Shape functions for Empirical-Mode type decompositions.

Ingrid Daubechies

Hau-Tieng Wu

1. Introduction :

   Empirical-mode type decompositions.

2. "Synchrosqueezing" as a tool to obtain EMTD.

3. Shape functions.

- Jianfeng Lu, Hautieng Wu & ID
- Hautieng Wu (& collaborators)

$$f(t) = \sum_k A_k \cos \omega_k t$$

Fourier series

quasi-harmonic series

Very standard! But not always appropriate...

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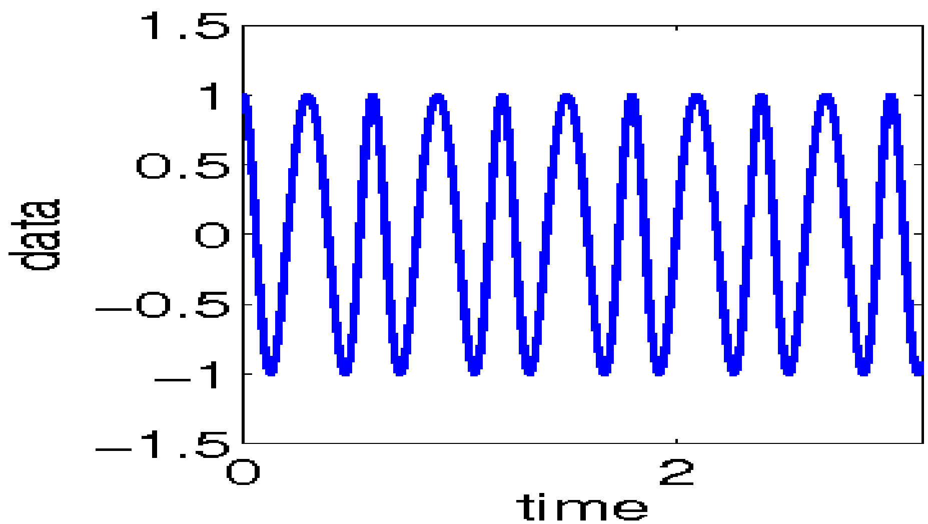

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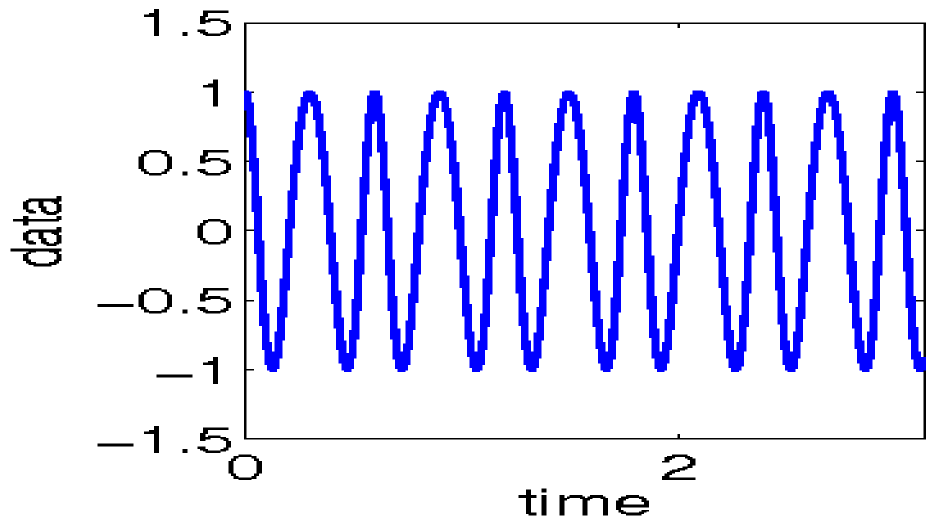

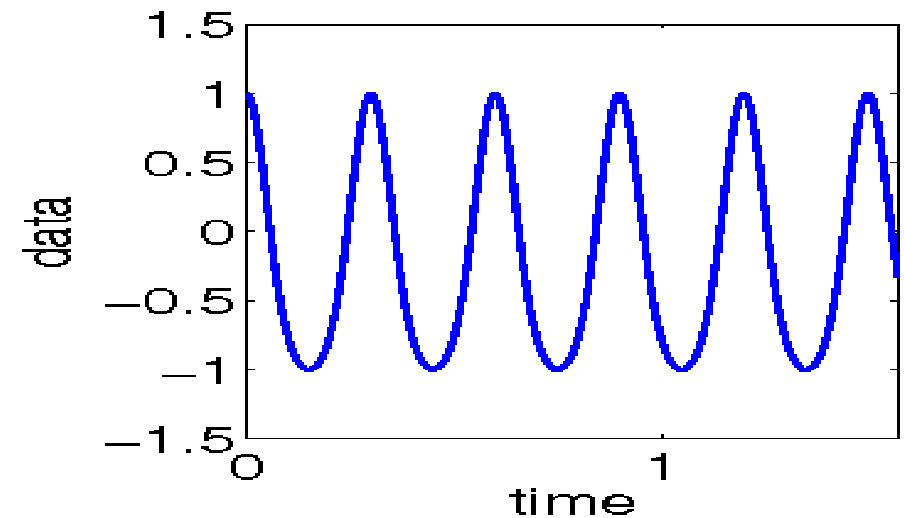

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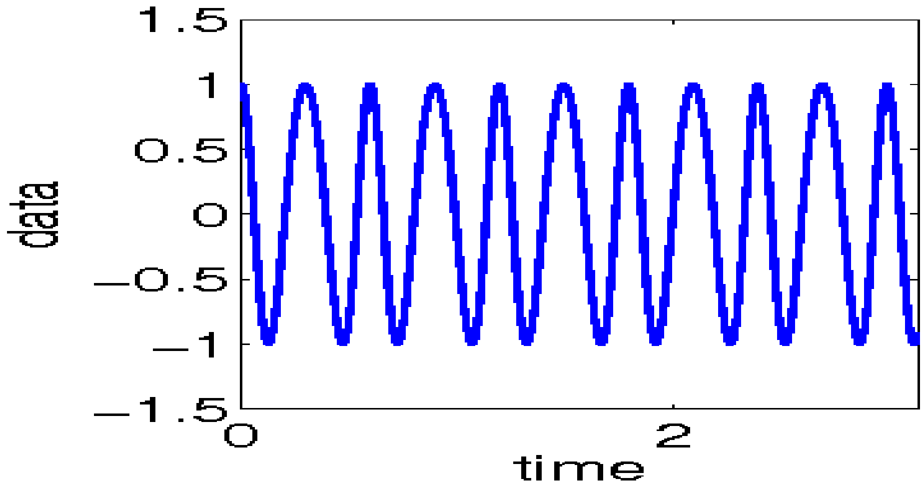

