Iterative Matching Pursuit and its Applications in Adaptive Time-Frequency Analysis

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Outline

- Review of the Data-Driven Time-Frequency Analysis.
- Convergence result of the Data-Driven Time-Frequency Analysis.
- Method for the data with sparse samples and convergence analysis.
- Method for the data with poor scale separation.
- Method for the data with intra-wave frequency modulation and nonlinearity analysis.
- Conclusion remark.

General Framework: sparsest decomposition

Basic idea: Looking for the sparest decomposition over a huge dictionary, $\mathcal{D}$.

$$\begin{aligned} &\textit{Minimize} && M \\ &\textit{Subject to:} && f(t) = \sum_{k=1}^{M} a_k(t)\cos\theta_k(t), \quad a_k(t)\cos\theta_k(t) \in \mathcal{D}, \end{aligned} \tag{1}$$

General Framework: Dictionary

- The dictionary is

$$\mathcal{D} = \{a(t)\cos\theta(t) : \ \theta'(t) \geq 0, \ a(t) \in V(\theta)\}, \qquad (2)$$

where $V(\theta)$ is a linear space consisting of functions smoother than $\cos\theta(t)$.

- $V(\theta)$ is chosen to be

$$V(\theta) = \text{span}\left\{1, \cos\left(\frac{k\theta}{2L_\theta}\right), \sin\left(\frac{k\theta}{2L_\theta}\right) : k = 1, \cdots, 2\lambda L_\theta\right\},$$

where $\lambda \leq 1/2$ is a parameter to control the smoothness of functions in $V(\theta)$ and $L_\theta = (\theta(T) - \theta(0))/2\pi$..

Algorithm based on l^1 regularized nonlinear least square

- $r^0 = f(t)$.

Step 1: Solve the following constraint nonlinear least-square problem (P_2):

$$\begin{aligned} \text{Minimize} \quad & \gamma \|\widehat{a}_k\|_1 + \|r^{k-1} - a_k(t)\cos\theta_k(t)\|_2^2 \qquad (3) \\ \text{Subject to:} \quad & \theta_k' \geq 0, \quad a_k(t) \in V(\theta_k). \end{aligned}$$

where $\widehat{a}_k$ is the representation of a_k in $V(\theta_k)$ space

Step 2: Update residual

$$r^k = f(t) - \sum_{j=1}^{k} a_j(t)\cos\theta_j(t) \qquad (4)$$

Step 3: If $\|r^k\|_2 < \epsilon_0$, stop. Otherwise, go to Step 1.

l_1 regularized Gauss-Newton iteration

- $\theta_k^0 = \theta_0$.

Step 1: Get a_k^{n+1} and b_k^{n+1} by solving following least square problem:

$$\text{Minimize} \quad \|r^{k-1} - a_k^{n+1}\cos\theta_k^n(t) - b_k^{n+1}\sin\theta_k^n\|_2^2 + \gamma(\|\widehat{a}_k^{n+1}\|_1 + \|\widehat{b}_k^{n+1}\|_1), \tag{5}$$

$$\text{Subject to} \quad a_k^{n+1}(t),\ b_k^{n+1}(t) \in V(\theta_k^n).$$

where $\widehat{a}_k^{n+1}, \widehat{b}_k^{n+1}$ is the coefficients of a_k^{n+1}, b_k^{n+1} in $V(\theta_k^n)$ space.

Step 2: Update θ_k^n:

$$\theta_k^{n+1} = \theta_k^n - \lambda \arctan\left(\frac{b_k^{n+1}}{a_k^{n+1}}\right), \tag{6}$$

where $\lambda \in [0,1]$ is chosen to ensure that θ_k^{n+1} is monotonically increasing,

$$\lambda = \max\left\{\alpha \in [0,1] : \frac{d}{dt}\left(\theta_k^n - \alpha \arctan\left(\frac{b_k^{n+1}}{a_k^{n+1}}\right)\right) \geq 0\right\}. \tag{7}$$

Step 3: If $\|\theta_k^{n+1} - \theta_k^n\|_2 < \epsilon_0$, stop. Otherwise, go to Step 1.

Fast algorithm for periodic data

The least square problem can be solved approximately by FFT in θ space, then we get the following more efficient algorithm:

Step 1: Interpolate r_{k-1} from $\{t_i\}_{i=1}^N$ in the physical space to a uniform mesh in the θ_k^n-coordinate to get $r_{\theta_k^n}$ and compute the Fourier transform $\widehat{r}_{\theta_k^n}$:

$$r_{\theta_k^n,j} = \text{Interpolate}\ \left(r_{k-1}, \theta_{k,j}^n\right), \tag{8}$$

where $\theta_{k,j}^n,\ j = 0, \cdots, N-1$ are uniformly distributed in the θ_k^n-coordinate,i.e. $\theta_{k,j}^n = 2\pi L_{\theta_k^n}\, j/N$. And the Fourier transform of $r_{\theta_k^n}$ is given as follows

$$\widehat{r}_{\theta_k^n}(\omega) = \frac{1}{N}\sum_{j=1}^{N} r_{\theta_k^n,j}\, e^{-i2\pi\omega\overline{\theta}_{k,j}^n}, \quad \omega = -N/2+1, \cdots, N/2, \tag{9}$$

where $\overline{\theta}_{k,j}^n = \frac{\theta_{k,j}^n - \theta_{k,0}^n}{2\pi L_{\theta_k^n}}$.

