Ramanujan Complexes of Type $\tilde{A}_d$

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GL_2: Ramanujan Graphs

Def: X - k-regular graph

X is Ramanujan $\iff$ every λ of the adjacency operator A satisfies $\lambda = \pm k$ or $|\lambda| \le 2\sqrt{k-1}$.

K - global field

$K_{v_0} = F$

$\mathcal{O}_{v_0} = \mathcal{O}$ ring of integers

$\mathcal{B}_2 = PGL_2(F) / PGL_2(\mathcal{O})$

$G = PGL_2(F)$ acts on $\mathcal{B}_2$

$\Gamma \leq G$ discrete, cocompact

Prop. $\Gamma \curvearrowright \mathcal{B}_2$ is Ramanujan $\Longleftrightarrow$

every irr. spherical infinite dim. subrep ρ of $L^2(\Gamma \backslash G)$ is tempered.

Construction of Γ for Γ^{B_2} Ramanujan

D - quaternion algebra over K ramifies at T

$G' = D^* / Z(D^*)$ $\quad v_0 \notin T$

$R_0 = \{ x \in K : v(x) \geq 0 \quad \forall v \neq v_0 \}$

$I \triangleleft R_0$

$\Gamma = G'(R_0)$

$\Gamma(I) = \mathrm{Ker}\,(G'(R_0) \longrightarrow G'(R_0/I))$

Thm $\quad \Gamma(I)^{B_2}$ are Ramanujan

Proof:

show $\rho < L^2(\Gamma \backslash G(F))$ tempered

① Strong Approximation
(from local to global)

$$\exists \ \pi' = \otimes' \pi_v' < L^2(G'(k) \backslash G'(\mathbb{A}_k))$$

such that $\pi'_{v_0} = \rho$

Note - if ρ is infinite dim.
then π' is inf. dimensional

② Jacquet-Langlands Correspondence

$$JL: \pi' \longrightarrow \pi < L^2_{disc}\left({}_{PGL_2(k)}\backslash^{PGL_2(\mathbb{A})} \right)$$

such that $\pi'_v = \pi_v \quad \forall v \notin T$

Note - $L^2\left({}_{PGL_2(k)}\backslash^{PGL_2(\mathbb{A})} \right) = L^2_{disc} \oplus L^2_{cont}$

③ Ramanujan Conjecture for GL_2
(Deligne, Drinfeld)

$\pi = \bigotimes' \pi_v$ irr. infin. dimensional (cuspidal) $< L^2(GL_2(k) \backslash GL_2(\mathbb{A}))$

(Assume π_∞ is square integrable)
Then π_v is tempered $\forall v$.

$\rho < L^2(\Gamma \backslash G(F))$

① $\pi' < L^2(G'(k) \backslash G'(\mathbb{A}))$ $\pi'_{v_0} = \rho$

② $\pi < L^2(PGL_d(k) \backslash PGL_d(\mathbb{A}))$ $\pi_{v_0} = \pi'_{v_0} = \rho$

③ $\pi_{v_0} = \rho$ is tempered

PGL_d

$$\mathcal{B}_d = PGL_d(F) / PGL_d(\mathcal{O})$$

$A_1, \ldots, A_{d-1}$ - colored adjacency operators

Def: $\Gamma \backslash \mathcal{B}_d$ is Ramanujan

if $\mathrm{Spec}_{\Gamma \backslash \mathcal{B}_d} (A_1 \ldots A_{d-1})$

$\subseteq \mathrm{Spec}_{\mathcal{B}_d} (A_1 \ldots A_{d-1})$

i.e.

$A_k f = \lambda_k f \qquad \forall k$

$\lambda_k = q^{k(d-k)/2} \sigma_k(z_1, \ldots, z_d)$

$|z_i| = 1 \qquad z_1 \cdots z_d = 1$

σ_k - k^{th} elem. symm function

Prop: $\Gamma \backslash \mathcal{B}_d$ is Ramanujan

$\Longleftrightarrow$ every irr. spherical inf. dim subrep ρ of $L^2(\Gamma \backslash PGL_d(F))$ is tempered

Construction of Γ for $\Gamma \backslash \mathcal{B}_d$ Ramanujan

D - division algebra of deg d over K

ramifies at T with invariants $\frac{a}{d}$: $(a,d)=1$

$v_0 \notin T$

$G' = D^* / Z(D^*)$

$\Gamma = G'(R_0)$, $\Gamma(I)$

Conjecture: [CSZ]