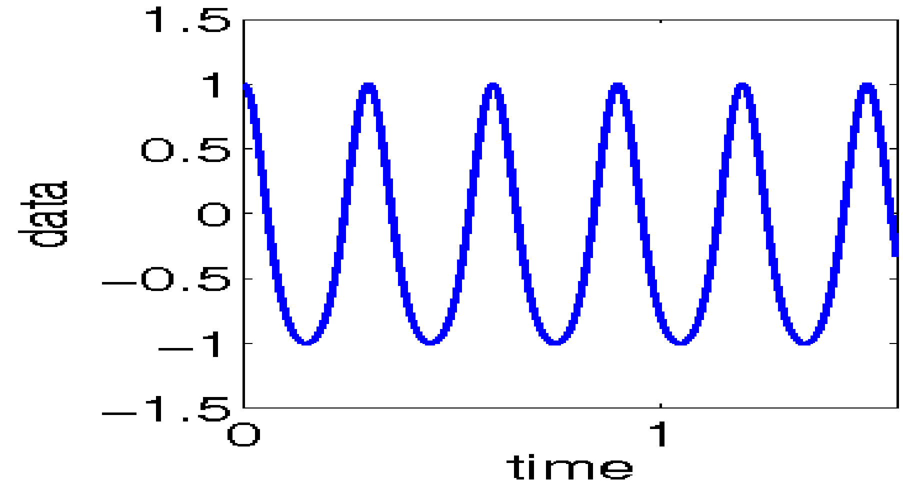

$$f(t) = \sum_{k=1}^{K} A_k(t) \cos(\varphi_k(t))$$

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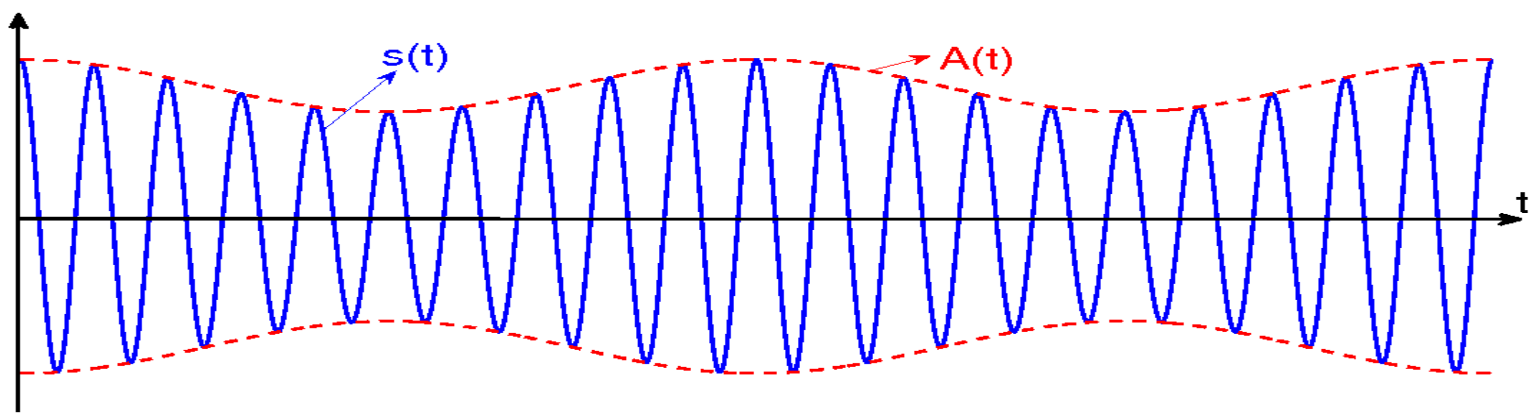
Want to extract $A_k(t)$
$\varphi_k(t)$

Note : non uniqueness

Ex:



but ... can also be written as sum of 3 cosines

$$f(t) = \sum_{k=1}^{K} A_k(t) \cos(\varphi_k(t))$$

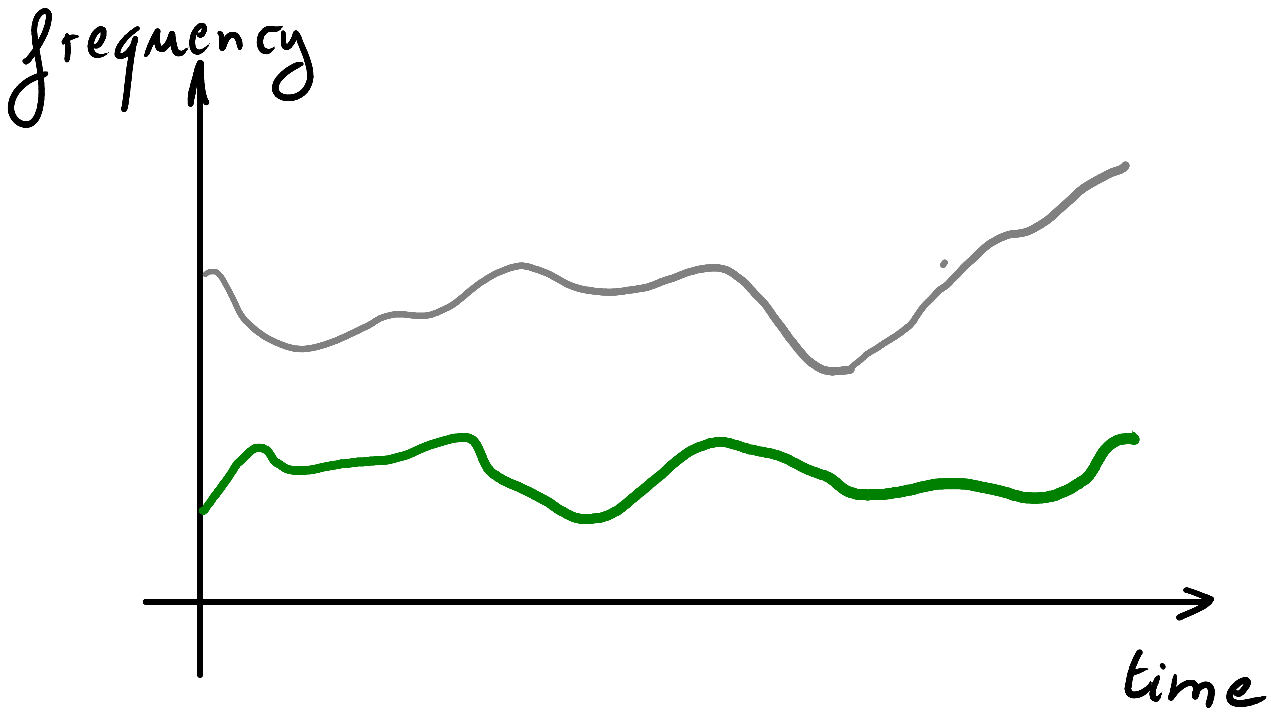
Want to extract $A_k(t)$
$\varphi_k(t)$

→ Sparsity will play a role

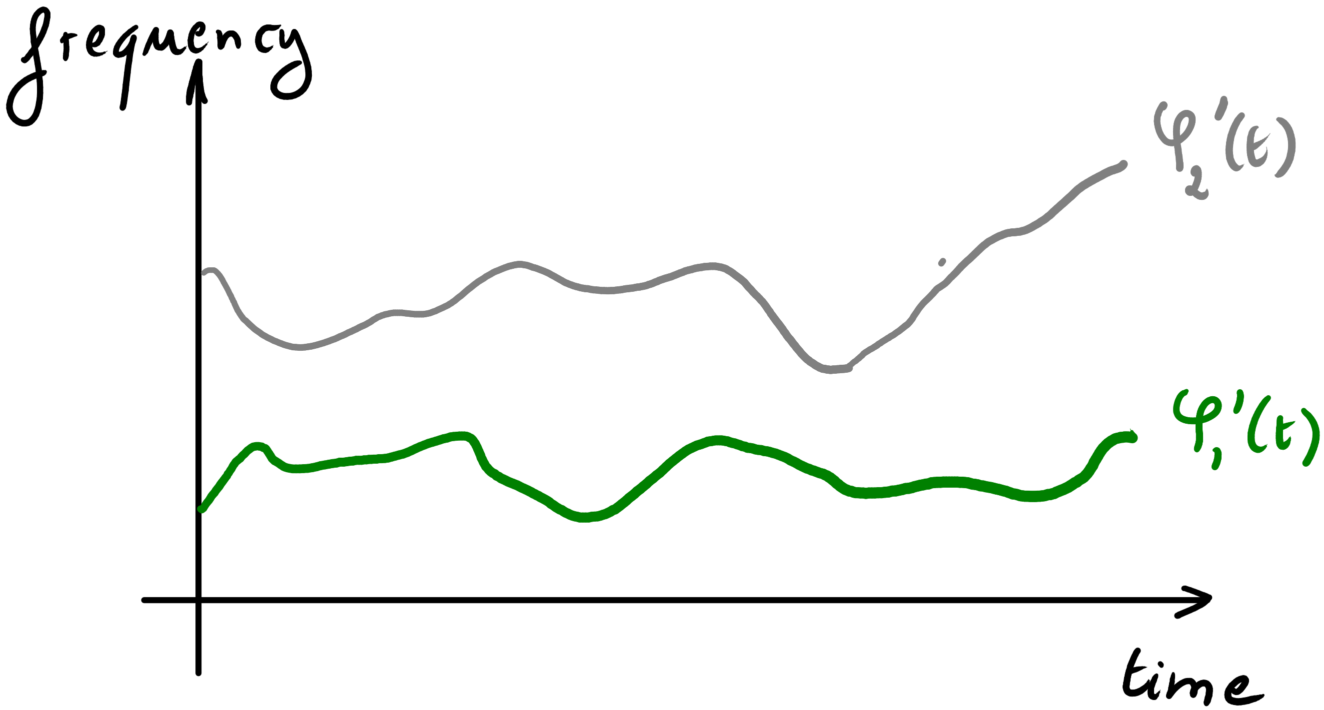
How "separate" role of $A_k(t)$, $\varphi_k(t)$?