Fast algorithm for periodic data

Step 2: Apply a cutoff function to the Fourier Transform of $r_{\theta_k^n}$ to compute a and b on the mesh of the θ_k^n-coordinate, denoted by $a_{\theta_k^n}$ and $b_{\theta_k^n}$:

$$\begin{aligned} a_{\theta_k^n} &= \mathcal{F}^{-1}\left[\left(\widehat{r}_{\theta_k^n}\left(\omega + L_{\theta_k^n}\right) + \widehat{r}_{\theta_k^n}\left(\omega - L_{\theta_k^n}\right)\right) \cdot \chi_\lambda\left(\omega / L_{\theta_k^n}\right)\right], \\ b_{\theta_k^n} &= \mathcal{F}^{-1}\left[i \cdot \left(\widehat{r}_{\theta_k^n}\left(\omega + L_{\theta_k^n}\right) - \widehat{r}_{\theta_k^n}\left(\omega - L_{\theta_k^n}\right)\right) \cdot \chi_\lambda\left(\omega / L_{\theta_k^n}\right)\right]. \end{aligned}$$

$\mathcal{F}^{-1}$ is the inverse Fourier transform defined in the θ_k^n coordinate:

$$\mathcal{F}^{-1}\left(\widehat{r}_{\theta_k^n}\right) = \frac{1}{N} \sum_{\omega=-N/2+1}^{N/2} \widehat{r}_{\theta_k^n} e^{i 2\pi \omega \overline{\theta}_{k,j}^n}, \quad j = 0, \cdots, N-1.$$

Step 3: Interpolate $a_{\theta_k^n}$ and $b_{\theta_k^n}$ from the uniform mesh $\{\theta_{k,j}^n\}_{j=1}^N$ in the θ_k^n-coordinate back to the physical grid points $\{t_i\}_{i=1}^N$:

$$\begin{aligned} a(t_i) &= \text{Interpolate}\left(a_{\theta_k^n}, t_i\right), \quad i = 0, \cdots, N-1, \\ b(t_i) &= \text{Interpolate}\left(b_{\theta_k^n}, t_i\right), \quad i = 0, \cdots, N-1,. \end{aligned}$$

Fast algorithm for periodic data

Step 4: Update θ^n in the t-coordinate:

$$\Delta\theta' = P_{V(\theta;\,\eta)}\left(\frac{d}{dt}\left(\arctan\left(\frac{b_k^{n+1}}{a_k^{n+1}}\right)\right)\right),\ \Delta\theta(t) = \int_0^t \Delta\theta'(s)ds,$$
$$\theta_k^{n+1} = \theta_k^n - \beta\Delta\theta,$$

where $\beta \in [0,1]$ is chosen to make sure that θ_k^{n+1} is monotonically increasing:

$$\beta = \max\left\{\alpha \in [0,1] : \frac{d}{dt}\left(\theta_k^n - \alpha\Delta\theta\right) \geq 0\right\}.$$

and $P_{V_p(\theta;\,\eta)}$ is the projection operator to the space $V_p(\theta;\,\eta)$.

Step 5: If $\|\theta_k^{n+1} - \theta_k^n\|_2 < \epsilon_0$, go to step 6. Otherwise, set $n = n+1$ and go to Step 1.

Step 6: If $\eta \geq \lambda$, stop. Otherwise, set $\eta = \eta + \Delta\eta$ and go to step 1.

Convergence analysis

Theorem (Exact recovery)

For the data $f = f_0 + f_1 \cos\theta + \sigma(t)$, $|\sigma(t)| \leq \epsilon$ which is noise. Assume that the instantaneous frequency θ' is M_0-sparse over the Fourier basis in physical space, $\theta' \in V_{M_0} = span\left\{e^{i2k\pi t/T}, k = -M_0, \cdots, 1, \cdots, M_0\right\}$, mean f_0 and envelop f_1 are M_1-sparse over the Fourier basis in θ-space,

$$\widehat{f}_{0,\theta}(k) = \widehat{f}_{1,\theta}(k) = 0, \quad \forall |k| > M_1.$$

If the initial guess of θ satisfies

$$\|\mathcal{F}\left((\theta^0 - \theta)'\right)\|_1 \leq \pi M_0/2,$$

then there exist $\eta_0 > 0$ such that

$$\|\mathcal{F}\left((\theta^{m+1} - \theta)'\right)\|_1 \leq \frac{1}{2}\left\|\mathcal{F}\left((\theta^m - \theta)'\right)\right\|_1 + C \cdot \epsilon,$$

provided $L = \frac{\theta(T) - \theta(0)}{2\pi} \geq \eta_0(M_0, M_1, \epsilon)$, where $\eta_0(M_0, M_1, \epsilon)$ is a constant determined by M_0, M_1, ϵ .

Numerical Validation

$$\theta = 20\pi t + 2\cos 2\pi t + 2\sin 4\pi t, \quad \overline{\theta} = \theta/10$$

$$f_0 = 2 + \cos\overline{\theta} + 2\sin 2\overline{\theta} + \cos 3\overline{\theta}, \quad f_1 = 3 + \cos\overline{\theta} + \sin 3\overline{\theta}$$

$$f = f_0 + f_1\cos\theta.$$

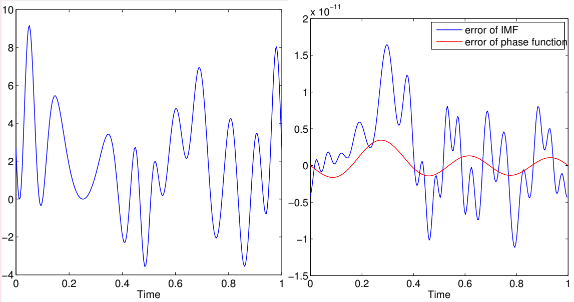


Figure: Original data and errors of IMF and phase function.

