q$-linear map on $\mathbb{F}_{q^d}$ by sending x to $\sum a_i x^{q^i}$, that is, regarding τ as the Frob. map on $\mathbb{F}_{q^d}$. Given a complete filtration

$$0 \subset \{v_1\} \subset \langle v_1, v_2 \rangle \subset \cdots \subset \langle v_1, \cdots, v_d \rangle = \mathbb{F}_{q^d}$$

of $\mathbb{F}_{q^d}$ with v_i perpendicular to $v_1, \cdots, v_{i-1}$, we see that $\langle v_1 \rangle$ is the kernel of $1 - v_1^{1-q}\tau$, $\langle v_1, v_2 \rangle$ is the kernel of

$$\Big(1 - \underbrace{(v_2 - v_1^{1-q} v_2^{q})^{1-q}}_{\alpha_2}\tau\Big)\Big(1 - \underbrace{v_1^{1-q}}_{\alpha_1}\tau\Big),$$

. . .

$\langle v_1, v_2, \cdots, v_i \rangle$ is the kernel of

$(1-\alpha_i \tau)(1-\alpha_{i-1}\tau) \cdots (1-\alpha_1 \tau)$

with unique $\alpha_1, \cdots, \alpha_i \in N_{d,1}$ and for $i = 1, \cdots, d$.

Since the product

$(1-\alpha_d \tau)(1-\alpha_{d-1}\tau) \cdots (1-\alpha_1 \tau)$

is a poly of deg d in τ with const. 1 which vanishes on $\mathbb{F}_{q^d}$, it is equal to $1-\tau^d$.

Sarveniazi showed that: there is a 1-1 correspondence between the factorization $1-\tau^d = (1-\alpha_d \tau) \cdots (1-\alpha_1 \tau)$ with $\alpha_i \in N_{d,1}$ and the complete filtrations of $\mathbb{F}_{q^d}$.

At the place $t-1$, the Hecke operators $T_1, \cdots, T_{d-1}$ are given by the distinct factors $(1-\alpha_1\tau)$, $(1-\alpha_2\tau)(1-\alpha_1\tau)$, $\cdots$, $(1-\alpha_{d-1}\tau) \cdots (1-\alpha_1\tau)$ of $1-\tau$

Sarveniazi showed:

Let $f(t)$ be an irred. poly over $\mathbb{F}_q$ of $\deg = dn$. Then the group gen. by $1 - \alpha\tau$, $\alpha \in N_{d,1}$, mod $f(t)$ is isom. to

$PGL_n(\mathbb{F}_{q^{dn}})$ if $1-t$ is not a d-th power mod $f(t)$

$PSL_n(\mathbb{F}_{q^{dn}})$ if is

They give rise to $(q+1)$-reg. d-hypergp.

They are Ramanujan for $d=3$, prime or whenever JL corresp. for the div. alg $H^\times$/center holds.

$$\alpha\tau \leftrightarrow \begin{pmatrix} 1 & \alpha & & & \\ & 1 & \alpha^q & & \\ & & \ddots & \ddots & \\ & & & & \alpha^{q^{d-2}} \\ \alpha^{q^{d-1}} t & & & & 1 \end{pmatrix}$$

Let $D = H^\times/\text{center}$ and consider

$$\tilde{X} = D(K) \backslash D(A_K) / D(K_\infty)(1+P_t^d) \prod_{w \neq \infty, t} D(\mathcal{O}_w).$$

The elements in $\tilde{X}$ can be represented by

$$1 + a_1\tau + \cdots + a_{d-1}\tau^{d-1}, \quad a_1, \ldots, a_{d-1} \in \mathbb{F}_{q^d},$$

with respect to the same product rule and $\tau^d = 0$.