$\varphi_k'(t) = \omega_k(t) > 0$ instantaneous frequency

$|A_k'(t)|, |\varphi_k''(t)| << \varphi_k'(t)$



$$f(t) = A_1(t)\cos\varphi_1(t) + A_2(t)\cos\varphi_2(t)$$



$$f(t) = A_1(t)\cos\varphi_1(t) + A_2(t)\cos\varphi_2(t)$$

$$= \mathrm{Re}\left[A_1(t)\, e^{i\varphi_1(t)} + A_2(t)\, e^{i\varphi_2(t)} \right]$$

$$= \mathrm{Re}\left[\tilde{A}_1(t)\, e^{i\omega_1(t)\, t} + \tilde{A}_2(t)\, e^{i\omega_2(t)\, t} \right]$$

$$= \mathrm{Re}\left[\int e^{i\omega t} F(t,\omega)\, d\omega \right]$$

where $F(t,\omega) = \tilde{A}_1(t)\,\delta(\omega - \omega_1(t)) + \tilde{A}_2(t)\,\delta(\omega - \omega_2(t))$

This suggests a variational minimization :

given $$f(t) = \sum_{k=1}^{K} A_k(t) \cos \varphi_k(t) + \text{noise}$$

with $|\varphi_k''|, |A_k'| \ll \varphi_k'$

solve

$$\operatorname{argmin} \int_0^T \left| \mathrm{Re}\left[\int F(t,\omega) e^{i\omega t} d\omega \right] - f(t) \right|^2 dt$$
$$+ \lambda \int_0^T \int | F(t,\omega) | \, dt \, d\omega + \mu \iint | \partial_t F(t,\omega) |^2 \, dt \, d\omega$$

. Tackling this straight on was too hard for us; we used a detour.

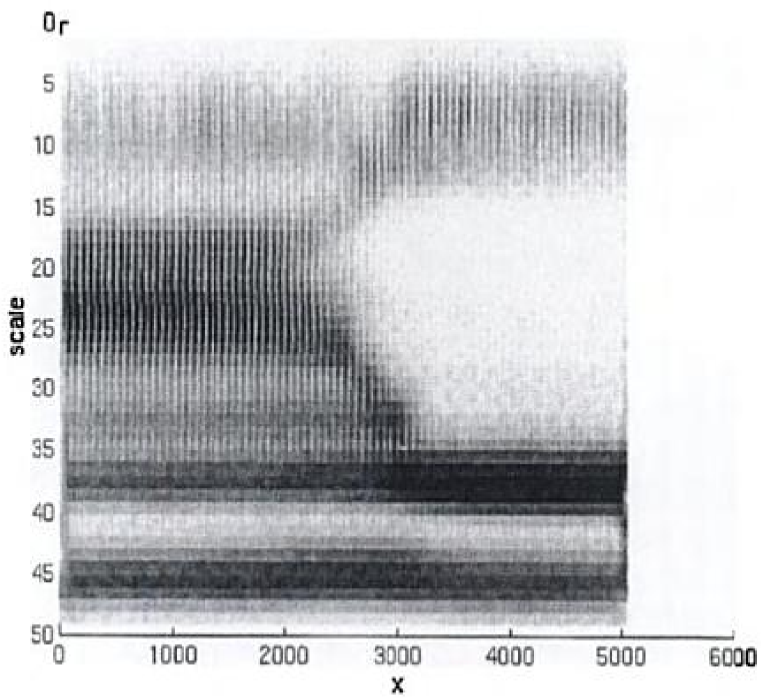
Jianfeng Lu, Hautieng Wu & ID

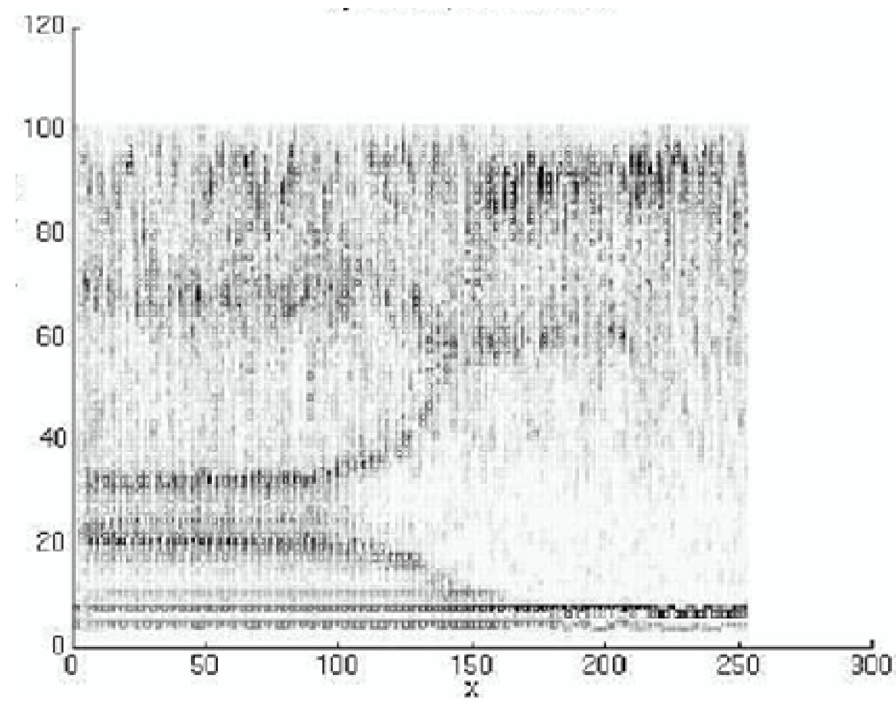
However, new results by Mathieu & Wu!

Synchrosqueezing.

Motivation & original use:

- speaker identification w. S. Maes (early 90s)

- 



a-a-a-a-e-e-e-e

continuous wavelet tr.

synchrosqueezed transform

$$f(x) \xrightarrow[\text{c.w.t.}]{} (Wf)(a,b) = \int f(x) \frac{1}{\sqrt{a}} \overline{\psi\left(\frac{x-b}{a}\right)}\, dx$$

$$s_{\omega_0}(x) = e^{i\omega_0 x} \longrightarrow (Ws_{\omega_0})(a,b) = \sqrt{a}\, e^{i\omega_0 b} \underbrace{\overline{\hat{\psi}(a\omega_0)}}$$

this oscillates with freq. exactly ω_0.

if $\hat{\psi}(\xi)$ has peak at $\xi = 1$, then this is concentrated around $a = 1/\omega_0$

$$s(x) = \sum_{k=1}^{K} A_k e^{i\omega_k(x)} \longrightarrow (Ws)(a,b) = \sum_{k=1}^{K} \sqrt{a}\, A_k e^{i\omega_k b} \overline{\hat{\psi}(a\omega_k)}$$

if components are well-separated $\Rightarrow$ get ω_k!

In practice:

$f(x) \longrightarrow Wf(a,b) \longrightarrow$ for each b, consider

$$\Omega(b) = \{a \,,\ |Wf(a,b)| > \text{t.h.}\}$$

$$\downarrow$$

for $a \in \Omega(b)$, define

$$\omega_f(a,b) = \frac{i\partial_b Wf(a,b)}{Wf(a,b)}$$

(note: $\partial_b Wf$ is computed by w.transf. w.r.t. ψ')

binning: $\Omega_\ell(b) = \{a \,;\ |\omega_f(a,b) - \ell\cdot\omega_0| < \frac{\omega_0}{2}\}$

$$f(x) = C_\psi \int Wf(a,x)\, a^{-3/2}\, da$$

$$= C_\psi \sum_\ell \underbrace{\int_{a\in\Omega_\ell(b)} Wf(a,x)\, a^{-3/2}\, da}_{\sim A_\ell(x)\, e^{i\ell\omega_0 x}}$$

Theorem.