$\Gamma(I) \backslash \mathcal{B}_d$ are Ramanujan

Show $\rho < L^2(\Gamma(I) \backslash PGL_d(F))$ tempered

① Strong Approximation ✓

② Jacquet - Langlands Corr.

$$JL: \pi' \longrightarrow \pi < L^2_{disc}(PGL_d(k) \backslash PGL_d(\mathbb{A}))$$

$$\pi'_v = \pi_v \quad \forall v \notin T$$

Note: $L^2(PGL_d(k) \backslash PGL_d(\mathbb{A})) = L^2_{disc} \oplus L^2_{cont}$

$$L^2_{disc} = L^2_{cusp} \oplus L^2_{res.}$$

③ Ramanujan Conjecture GL_d

(Lafforgue) char $k > 0$

$$\pi < L^2_{cusp}(GL_d(k) \backslash GL_d(\mathbb{A}))$$

then π_v is tempered

~~Non-Ramanujan complexes~~

Theorem 14 (Mœglin and Waldspurger). *The residual spectrum of $L^2(\mathrm{GL}_d(k)\backslash\mathrm{GL}_d(\mathbb{A}))$ is composed of the unique irreducible sub-representations of*

$$\tilde{M}_s(\pi) = Ind_{P_s(\mathbb{A})}^{\mathrm{GL}_d(\mathbb{A})}(|det|^{\frac{1-s}{2}}\pi \oplus \cdots \oplus |det|^{\frac{s-1}{2}}\pi)$$

where s is a proper divisor of d and π is a cuspidal representation of $\mathrm{GL}_{d/s}(\mathbb{A})$.

Cor: When d is prime, L^2_{res} is composed of one-dim. representations

- ~~The discrete spectrum of automorphic representations of $GL_d(\mathbb{A})$ is composed of cuspidal representations and residual representations.~~
- ~~All one dimensional representations are residual. When d is prime, these are the only residual representations.~~

Our First Main Theorem

Assume char $F > 0$.

Theorem 1. *Let d be prime. For any ideal $I \triangleleft R_0$, let $\Gamma(I) = \mathrm{Ker}(G'(R_0) \rightarrow G'(R_0/I))$, a congruence subgroup of inner type. Then $\Gamma(I) \backslash \mathcal{B} = \Gamma(I) \backslash PGL_d(F)/PGL_d(O)$ is a Ramanujan complex.*

- Strong Approximation
- Jacquet-Langlands Correspondence
- Ramanujan Conjecture

Thm 2. For any d, if I is prime to at least one $\sigma \in T$, then

$$\Gamma(I)\backslash B_d$$ is Ramanujan.

Proof: show $\rho < L^2(\Gamma(I)\backslash PGL_d(F))$ is tempered

① SA $\pi' < L^2(G'(k)\backslash G'(\mathbb{A}))$

$\pi'_{v_0} = \rho$ $\quad$ π'_σ - 1dim. character

② JL Corr. $\pi < L^2_{disc}(PGL_d(k)\backslash PGL_d(\mathbb{A}))$

$\pi_{v_0} = \pi'_{v_0} = \rho$ $\quad$ π_σ - tempered

$\Rightarrow$ π is cuspidal

③ Ramanujan Conj.

$\pi_{v_0} = \rho$ is tempered

Thm 3. If d is not prime, $\Gamma(I) \backslash B$ is not Ramanujan for infinitely many I.

Proof.

- show $\exists\, \rho < L^2(\Gamma(I) \backslash PGL_d(F))$ which is not tempered
- Construct cuspidal $\pi < L^2(PGL_{d/s}(k) \backslash PGL_{d/s}(\mathbb{A}))$
  unramified at π_{v_0}
  supercuspidal at π_α $\alpha \in T$
- $\tilde{M}_s(\pi) < L^2_{res}(PGL_d(k) \backslash PGL_d(\mathbb{A}))$
  $\tilde{M}_s(\pi)_v$ - not tempered
- JL corr. $\exists \pi' < L^2(G'(k) \backslash G'(\mathbb{A}))$
  some D, ramified at T
  $\pi'_{v_0} = \tilde{M}_s(\pi)_{v_0}$
- SA $\pi'_{v_0} = \rho < L^2(\Gamma(I) \backslash PGL_d(F))$
  for some $\Gamma(I)$
- ρ not tempered