Theorem (Approximate recovery)

Assume that the Fourier coefficients of the instantaneous frequency θ', local mean f_0 and envelop f_1 all have fast decay, i.e. there exists $C_0 > 0,\ p \geq 4$ such that

$$|\mathcal{F}(\theta')(k)| \leq C_0|k|^{-p}, \quad |\mathcal{F}_\theta(f_0)(k)| \leq C_0|k|^{-p}, \quad |\mathcal{F}_\theta(f_1)(k)| \leq C_0|k|^{-p}$$

If L is large enough and the intial guess satisfies

$$\|\mathcal{F}\left((\theta^0-\theta)'\right)\|_1 \leq \pi M_0/2,$$

then, we have

$$\|\mathcal{F}\left((\theta^{m+1}-\theta)'\right)\|_1 \leq \Gamma_0(L/4)^{-p+2} + C_0 M_0^{-p+1} + \frac{1}{2}\left\|\mathcal{F}\left((\theta^m-\theta)'\right)\right\|_1,$$

where $\Gamma_0 > 0$ is a constant determined by C_0, M_0 and f_1.

$$a_0 = \frac{1}{1.1 + \sin(2\pi t)}, \quad a_1 = \frac{1}{1.1 + \cos(2\pi t)},$$
$$\theta = 10\sin(2\pi t) + 40\pi t, \quad f = a_0 + a_1 \cos\theta$$

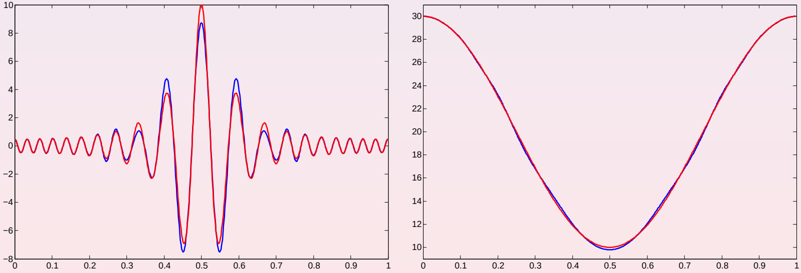


Figure: IMF and instantaneous frequency with bad scale seperation.

Numerical Validation

$f(t) = a_0(t) + a_1(t)\cos(2\theta(t))$, where $a_0(t)$, $a_1(t)$ and $\theta(t)$ are the same as the previous example.

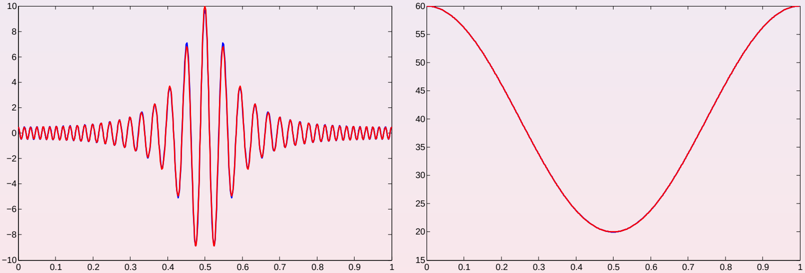


Figure: IMF and instantaneous frequency with good scale seperation.

Another Formulation: Basis Pursuit

From another point of view, the optimization problem can be seen as the nonlinear version of the L_0 minimization problem:

$$\min_{\mathbf{x},\theta_1,\cdots,\theta_M} \|\mathbf{x}\|_0, \quad \text{subject to} \quad [\mathbf{A}_{\theta_1},\cdots,\mathbf{A}_{\theta_M}]\,\mathbf{x} = \mathbf{f}.$$

where $\mathbf{A}_{\theta_j}$ is the collection of the (overcomplete) Fourier basis in θ_j coordinate.

One natrual idea to solve above optimization problem is to solve a l^1 optimization problem with fixed $\theta_1,\cdots,\theta_M$ and update $\theta_1,\cdots,\theta_M$ iteratively.

- Solve

$$\min_{\mathbf{x}} \|\mathbf{x}\|_1, \quad \text{subject to} \quad \left[\mathbf{A}_{\theta_1^n},\cdots,\mathbf{A}_{\theta_M^n}\right]\mathbf{x} = \mathbf{f}.$$

- Update $\theta_1^n,\cdots,\theta_M^n$ to get $\theta_1^{n+1},\cdots,\theta_M^{n+1}$.

Another Formulation: Basis Pursuit

Corresponding to the algorithm based on the matching pursuit, the phase function is calculated one by one to reduce the complexity of the method,

$$\min_{\mathbf{x},\theta} \|\mathbf{x}\|_0, \quad \text{subject to } \mathbf{A}_\theta \mathbf{x} = \mathbf{f}.$$

where $\mathbf{A}_\theta$ is the collection of the (overcomplete) Fourier basis in θ coordinate.

The algorithm should be

- Solve

$$\min_{\mathbf{x}} \|\mathbf{x}\|_1, \quad \text{subject to } \mathbf{A}_{\theta^n} \mathbf{x} = \mathbf{f}.$$

- Update θ^n to get θ^{n+1}.