If $f(x) = \sum_{k=1}^{K} A_k(x)\, e^{i\phi_k(x)}$, with $\phi_k'(x) \geq c > 0$

$$\phi_{k+1}'(x) - \phi_k'(x) > \frac{d}{2}\left[\phi_{k+1}'(x) + \phi_k'(x)\right]$$

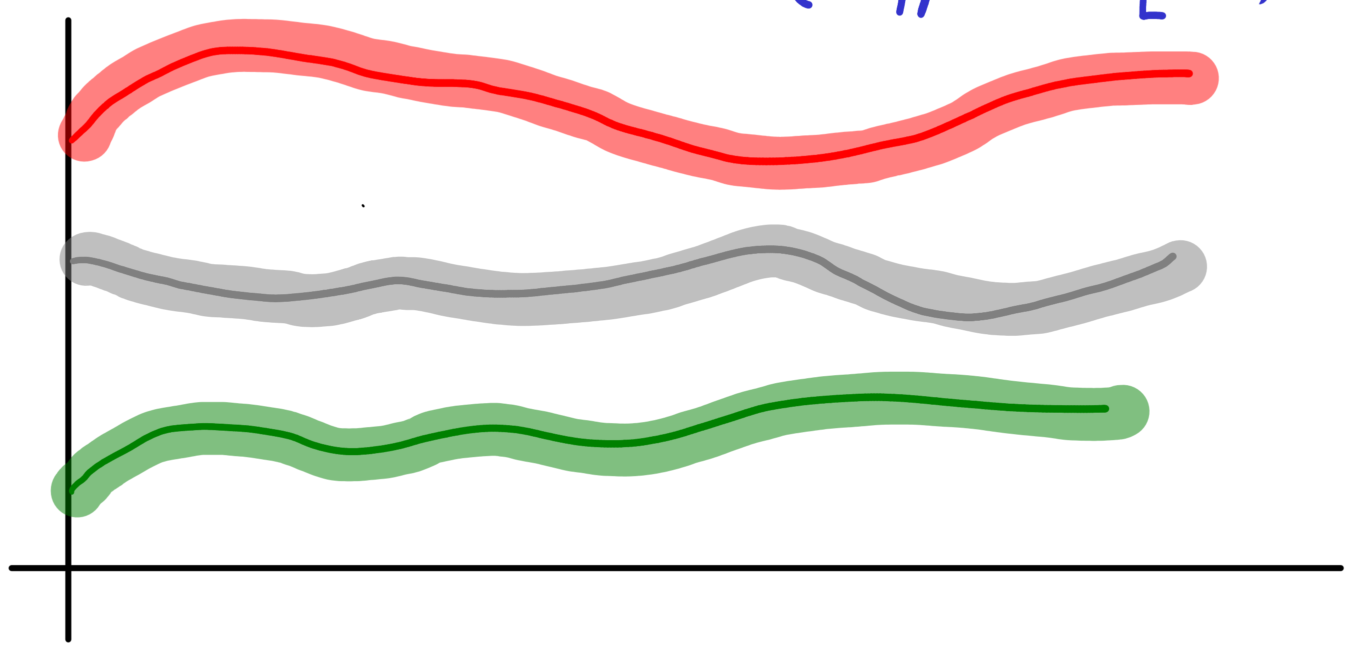
$$|A_k'(x)|, |\phi_k''(x)| < \varepsilon\, |\phi_k'(x)|$$

$$|\phi_k''(x)| \leq M$$

and ψ satisfies some technical conditions

(supp $\hat{\psi} \subset [1-\Delta, 1+\Delta]$, $\Delta = \frac{1-d}{1+d}$)

then.



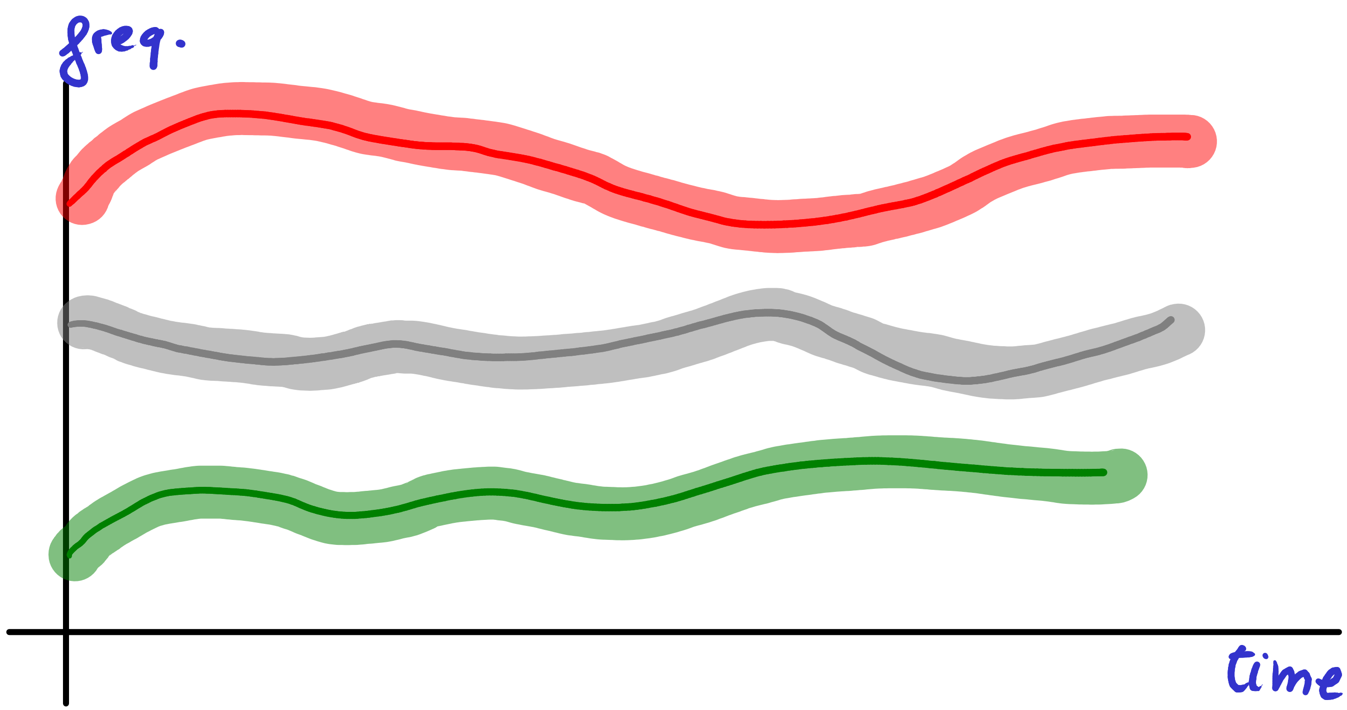
- regions where $|Wf(a,b)| > \varepsilon^{1/3}$ are disjoint

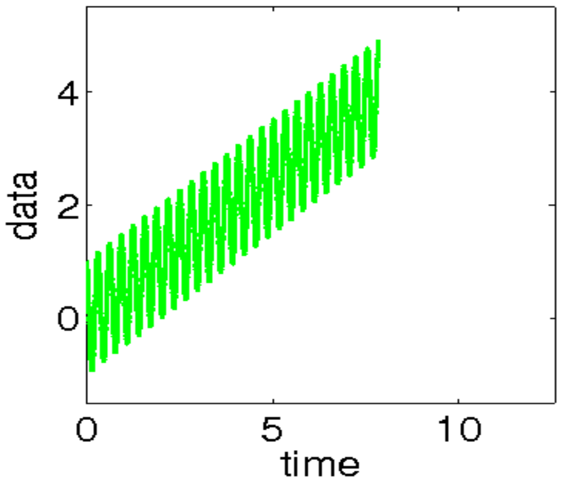
  all located so that $|a\,\phi_k'(b) - 1| < \Delta$

- for $a \in$ k-th region, $|\omega_f(a,b) - \phi_k'(b)| < \varepsilon^{1/3}$

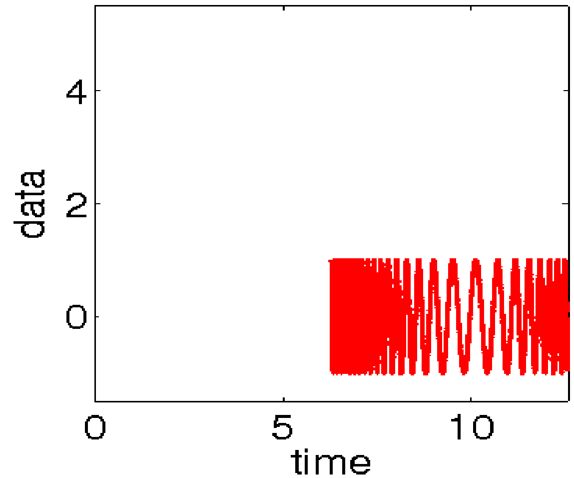
- $\left| A_k(b) - e^{-i\phi_k(b)} \int_{a \in k\text{-th region}} a^{-3/2}\, Wf(a,b)\, da \right| < C\,\varepsilon^{1/3}$.

then.