$\tilde{X}$ yields a $(q+1)$-regular d-hypergraph X_d with vertex set d copies of $\tilde{X}$, marked by $(\tilde{X}, 1), \ldots, (\tilde{X}, d)$, with the second component denoting the type. Elements $(x_1, 1), (x_2, 2), \ldots, (x_d, d)$ form a hyperedge if and only if there is a factorization

$$1 - \tau^d = (1 - \alpha_d \tau) \cdots (1 - \alpha_1 \tau) \quad \text{with } \alpha_j \in N_{d,1}$$

such that

$$x_2 = (1 - \alpha_1 \tau)\, x_1$$

$$x_3 = (1 - \alpha_2 \tau)(1 - \alpha_1 \tau)\, x_1 = (1 - \alpha_2 \tau)\, x_2$$

$$\vdots$$

$$x_1 = (1 - \alpha_d \tau) \cdots (1 - \alpha_1 \tau)\, x_1 = (1 - \alpha_d \tau)\, x_d$$

In other words, nbhrs of (x, i) with type difference j are

$$\big((1 - \alpha_j \tau) \cdots (1 - \alpha_1 \tau)\, x,\ i + j \bmod d\big),$$

where $(1 - \alpha_j \tau) \cdots (1 - \alpha_1 \tau)$ runs thru' all elements occurring in T_j.

For example, when $d = 3$, X_3 is a $(8+1)$-reg. Ramanujan 3-hypergraph since rep'ns of the division alg of deg 9 do correspond to rep'ns of GL_3 via the converse thm so that one may apply Lafforgue.

Note that X_3 mod $1+\mathcal{P}_{t-1}^2$ yields another $(q+1)$-reg. Ramanujan hypergraph X_3' which can be described more easily:

The vertex set is $(\mathbb{F}_{q^3}, 1) \cup (\mathbb{F}_{q^3}, 2) \cup (\mathbb{F}_{q^3}, 3)$ with the send component marking the type; the nbhrs of type diff. 1 of (x, i) are

$(x+\alpha, i+1 \bmod 3)$, $\alpha \in N_{3,1}$,

the nbhrs of type diff 2 of (x, i) are

$(x+\beta, i+2 \bmod 3)$, $\beta \in N_{3,-1}$.

The Ramanujan property of X_3' can be verified not using Lafforgue and repn theory. The nontrivial eigenvalues of A_1, up to twists by ζ_3, are the generalized Kloosterman sums

$$Kl_3(\mathbb{F}_q, a) := \sum_{z \in N_{3,a}} \psi_p \circ Tr_{\mathbb{F}_{q^3}/\mathbb{F}_p}(z) \quad , \quad a \in \mathbb{F}_q^{*}$$

$$= kl_3(\mathbb{F}_q, a) := \sum_{x_1, x_2 \in \mathbb{F}_q^{*}} \psi_p \circ Tr_{q/p}\left(x_1 + x_2 + \frac{a}{x_1 x_2}\right)$$

and Deligne showed that

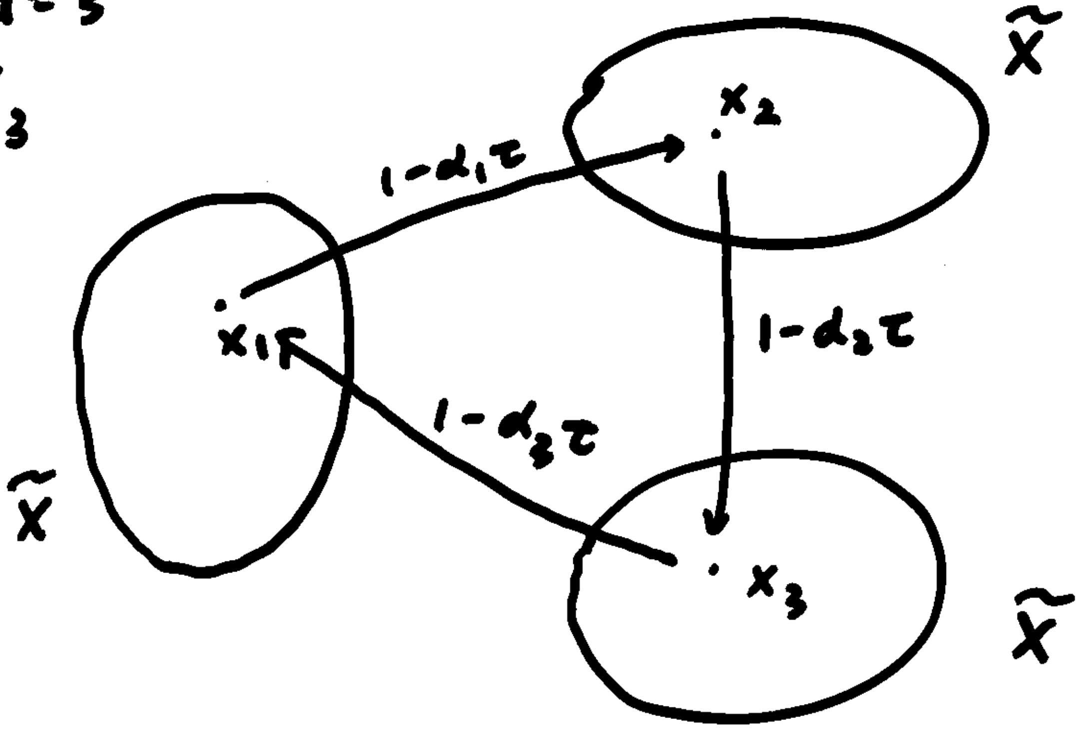
$$kl_3(\mathbb{F}_q, a) = (z_1 + z_2 + z_3)\, q$$