This gives us the immediate generalization for the data with sparse samples.

Data with sparse samples

To deal with the data with sparse samples, we only need to replace the interpolation-FFT by solving a l^1 minimization problem.

- Solve the l_1 minimization problem to get the Fourier transform of the signal f in θ^m coordinate:

$$\widehat{f}_{\theta^m} = \arg \min_{x \in \mathbb{R}^{N_b}} \|x\|_1, \quad \text{subject to} \quad \mathbf{A}_{\theta^m} \cdot x = f$$

where $\mathbf{A}_{\theta^m} \in \mathbb{R}^{N_s \times N_b}$, $N_s < N_b$, N_s is the number of samples and N_b is the number of Fourier basis.

$\mathbf{A}_{\theta^m}(j,k) = e^{i2\pi k \overline{\theta}^m(t_j)}, \quad j = 1, \cdots, N_s, \ k = -N_b/2+1, \cdots, N_b/2$ and $\overline{\theta}^m = \frac{\theta^m - \theta^m(0)}{\theta^m(T) - \theta^m(0)}$.

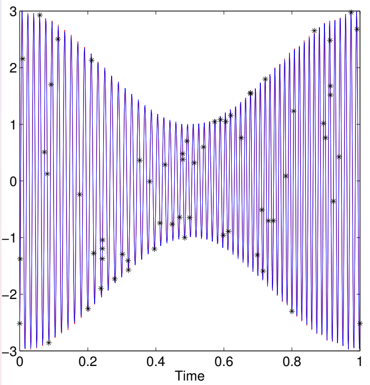
Data with sparse samples

$$\theta(t_i) = 120\pi t_i + 10\cos(2\pi t_i), \quad a(t_i) = 2 + \cos(2\pi t_i), \quad f(t_i) = a(t_i)\cos\theta(t_i),$$

and $i = 1, 2, \cdots, N$.

In this example, the number of samples is 64, approximately one sample point in one period of the signal. The location t_i is chosen randomly in $[0, 1]$.

Data with sparse samples



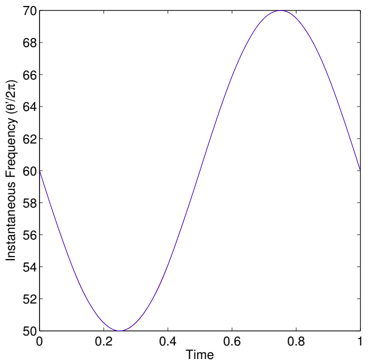


Figure: Left: original samples, red: exact; blue: recovered; '*' represent the sample points. Right: instantaneous frequency, red: exact; blue: numerical.

Data with sparse samples

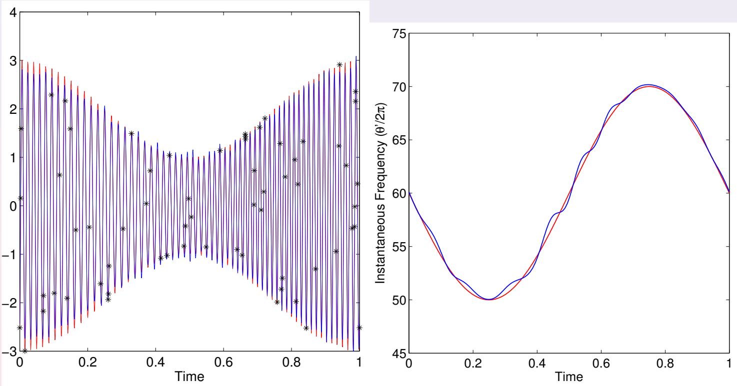


Figure: Left: original samples, $f(t) + 0.2X(t)$, red: exact; blue: recovered from the noised data; '*' represent the sample points. Right: instantaneous frequency, red: exact; blue: numerical.

Data with sparse samples

Theorem

If the samples points are selected at random, under the same assumption in Theorem 1, we have there exist $\eta_0 > 0$, $\eta_1 > 0$, *such that with overwhelming probability*

$$\|\mathcal{F}\left(\left(\theta^{m+1} - \theta\right)'\right)\|_1 \leq \frac{1}{2}\left\|\mathcal{F}\left(\left(\theta^{m} - \theta\right)'\right)\right\|_1, \tag{11}$$

provided $L \geq \eta_0$ *and* $N_s \geq \eta_1 \max(\overline{\theta})' \left(\log N_b\right)^6$, N_s *is the number of samples,* N_b *is the number of basis.*

Data with incomplete samples

Using the assumption that $a(t)$ is smooth, we put a weight in the l^1 term to penalize the high wave number components,

- Solve
$$\min_{\mathbf{x}} \|\omega \mathbf{x}\|_1, \quad \text{subject to} \;\; \mathbf{A}_{\theta^n}\mathbf{x} = \mathbf{f}.$$
where ω is a weight vector.
- Update θ^n to get θ^{n+1}.

We can get better theoretical estimate for this formulation.

Data with sparse samples

Theorem

If the samples points are selected at random, under the same assumption in Theorem 1, we have there exist $\eta_0 > 0$, $\eta_1 > 0$, such that with probability more than $1 - \delta$

$$\|\mathcal{F}\left(\left(\theta^{m+1} - \theta\right)'\right)\|_1 \leq \frac{1}{2}\left\|\mathcal{F}\left(\left(\theta^{m} - \theta\right)'\right)\right\|_1, \tag{12}$$

provided $L \geq \eta_0$ and $N_s \geq \eta_1 \max(\overline{\theta})' \max\{\log N_b, -\log\delta\}$, N_s is the number of samples, N_b is the number of basis.