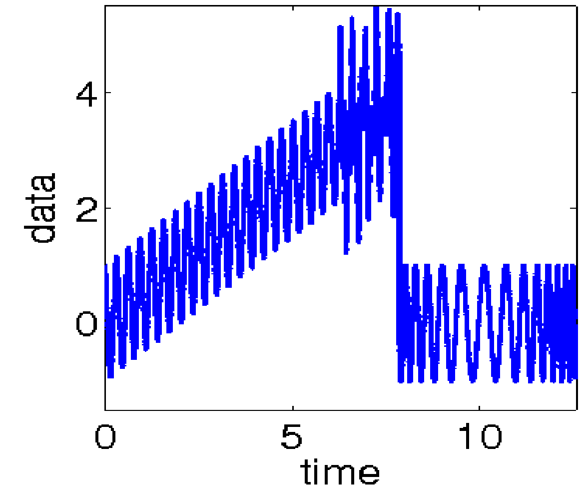




$s_1(t)$

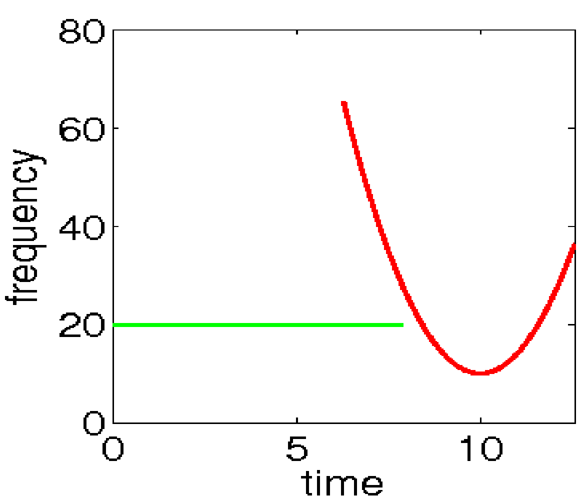


$s_2(t)$

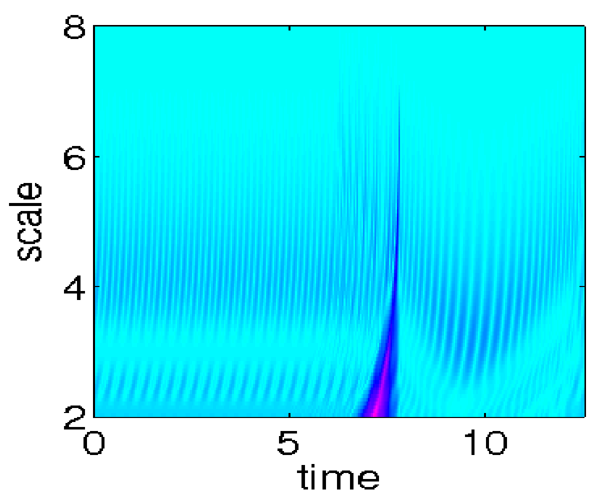


$f(t) = s_1(t) + s_2(t)$

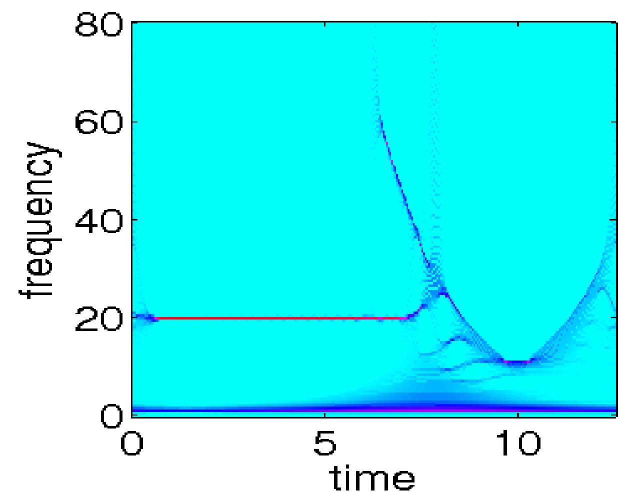
$$\boxed{Sf(t,\omega) = \int a^{-3/2}\, Wf(a,t)\, \delta(\omega(a,b) - \omega)\, da}$$



"ideal" T.F-localiz.

