Ramanujan Complexes of Type $\tilde{A}_d$

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GL₂: Ramanujan Graphs

**Def:** $X$ - $k$-regular graph

$X$ is Ramanujan $\iff$

every $\lambda$ of the adjacency operator $A$ satisfies $\lambda = \pm k$
or $|\lambda| \leq 2 \sqrt{k-1}$. 
$K$ - global field
$K^0 = F$
$O^0 = O$ ring of integers

$B_2 = \text{PGL}_2(F) / \text{PGL}_2(O)$

$G = \text{PGL}_2(F)$ acts on $B_2$

$\Gamma \leq G$ discrete, cocompact

Prop. $\Gamma \backslash B_2$ is Ramanujan $\iff$

every infinite dim. spherical subrep $\pi$ of $L^2(\pi \mathbb{G})$ is tempered.
Construction of \( \Gamma \)
for \( n B_z \) Ramanujan

\[ D - \text{quaternion algebra over } k \]
\[ G' = D^*/Z(D^*) \]
\[ R_0 = \{ x \in k^* : v(x) > 0 \text{ \( \forall v \neq v_0 \) } \}
\[ I A R_0 \]
\[ \Gamma = G'(R_0) \]
\[ \Gamma(I) = \ker (G'(R_0) \rightarrow G'(R_0/I)) \]

\text{Thm} \quad \Gamma(I) B_z \text{ are Ramanujan}
Proof:

show \( p < L^2(\mathfrak{m}^\mathfrak{G}(F)) \) tempered

1. Strong Approximation
   (from local to global)

   \[
   \exists \quad \Pi' = \otimes' \Pi'_{v'} < L^2 \left( G'(k) \left| G'(A_k) \right. \right)
   \]

   such that \( \Pi' \Pi'_{v_0} = p \)

Note: if \( p \) is infinite dim.
then \( \Pi' \) is inf. dimensional
Jacquet–Langlands Correspondence

\[ \mathcal{JL}: \Pi' \mapsto \Pi < L^2_{\text{disc}} \left( \frac{PGL_2(A)}{PGL_2(k)} \right) \]

such that \[ \Pi'_v = \Pi_v \quad \forall v \in T \]

Note: \[ L^2 \left( \frac{PGL_2(A)}{PGL_2(k)} \right) = L^2_{\text{disc}} \oplus L^2_{\text{cont}} \]
Ramanujan Conjecture for $GL_2$

$(\text{Deligne, Drinfeld})$

$\mathcal{H} = \otimes_v \mathcal{H}_v$ is a finite-dimensional (cuspidal) $L^2(\mathbb{A}_{GL_2}(\mathbb{A}))$

(Assume $\mathcal{H}_\infty$ is square integrable)

Then $\mathcal{H}_v$ is tempered $\mathbb{A}_v$

$p \leq L^2(\mathfrak{p} \mathfrak{g}(F))$

1. $\mathcal{H}' \leq L^2(\mathbb{G}(\mathbb{A})^\prime, \mathfrak{g}(\mathfrak{a}))$ $\mathcal{H}_0' = p$

2. $\mathcal{H} \leq L^2(\mathbb{PGL}_d(\mathbb{A}))$ $\mathcal{H}_0 = \mathcal{H}'_0 = p$

3. $\mathcal{H}_0 = p$ is tempered
\( PGL_d \)

\[ B_d = \frac{PGL_d(F)}{PGL_d(O)} \]

\( A_1, \ldots, A_d \) - colored adjacency operators

**Def:** \( \Pi \backslash B_d \) is Ramanujan if

\[ \text{Spec}_{B_d} (A_1, \ldots, A_d) \leq \text{Spec}_{B_d} (A_1, \ldots, A_d) \]

i.e.

\[ A_k f = \mathcal{L}_k f \quad \forall f \]

\[ \mathcal{L}_k = 2^{k(d-k)/2} \sigma_k (z_1, \ldots, z_d) \]

\[ |z_i| = 1 \quad z_1, \ldots, z_d = 1 \]

\( \sigma_k \) - \( k \)-th elem. symm function
Prop: $\mathcal{P}\backslash \mathcal{B}_d$ is Ramanujan

$\iff$ every irr. spherical int. dm subrep $\pi$ of $L^2(\mathcal{P}\backslash \text{PGl}_d(F))$ is tempered.
Construction of $\Gamma$

for $\mathfrak{p} / \mathcal{B}_d$ Ramarajan

$D$ - division algebra of degree $d$ over $K$

ramifies at $T$ with invariants $\frac{a}{d} : (a, d) = 1$

$v_0 \not\in T$

$G^* = D_e / Z(D_e)$

$\Gamma = G^*(R_0)$, $\Gamma (I)$

Conjecture: [CSZ]