Data with incomplete samples

$$\theta(t) = 120\pi t + 10\cos(4\pi t), \; a(t) = 2 + \cos(2\pi t),$$
$$f(t) = a(t)\cos\theta(t), \qquad t \in [0, 0.4] \cup [0.6, 1].$$

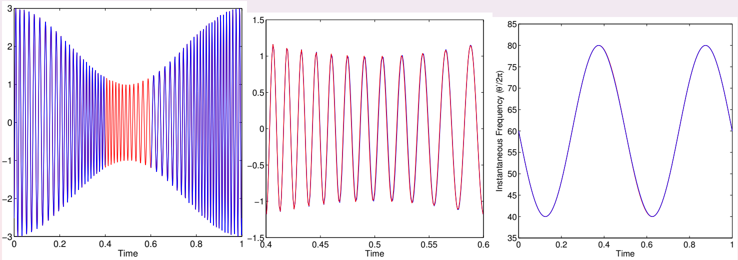


Figure: Left: blue: original incomplete data, the gap is $(0.4, 0.6)$; red: missing data obtained by our method; Middle: recovered missing data, red: exact; blue: numerical. Right: recovered instantaneous frequency, red: exact; blue: numerical.

Data with incomplete samples

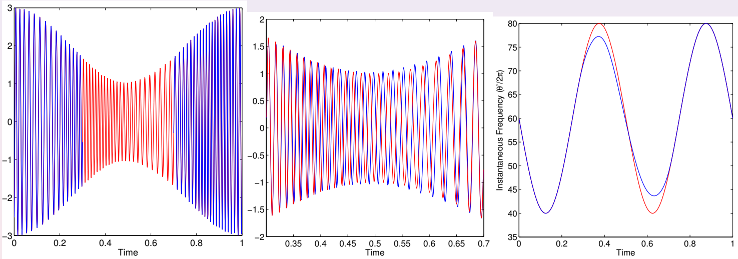


Figure: Left: blue: the original incomplete data, the gap is $(0.3, 0.7)$; red: the missing data recovered by our method; Middle: the recovered missing data, red: exact; blue: numerical. Right: the instantaneous frequencies, red: exact; blue: numerical.

Data without scale-separation

$$f(t) = \cos(20\pi t + 40\pi t^2 + \sin(2\pi t)) + \cos(40\pi t) \tag{13}$$

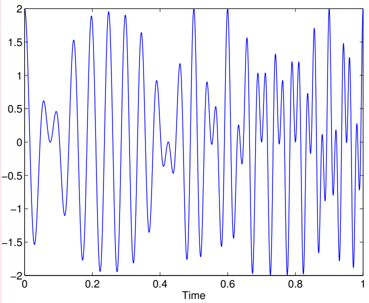


Figure: Original data.

Data without scale-separation

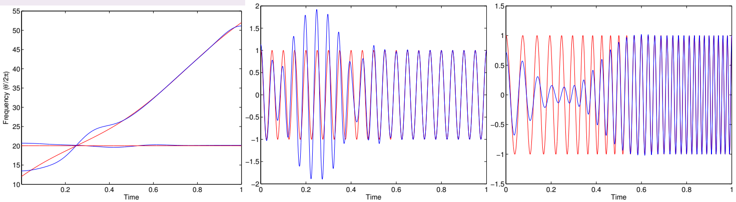


Figure: Left: Instantaneous frequencies; red: exact frequencies; blue: numerical results. Middle and Right: IMFs extracted by previous method .

Data without scale-separation

For this kind of signals, we have to decompose these components simultaneously, since they have strong correlation.

- Solve

$$\min_{\mathbf{x}} \|\mathbf{x}\|_1, \quad \text{subject to} \quad \left[\mathbf{A}_{\theta_1^n}, \cdots, \mathbf{A}_{\theta_M^n}\right] \mathbf{x} = \mathbf{f}.$$

- Update $\theta_1^n, \cdots, \theta_M^n$ to get $\theta_1^{n+1}, \cdots, \theta_M^{n+1}$.

Combining the method to update the phase function, we can get following algorithm.

Data without scale-separation

Initialize: $n=0, \quad \eta=0.$

Step 1: Solve the following l^1 regularized least-square problem:

$$\left(a_k^{n+1}, b_k^{n+1}\right) \in \operatorname*{Argmin}_{a_k, b_k} \quad \gamma \sum_{k=1}^{M} \left(\|\widehat{a}_k\|_{l^1} + \|\widehat{b}_k\|_{l^1} \right) + \|f - \sum_{k=1}^{M} (a_k \cos\theta_k^n + b_k \sin\theta_k^n)\|_{l^2}^2$$

$$\text{Subject to:} \quad a_k \in V(\theta_k^n),\ b \in V(\theta_k^n),$$

where $\widehat{a}_k, \widehat{b}_k$ are the representations of a_k, b_k in the $V(\theta_k^n)$ space.