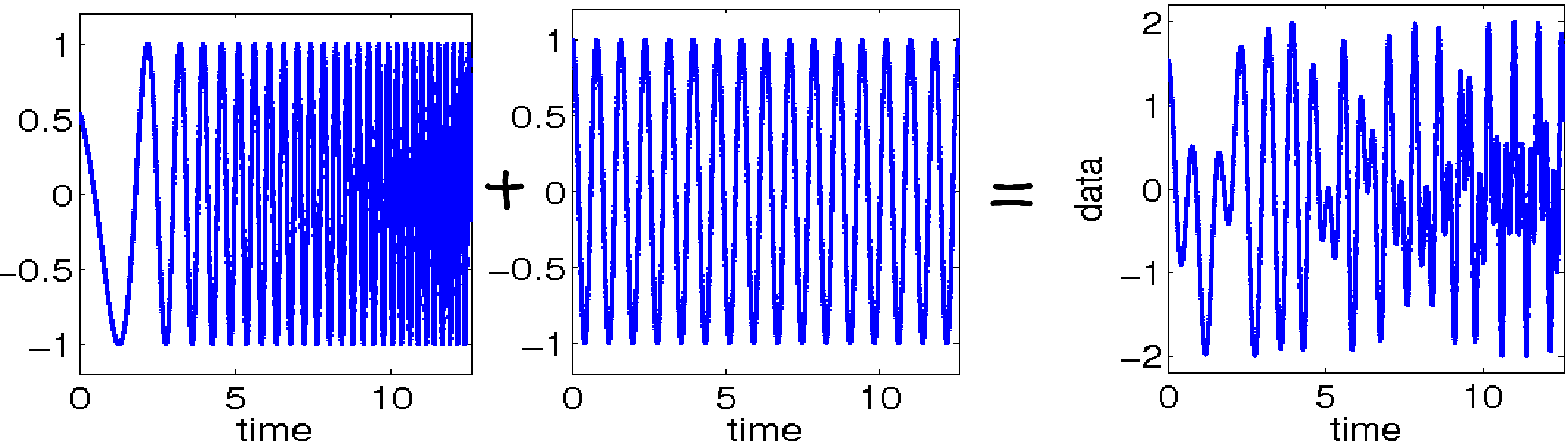


wavelet transform

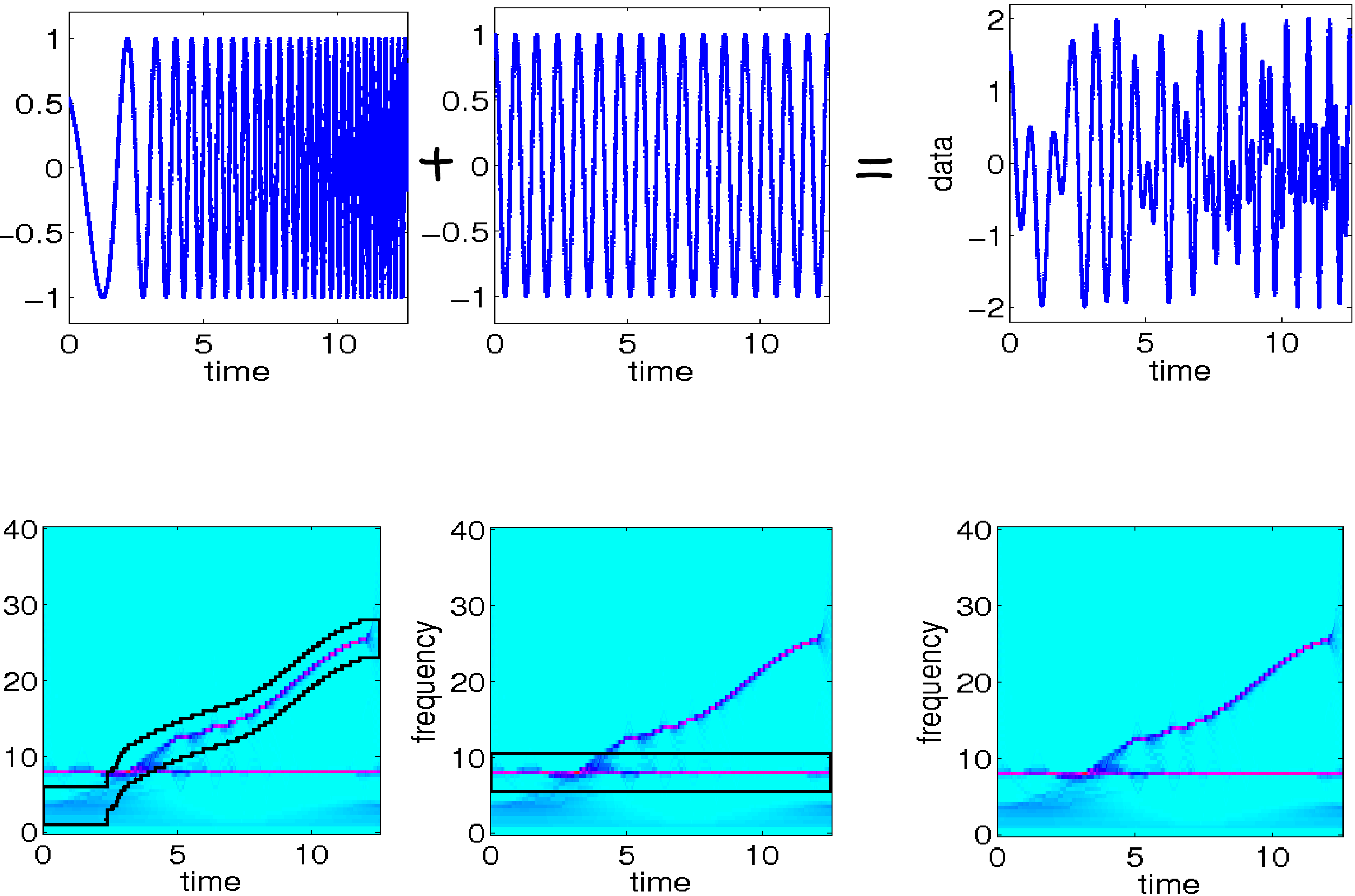


synchrosq. trsf.

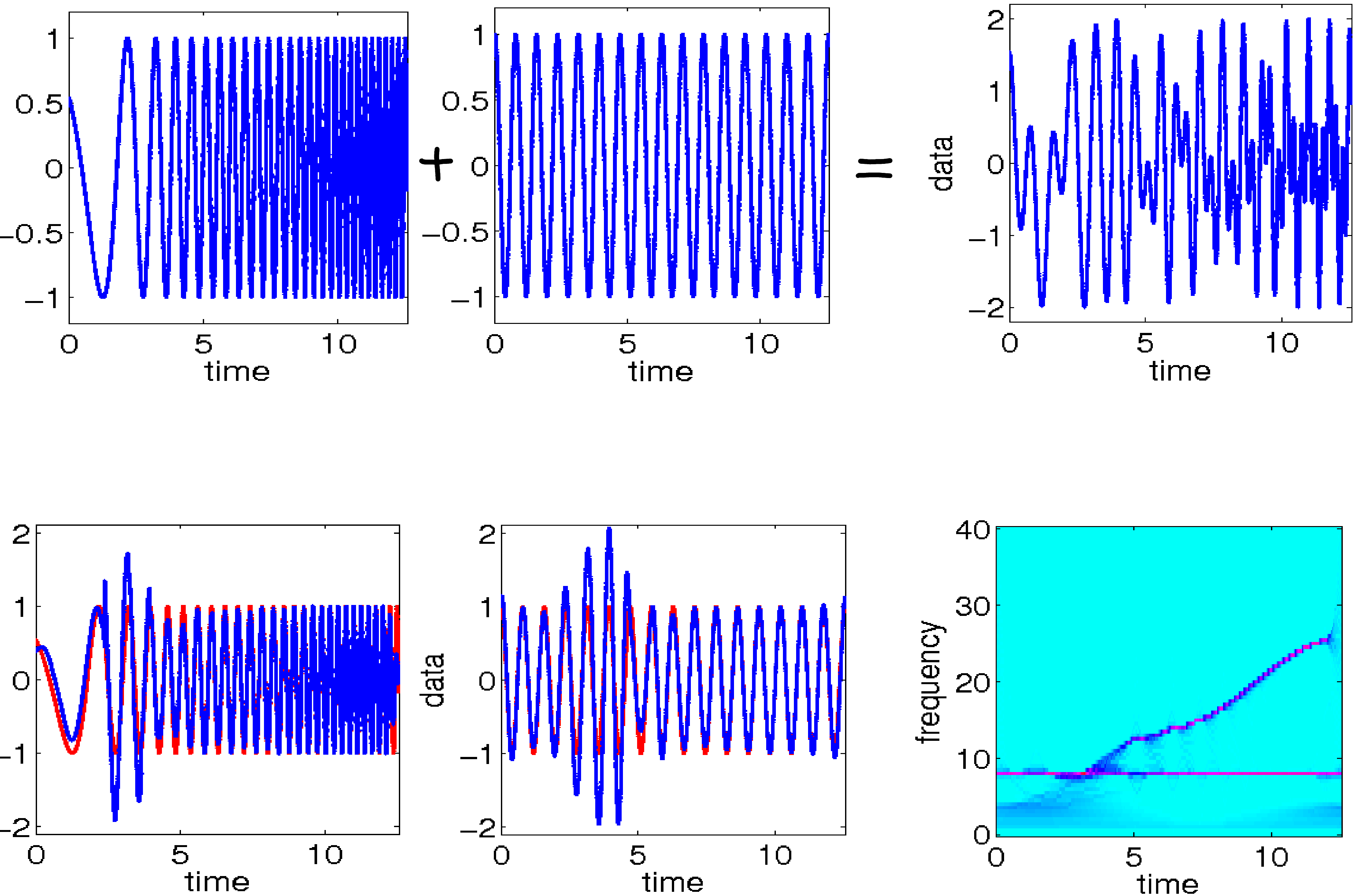
Synchrosq - 7.



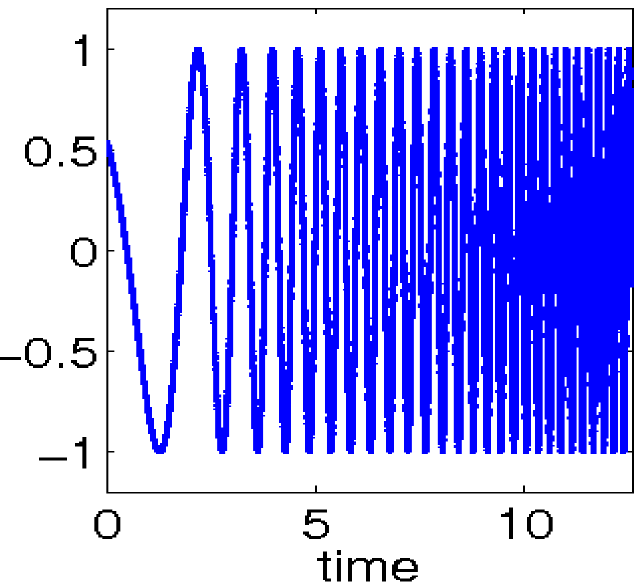
Synchrosq - 8.



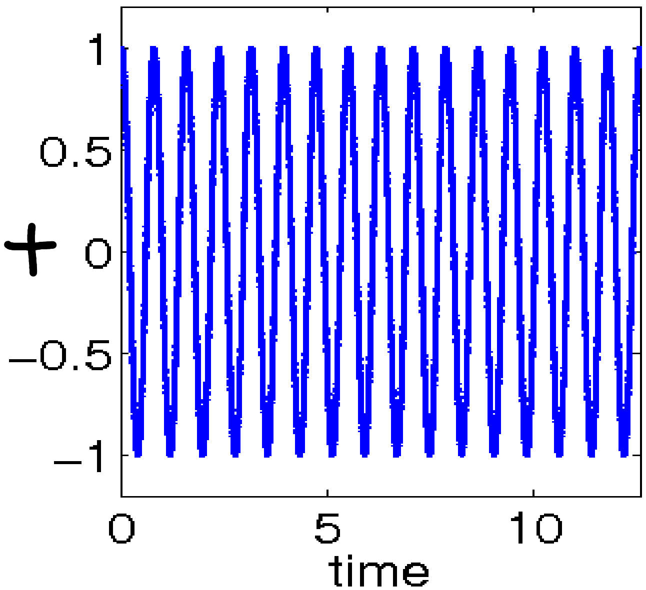
Synchrosq - 9.



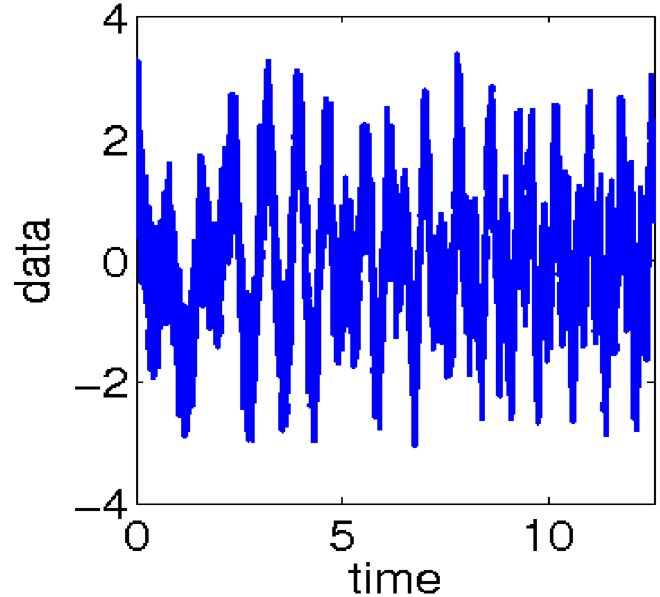
Synchrosq - 10.