for some $z_1, z_2, z_3 \in S^1$ and $z_1 z_2 z_3 = 1$.

The eigenvalues of A_2 are the conjugates of those of A_1.

$d=3$

X_3



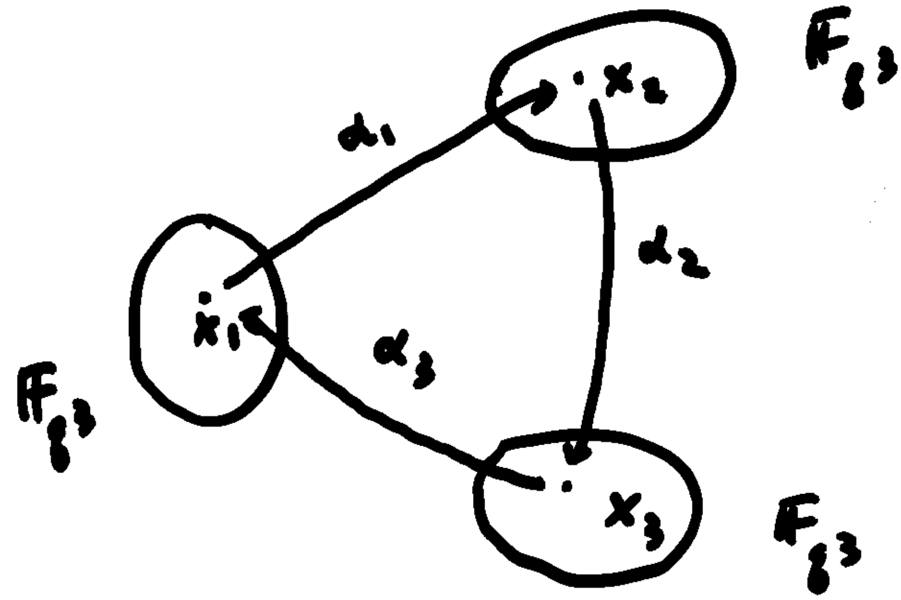
$$(1-\alpha_3\tau)(1-\alpha_2\tau)(1-\alpha_1\tau) = 1-\tau^3 = 1 \quad \text{on } \tilde{X}.$$

In terms of matrices,

$$1 + a\tau + b\tau^2 \longleftrightarrow \begin{pmatrix} 1 & a & b \\ b^q & 1 & a^q \\ 0 & 0 & 1 \end{pmatrix}, \quad a, b \in \mathbb{F}_{q^3}$$

Ramanujan by converse thms and Lafforgue

X_3'



$$\alpha_1 + \alpha_2 + \alpha_3 = 0, \quad \alpha_i \in N_{3,1}$$

Ramanujan since the nontrivial eigenvalues of A_1 are generalized Kloosterman sums.

$kl_3(\mathbb{F}_q, a)$, $a \in \mathbb{F}_q^\times$.

Deligne showed

$$kl_3(\mathbb{F}_q, a) = (z_1 + z_2 + z_3)\, q,$$

where $z_1, z_2, z_3 \in S^1$ with $z_1 z_2 z_3 = 1$.