Data without scale-separation

Step 2: Update θ_k^n:

$$\Delta\theta_k' = P_{V(\theta_k^n;\,\eta)}\left(\frac{d}{dt}\left(\arctan\left(\frac{b_k^{n+1}}{a_k^{n+1}}\right)\right)\right), \tag{14}$$

$$\Delta\theta_k = \int_0^t \Delta\theta_k'(s)ds, \quad \theta_k^{n+1} = \theta_k^n - \beta_k\Delta\theta_k, \tag{15}$$

where $\beta_k \in [0,1]$ is chosen to make sure that θ_k^{n+1} is monotonically increasing:

$$\beta_k = \max\left\{\alpha \in [0,1] : \frac{d}{dt}\left(\theta_k^n - \alpha\Delta\theta_k\right) \geq 0\right\}. \tag{16}$$

and $P_{V(\theta_k^n;\,\eta)}$ is the projection operator to the space $V(\theta_k^n;\eta)$.

Step 3: If $\sum_{k=1}^{M} \|\theta_k^{n+1} - \theta_k^n\|_2 > \epsilon_0$, set $n = n+1$ and go to Step 1. Otherwise, go to step 4.

Step 4: If $\eta \geq \lambda$, stop. Otherwise, set $\eta = \eta + \Delta\eta$ and go to step 1.

Data without scale-separation

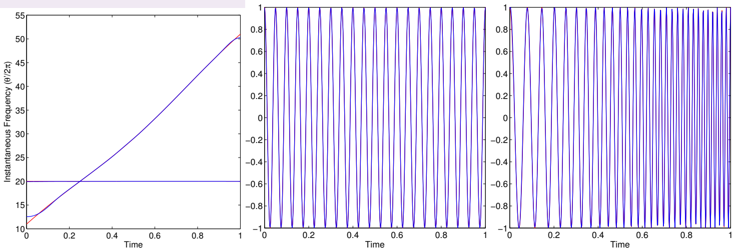


Figure: Left: Instantaneous frequencies; Middle and right: IMFs extracted by extracting two IMFs together. red: exact results; blue: numerical results.

Data with Intra-wave frequency modulation

In the previous approach, we want to decompose the signal to the form

$$f(t) = \sum_{k=1}^{M} a_k(t) \cos \theta_k(t). \tag{17}$$

For some applications, decompositions of above form are too restrictive. For example, the decomposition of following ECG data is not sparse at all.

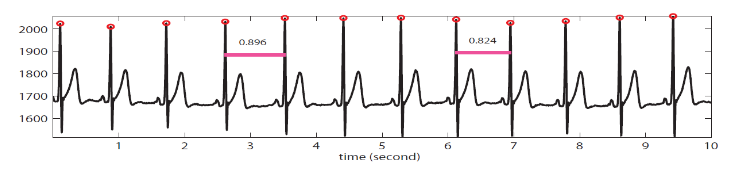


Figure: Typical ECG data

Data with Intra-wave frequency modulation

Using the concept of the shape function introduced by Prof. Daubechies and Wu, we decompose the signal to the following form

$$f(t) = \sum_{k=1}^{M} a_k(t) s_k(\theta_k(t)). \tag{18}$$

where $s_k(\cdot)$ are 2π-period function.

a_k, θ_k, s_k can be obtained by solving following optimization problem:

$$\min_{a_k,\theta_k,s_k} \|f(t) - \sum_{k=1}^{M} a_k(t) s_k(\theta_k(t))\|_2^2,$$

$$\text{subject to:} \quad a_k, \theta_k \text{ are smoother than } \cos\theta_k,$$

$$s_k(\cdot) \text{ is } 2\pi\text{-periodic.}$$

Data with Intra-wave frequency modulation

First, we consider a simple case, let $M = 1$, then we only need to solve

$$\min_{a,\theta,s} \|f(t) - a(t)s\left(\theta(t)\right)\|_2^2,$$
$$\text{subject to:} \quad a, \theta \in V(\theta), \quad s(\cdot) \text{ is } 2\pi\text{-periodic.}$$

The phase function θ can be obtained by the method introduce before, then

$$\min_{a,s} \|f(t) - a(t)s\left(\theta(t)\right)\|_2^2,$$
$$\text{subject to:} \quad a \in V(\theta), \quad s(\cdot) \text{ is } 2\pi\text{-periodic.}$$

Data with Intra-wave frequency modulation

Since s is periodic, it can be represented by Fourier basis,

$$s(\theta) = \sum_{k=-K}^{K} c_k e^{ik\theta} \tag{19}$$

Using this representation, the optimization problem becomes

$$\min_{a, c_k} \|f - a \sum_{k=-K}^{K} c_k e^{ik\theta}\|_{2,\theta}^2,$$

$$\text{subject to:} \quad a \in V(\theta).$$

Data with Intra-wave frequency modulation

Using the Parsarval equality,

$$\min_{\widehat{a},c_k} \|\widehat{f}_\theta(\omega) - \sum_{k=-K}^{K} c_k \widehat{a}(\omega + kL_\theta)\|_2^2,$$

$$\text{subject to:} \quad a \in V(\theta).$$

where $L_\theta = (\theta(T) - \theta(0))/2\pi$.

Using the constraint that $\widehat{a}(\omega) = 0, \ |\omega| > \lambda L_\theta$ and $\lambda \leq 1/2$.

$$\min_{\widehat{a},c_k} \sum_{k=-K}^{K} \sum_{|\omega|<\lambda L_\theta} |\widehat{f}_\theta(\omega + kL_\theta) - c_k \widehat{a}(\omega)|^2,$$

which can be solved by SVD.