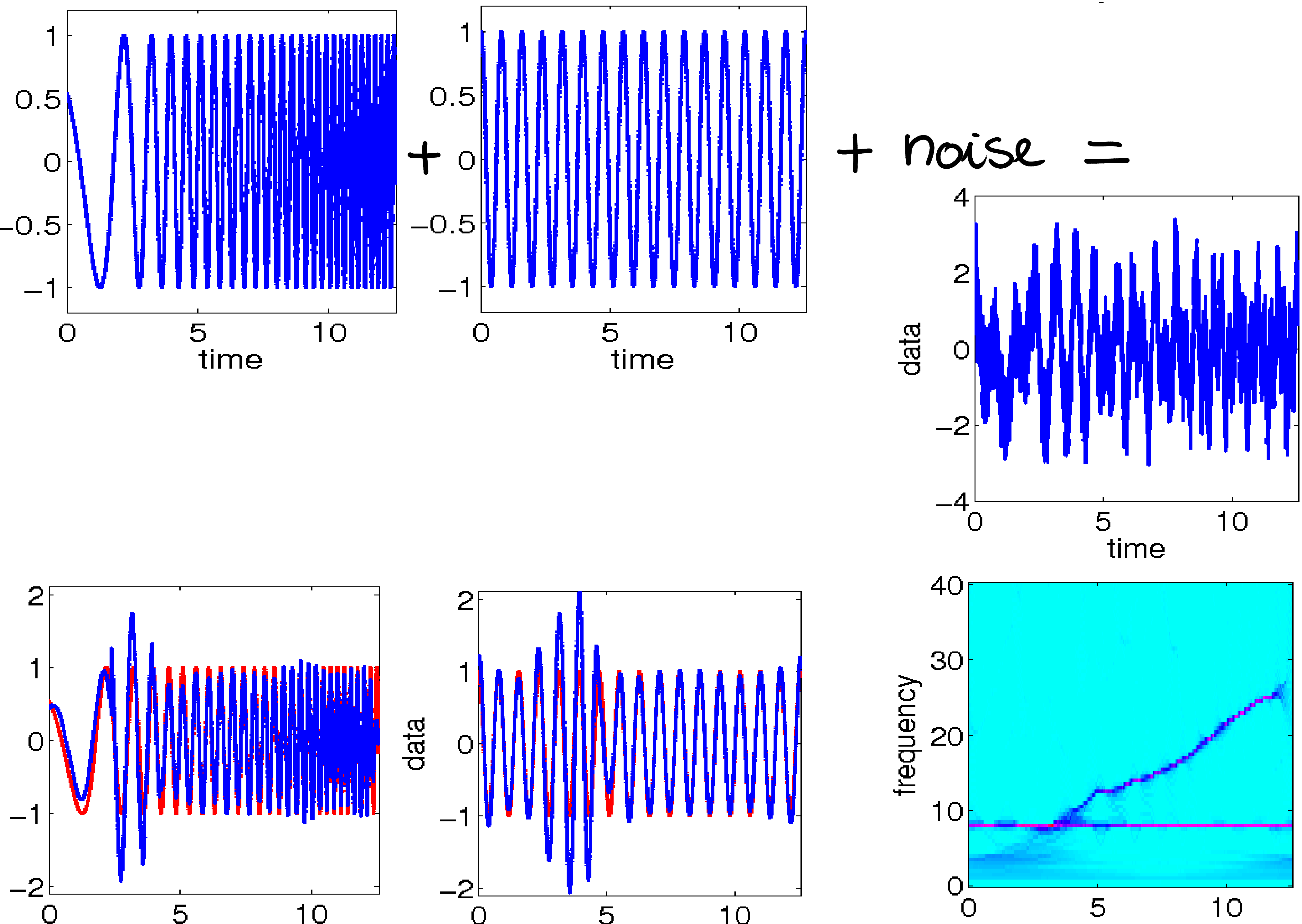
+



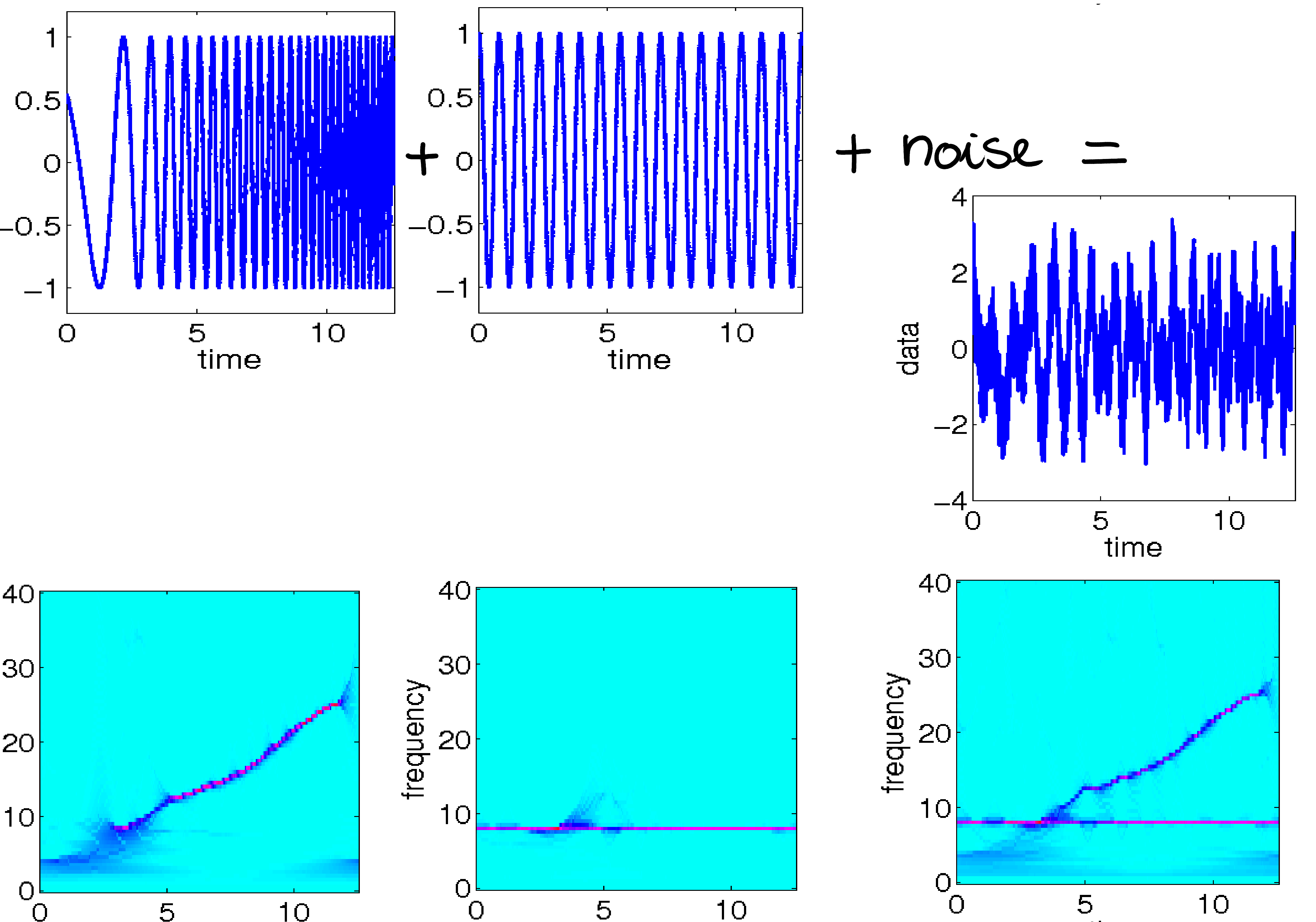
+ noise =



Synchrosq - II.