$\Gamma (I) / \mathcal{B}_d$ are Ramanujan
Show \( \rho < L^2 (\mathfrak{p}_{\mathbb{A}} \backslash \mathbb{PGL}_d (F)) \) tempered

1. Strong Approximation \( \checkmark \)


\[ \text{JL: } \pi^! \rightarrow \pi < L^2_{\text{disc}} (\mathbb{PGL}_d (\mathbb{A})) \]

\[ \pi_v^! = \pi_v \quad \forall v \notin \mathcal{T} \]

\( \text{Note: } L^2 (\mathbb{PGL}_d (\mathbb{A})) = L^2_{\text{disc}} \oplus L^2_{\text{cont}} \)

\[ L^2_{\text{disc}} = L^2_{\text{cusp}} \oplus L^2_{\text{res.}} \]

3. Ramanujan Conjecture GL\(_d\) (Lafforgue) \( \text{char } k > 0 \)

\[ \pi^! < L^2_{\text{cusp}} (\mathbb{GL}_d (\mathbb{A})) \]

then \( \pi_v \) is tempered
Theorem 14 (Möeglin and Waldspurger). The residual spectrum of $L^2(\text{GL}_d(k) \backslash \text{GL}_d(\mathbb{A}))$ is composed of the unique irreducible sub-representations of

$$\tilde{M}_s(\pi) = \text{Ind}_{P_s(\mathbb{A})}^{\text{GL}_d(\mathbb{A})} (|\det|^{\frac{1-s}{2}} \pi \oplus \cdots \oplus |\det|^{\frac{s-1}{2}} \pi)$$

where $s$ is a proper divisor of $d$ and $\pi$ is a cuspidal representation of $\text{GL}_{d/s}(\mathbb{A})$.

**Cor:** When $d$ is prime, $L^2_{\text{res}}$ is composed of one-dim. representations.
• The discrete spectrum of automorphic representations of $GL_d(A)$ is composed of cuspidal representations and residual representations.

• All one dimensional representations are residual. When $d$ is prime, these are the only residual representations.

Our First Main Theorem
Assume $\text{char } F > 0$.

Theorem 1. Let $d$ be prime. For any ideal $I \triangleleft R_0$, let $\Gamma(I) = \text{Ker}(G'(R_0) \to G'(R_0/I))$, a congruence subgroup of inner type. Then $\Gamma(I) \backslash B = \Gamma(I) \backslash PGL_d(F)/PGL_d(O)$ is a Ramanujan complex.

• Strong Approximation
• Jacquet-Langlands Correspondence
• Ramanujan Conjecture
Thm 2. For any \( d \), if \( I \) is prime to at least one \( \alpha \in T \), then \( \Gamma(X) \backslash \mathcal{B}_d \) is Ramanujan.

Proof: show \( \pi \leq L^2((\mathfrak{P}_{\alpha})^{PGL_d}(F)) \)

is tempered

1. SA \( \pi' \leq L^2(G'(k) \backslash G'(A)) \)
   \( \pi'_{v_0} = \pi \)
   \( \pi'_{v_0} \) - Idm. character

2. JL Corr. \( \pi \leq L^2_{disc}(\mathfrak{P}_{\alpha}(A)) \)
   \( \pi_{v_0} = \pi'_{v_0} = \pi \)
   \( \pi_{v_0} \) - tempered

\( \Rightarrow \) \( \pi \) is cuspidal

3. Ramanujan Corj.
   \( \pi_{v_0} = \pi \) is tempered
Thm 3. If \( d \) is not prime, 
\( r(\mathfrak{a})/B \) is not Ramanujan for infinitely many \( \mathfrak{a} \).

Proof. • show \( \exists \ p < L^2 (\text{PGL}_d (F)) \) 
which is not tempered
• Construct cuspidal \( \pi \) \( < L^2 (\text{PGL}_d (A)) \)
  unramified at \( \pi v_0 \) 
  supercuspidal at \( \pi v_0 \) \( \in \mathcal{D} \in T \)
• \( \tilde{\text{M}}_s (\pi) < L^2 (\text{res PGL}_d (A)) \)
  \( \tilde{\text{M}}_s (\pi) \) - not tempered
• JL corr. \( \exists \pi' \) \( < L^2 (G' (H) \backslash G' (A)) \) 
  some \( D \), ramified at \( T \) 
  \( \pi' v_0 = \tilde{\text{M}}_s (\pi) v_0 \)
• SA \( \pi' v_0 = P < L^2 (\text{PGL}_d (F)) \) 
  for some \( \mathfrak{a} (\Sigma) \)
• \( P \) not tempered