Data with Intra-wave frequency modulation

Algorithm to compute s and the envelop a,

- Compute the phase function θ.
- Interpolate the signal f to θ coordinate and apply FFT to get the Fourier coefficients of f over θ coordinate, $\widehat{f}_\theta$.
- Chop $\widehat{f}_\theta$ to several pieces to form a matrix $\widehat{\mathbf{F}}_\theta$.
- Apply the singular value decomposition on $\widehat{\mathbf{F}}_\theta$ to get c_k and $\widehat{a}$.

Data with intra-wave frequency modulation: Duffing equation

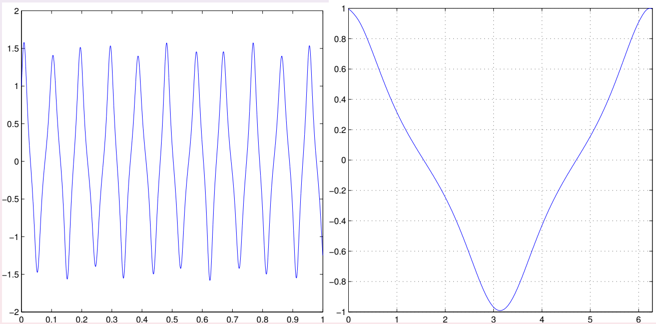


Figure: Left: The solution of the Duffing equation; Right: The shape function obtain by our method.

Data with intra-wave frequency modulation: Duffing equation

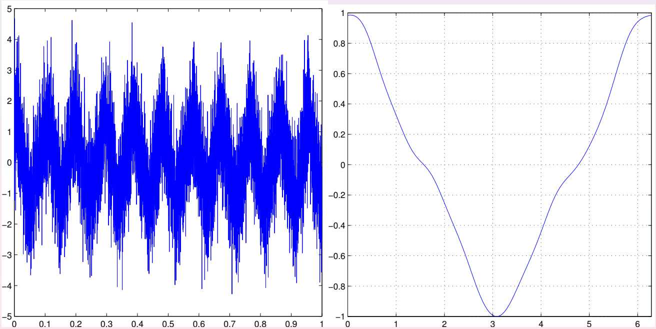


Figure: Left: The solution of the Duffing equation plus noise; Right: The shape function obtain by our method.

Data with intra-wave frequency modulation: ECG data

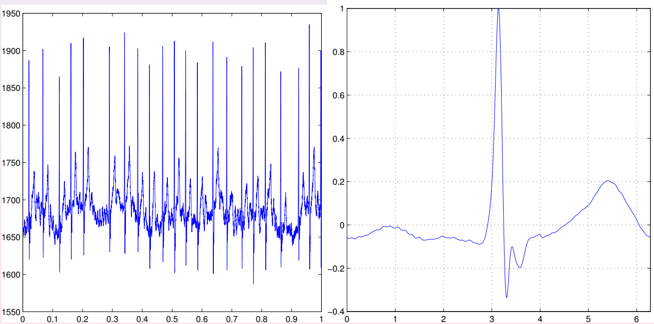


Figure: Left: The original ECG data; Right: The shape function obtain by our method for the ECG data.

Data with intra-wave frequency modulation: Nonlinear Dynamical System

For each IMF, there exists a second order ODE,

$$\ddot{x} + a(x,t)\dot{x} + b(x,t) = 0 \tag{20}$$

and $a(x,t)$, $b(x,t)$ are slowly varying with respect to t.

If we looking for $a(x,t)$ and $b(x,t)$ locally, they can be approximate by two functions which are independent on t, then the ODE becomes autonomous,

$$\ddot{x} + a(x)\dot{x} + b(x) = 0. \tag{21}$$

or we can rewrite it to the conservative form,

$$\ddot{x} + \dot{Q}(x) + b(x) = 0. \tag{22}$$

Data with intra-wave frequency modulation: Nonlinear Dynamical System

Then we can get weak formulation of the ODE,

$$< x, \ddot{\phi} > - < Q(x), \dot{\phi} > + < b(x), \phi >= 0, \quad \forall \phi \in C_0^\infty([0, T]). \quad (23)$$

In order to determine the ODE, we use polynomials to approximate $Q(x)$ and $b(x)$,

$$Q(x) = \sum_{k=0}^{M} q_k x^{k+1}, \quad b(x) = \sum_{k=0}^{M} b_k x^k \quad (24)$$

where M is the order of polynomials which is given *a prior*, q_k, b_k are constants.

Data with intra-wave frequency modulation: Nonlinear Dynamical System

Using this formulation, we can design following optimization problem to solve a_k and b_k,

$$
\begin{aligned}
(q_k, b_k) \quad = \quad & \arg\min_{\alpha_k,\beta_k} \gamma \sum_{k=1}^{M} (|\alpha_k| + |\beta_k|) \\
& + \sum_{i=1}^{N} \left| < x, \ddot{\phi}_i > - \sum_{k=0}^{M} \alpha_k < x^{k+1}, \dot{\phi}_i > + \sum_{k=0}^{M} \beta_k < x^k, \phi_i > \right|^2 .
\end{aligned}
$$

The test function we use is the following cosine type function,

$$
\phi_i(t) = \begin{cases} \frac{1}{2}(1+\cos(\pi(t-t_i)/\lambda)), & -\lambda < t - t_i < \lambda, \\ 0, & \textit{otherwise}. \end{cases} \qquad i = 1, \cdots, N
$$

where $t_i, i = 1, \cdots, N$ is the centers of the test functions and the parameter λ determines their support.