Synchrosq - 12



Multitaper

- result of synchrosqueezing is (mostly) independent of wavelet choice

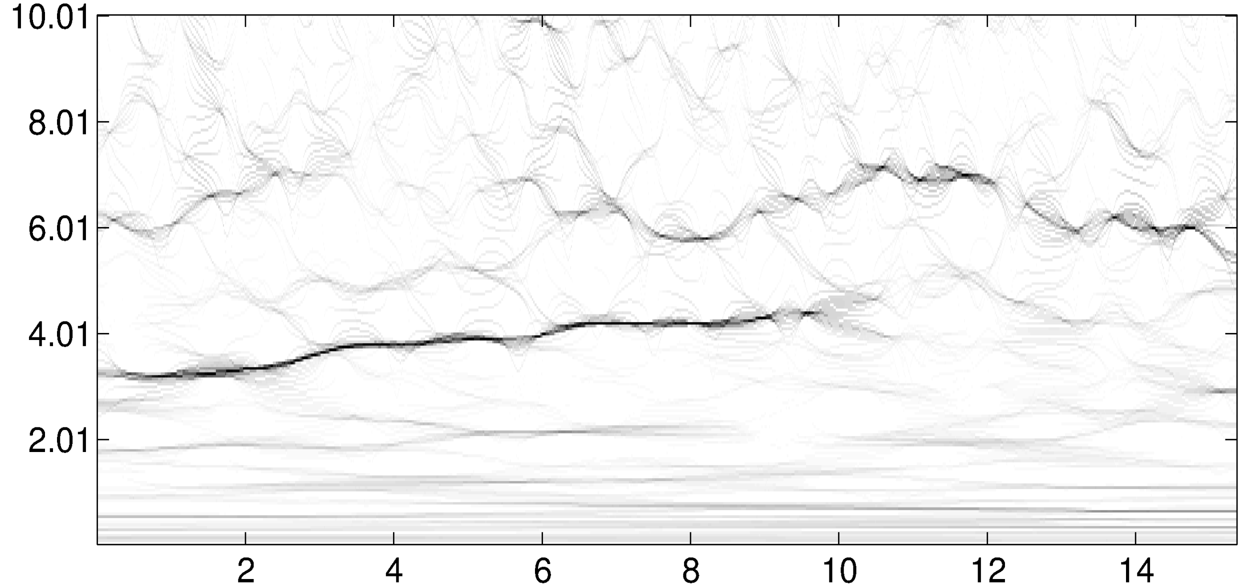
$$f(t) = \sum_{k=1}^{K} A_k(t) \cos \phi_k(t)$$

identify A_k, ϕ_k.

- reassignment
  ↳ also "reassigns" noise!

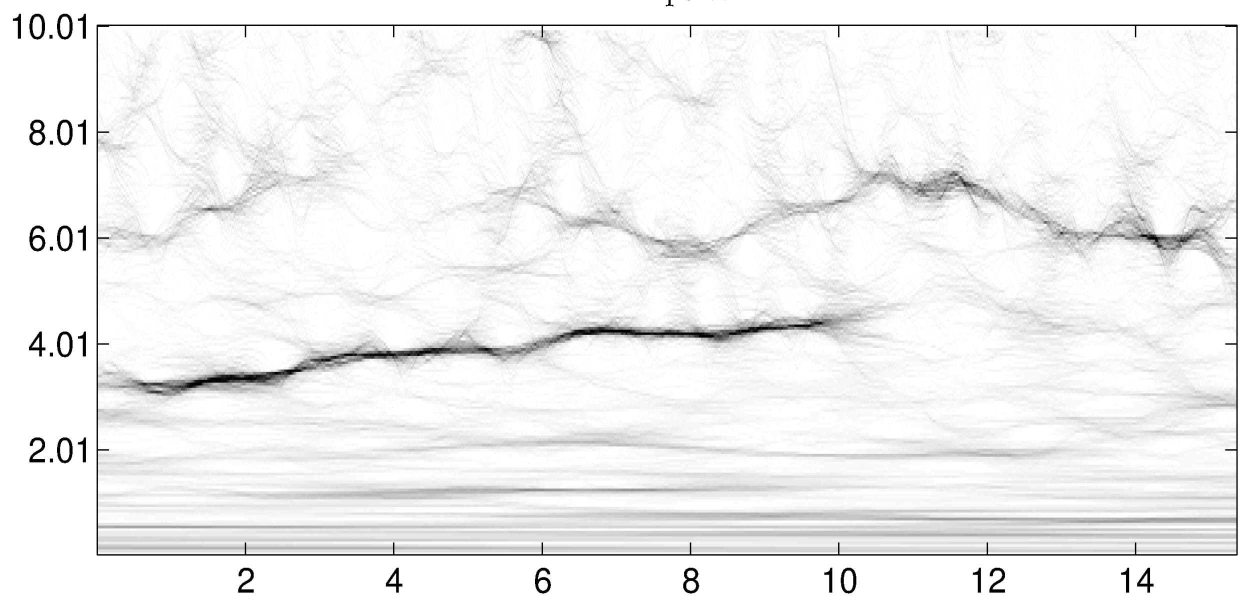
Synchrosq - 14

sqCWT



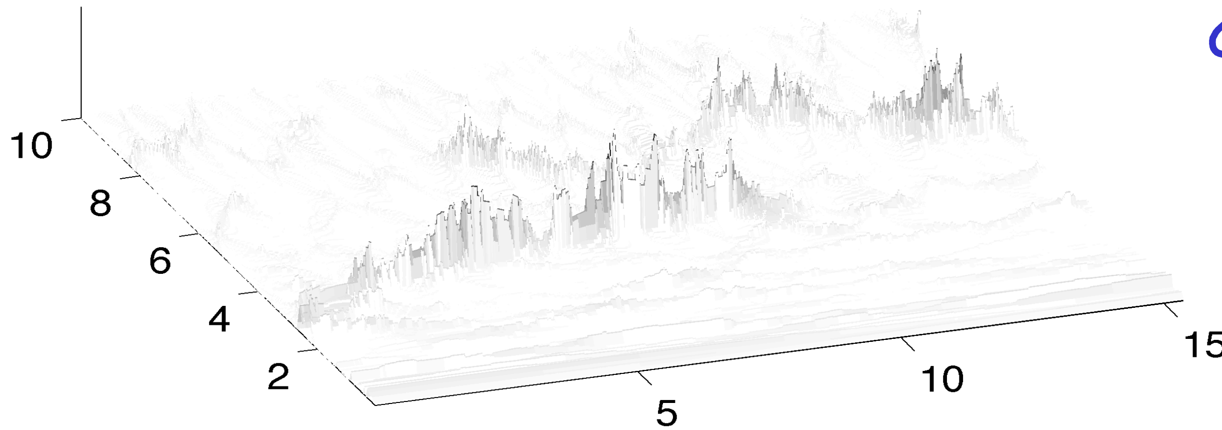
one ψ

mt sqCWT

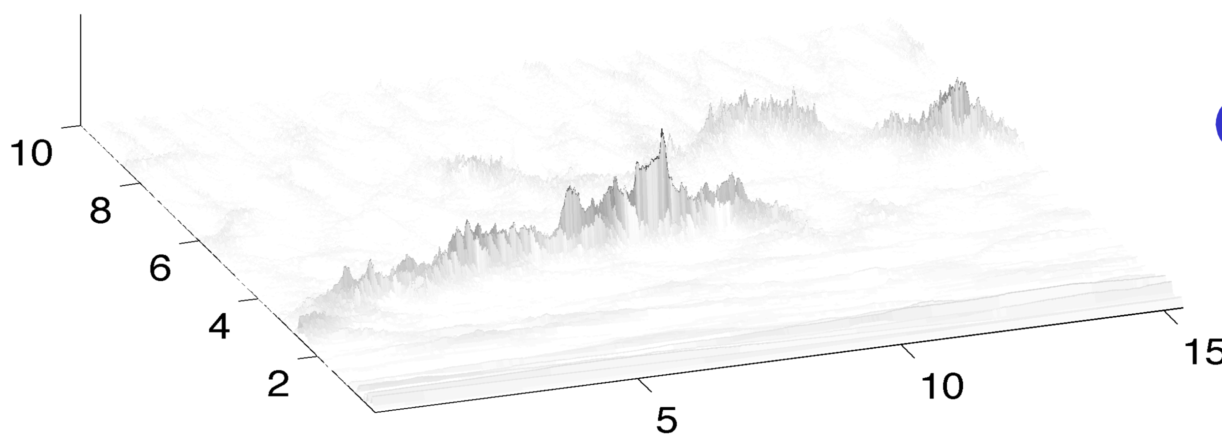


$\psi_1, \ldots \psi_6$

Synchrosq - 15



amplitud for one ψ



for average of results for $\psi_1, \dots \psi_6$.