Data with intra-wave frequency modulation: Nonlinear Dynamical System

The Duffing equation is a nonlinear ODE which has the following form:

$$\ddot{u} + u + u^3 = 0 \tag{25}$$

$u(0) = 1, \dot{u}(0) = 0.$

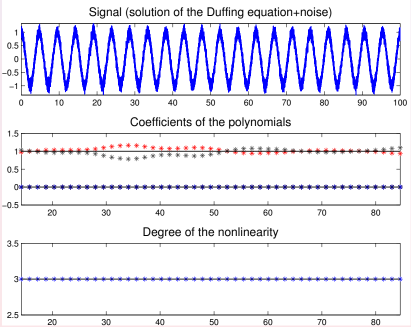


Figure: Top: the solution of the Duffing equation; Middle: Coefficients of polynomials; Bottom: Degrees of nonlinearity.

Data with intra-wave frequency modulation: Nonlinear Dynamical System

The Van der Pol equation is a nonlinear ODE which has the following form:

$$\ddot{u} + (u^2 - 1)\dot{u} + u = 0 \tag{26}$$

$u(0) = 1, u'(0) = 0.$

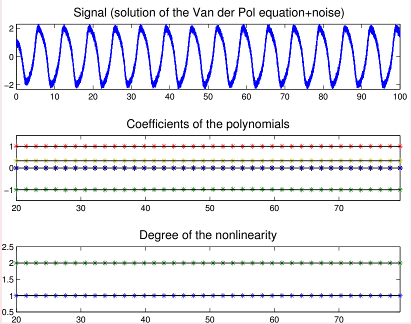


Figure: Top: the solution of the Van der Pol equation; Middle: Coefficients of polynomials; Bottom: Degrees of nonlinearity.

Data with intra-wave frequency modulation: Nonlinear Dynamical System

Consider a ODE with varying coefficients,

$$\ddot{u} + a(t)(u^2 - 1)\dot{u} + (1 - a(t))u^3 + u = 0 \tag{27}$$

where $a(t) = \frac{1}{2}\left(1 - \frac{t-100}{\sqrt{(t-100)^2+400}}\right)$. The initial condition is that $\dot{u}(0) = 0, u(0) = 1$ and the equation is solved over $t \in [0, 200]$.

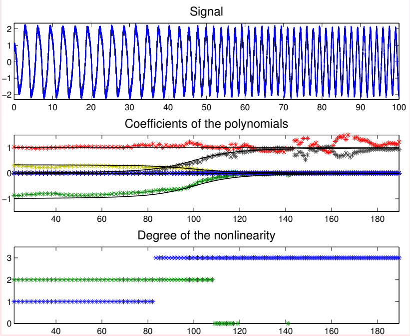


Figure:

Data with intra-wave frequency modulation: Nonlinear Dynamical System

Consider a ODE with varying coefficients,

$$\ddot{u} + a(t)(u^2 - 1)\dot{u} + (1 - a(t))u^3 + u = 0 \qquad (28)$$

where $a(t) = (1 - \text{sgn}(t - 100))/2$. The initial condition is that $\dot{u}(0) = 0, u(0) = 1$ and the equation is solved over $t \in [0, 200]$.

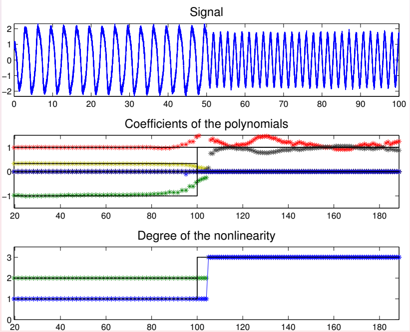


Figure:

Data with intra-wave frequency modulation: Nonlinear Dynamical System

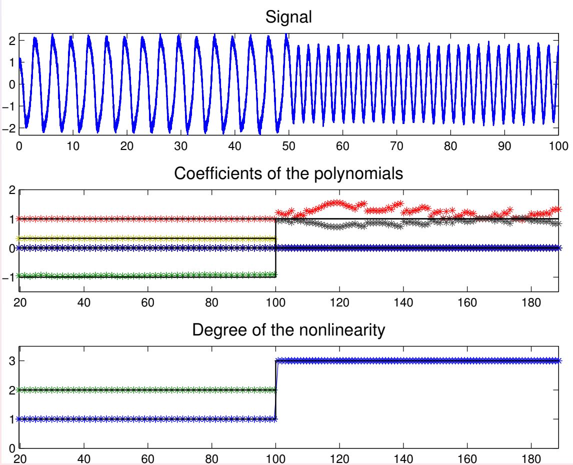


Figure:

Concluding Remarks

- We generalize the data-driven time-frequency analysis method proposed by Hou and Shi to several more complicated data sets.
- Convergence analysis has been performed under some scale separation assumption on the mutiscale data.
- One nonlinearity analysis method based on second order ODE has been developed to deal with the nonlinear data.
- Applications to biomedical problems, climate data, and geophysical problems are under investigation.

Reference:

- T. Y. Hou and Z. Shi, Adaptive Data Analysis via Sparse Time-Frequency Representation, *Advances in Adaptive Data Analysis*, **3**, pp. 1-28, 2011.
- T. Y. Hou and Z. Shi, Data-driven Time-Frequency Analysis, to appear in ACHA.

Thank you!